Mathematics

for the IB Middle Years Programme

MYP Years 4+5 Standard

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Contents

Course Structure

Mathematics can be fun!

'I can't do it!'

Have you ever thought or exclaimed these words when stumped by a mathematics problem? I bet every one of us has said these words to themselves at least once in their lifetime. And not just because of a mathematics problem! In order to be engaged as a learner, regardless of age, we like to experience things in a fun and interactive way. Not only that, learning only happens when we leave our comfort zone.

This series is dedicated to the idea that mathematics can be (and is) fun.

Our mission is to accompany you, dear learner, out of your comfort zone and towards the joy of mathematics. Do you accept this challenge?

Bearing in mind the latest research about learning mathematics, the driving ideas behind this series are the following:

- We believe that everyone can do mathematics. Of course there are a few that find it 'easier' than others, *but mathematics learning*, done right, *is for all*. We believe in you, the learner.
- *The essence of mathematics is solving problems*. We will work together to help you become a better problem solver. Problem solvers make mistakes. Plenty of them. With perseverance, they end up solving their problems. You can too. Making mistakes and learning from them is part of our education. Our approach is backed by research on *growth mindsets* and follows in the steps of George Pólya, the father of problem solving.
- Other than some special inventions, most of our societies' development is done by groups. That is why we will, with the help of your teachers, support you to achieve your goals within a group environment.
- Mathematicians' work, no matter how 'advanced' the result, starts with an exploration. Ideas do not magically materialise to a mathematician's mind by superpowers. Mathematicians work hard, and while working, discoveries are made. Whoever discovered gravitational forces did not sit back and then all of a sudden come out with the idea. It was observation first.
- Once you have an idea, you can investigate it to develop your understanding in more depth. We have included many opportunities for you to expand your knowledge further.

Fact

The legend is that Isaac Newton discovered gravity when he saw a falling apple while thinking about the forces of nature. Whatever really happened, Newton realised that some force must be acting on falling objects like apples because otherwise they would not start moving from rest.

It is also claimed that Indian mathematician and astronomer Brahmagupta-II (598–670) discovered the law of gravity over 1000 years before Newton (1642–1727) did. Others claim that Galileo discovered it 100 years before Newton.

How to use this book

No one can teach you unless you want to learn. We believe that, through this partnership with you, we can achieve our goals.

In this book, we have introduced each concept with an Explore. First and foremost, when you start a new concept, try to do the Explore. Have courage to make guesses but try to justify your guesses. Remember, it is ok to make mistakes. Work with others to analyse your mistakes.

Explore 9.4

Look back at the data for rolling a dice 36 times from Explore 9.3.

How would you represent this data to make it easy to read?

Throughout this book, you will find worked examples. When you are given a worked example, do not jump immediately to the solution offered. Try it yourself first. When you do look at the solution offered, be critical and ask yourself: could I have done it differently?

Worked example 8.3

Marta wants to cut 1 metre of ribbon into three equal lengths.

How long should each length be?

Give your answer to a suitable degree of accuracy.

Solution

$100 \div 3 = 33.333\ldots$ cm

It is not possible to measure 33.333… cm accurately.

Each length is 33.3 cm (to the nearest millimetre).

At the end of any activity, we encourage you to reflect on what you have done. There is always a chance to extend what you have learned to new ideas or different perspectives. Not only in studying mathematics, but in any task you should always take the opportunity to reflect on what you did. You will either feel that the task is completed, or you may find that you need to improve on some parts of it. This is true whether you are a student, a teacher, a parent, an engineer, or a business leader, to mention a few. You will find reflection boxes throughout the book to help you with this.

Reflect

In Worked example 8.3, how did it help to convert the measurements to cm?

Can you round 5.26 m to the nearest 10 cm without converting to cm first?

At the end of each section of the book, you will find practice questions. It is recommended that you do these, and more, until you feel confident that you have mastered the concept at hand.

Practice questions 8.2

1 Write 7.517 metres as ___ m ___ cm ____ mm

2 Write 3 m 24 cm 5 mm:

  a in centimetres, to the nearest mm

  b in metres, to the nearest mm.

Instead of summarising each chapter for you, we have you review what you learned from the chapter in a self-assessment. These self-assessments are checklists. Look at them, and if you feel you missed something, revisit the section covering it.

Self assessment

- [ ] I can identify natural numbers, integers and real numbers.
- [ ] I can identify and use the place value of digits in natural numbers up to hundreds of millions.
- [ ] I can identify and use the place value of digits in decimals.

Finally, at the end of every chapter, it is good practice to look back at the chapter as a whole and see whether you can solve problems. Each chapter contains check your knowledge questions for this purpose.

Check your knowledge questions

1 Complete these measurement conversions.

a $420\,\text{m} = \square\ \text{cm}$ b $530\,\text{cm} = \square\ \text{m}$

c $0.4\,\text{km} = \square\ \text{mm}$ d $546\,\text{mm} = \square\ \text{cm}$

e $2450\,\text{m} = \square\ \text{km}$ f $3.4\,\text{m} = \square\ \text{mm}$

2 How many books of width 18 mm will fit on a shelf 1 m long?

During your course, your teacher will help you work in groups. In group work, ask for help and help others when asked. The best way of understanding an idea is when you explain it to someone else.

Remember, mathematics is not a bunch of calculations. Mathematical concepts must be communicated clearly to others. Whenever you are performing a task, justify your work and communicate it clearly.

Additional features

Matched to the latest MYP Mathematics Subject Guide

Key concepts, related concepts and global contexts

Each chapter covers one key concept and one or more related concepts in addition to being set within a global context to help you understand how mathematics is applied in our daily lives.

KEY CONCEPT

Relationships

RELATED CONCEPTS

Patterns, Quantity, Representation, Systems

GLOBAL CONTEXT

Globalisation and sustainability

Statement of inquiry and inquiry questions

Each chapter has a statement of inquiry and inquiry questions that lead to the exploration of concepts. The inquiry questions are categorised as factual, conceptual and debatable.

Statement of inquiry

Using number systems allows us to understand relationships that describe our climate, so we are able to acknowledge human impact on global climate change.

Approaches to learning tags

We have identified activities and questions that have a strong link to specific approaches to learning to help you understand where you are using particular skills.

Thinking skills

Do you recall?

At the start of each chapter, you will find do you recall questions to remind you of the relevant prior learning before you start a new chapter. Answers to the do you recall questions can be found in the answers section at the back of the book.

Do you recall?

1 What are directed numbers?

2 What are the number operations?

3 What mental methods do you know for adding two 2-digit numbers?

4 What mental methods do you know for subtracting from a 2-digit number?

Investigations

Throughout the book, you will find investigation boxes. These investigations will encourage you to seek knowledge and develop your skills. They will often provide an opportunity for you to work with others.

Investigation 1.1

Collect magazine, newspaper or online articles that use global temperatures, sea levels and carbon dioxide emissions. Explain in each case what the data tells you.

Research temperatures and the amount of rainfall for five different locations on the same day or month each year for 20 years. What does your data show you?

Fact boxes

Fact boxes introduce historical or background information for interest and context.

Fact

A pescatarian is someone who eats fish, but does not eat any other meat.

Hint boxes

Hint boxes provide tips and suggestions for how to answer a question.

Hint Q9

Note the different units.

Reminders

These boxes are used to recap previous concepts or ideas in case you need a refresher.

Reminder

Always state how you have rounded the measurement in the final answer.

Connections

These boxes highlight connections to other areas of mathematics, or even other subjects.

Connections

You learned how to measure and draw angles accurately in Chapter 4.

Challenge Q12

Challenge tags

We have identified challenging questions that will help you stretch your understanding.

This series has been written with inquiry and exploration at its heart. We aim to inspire your imagination and see the power of mathematics through your eyes.

We wish you courage and determination in your quest to solve problems along your your MYP mathematics journey. Challenge accepted.

Ibrahim Wazir, Series Editor

A note for teachers

Alongside the textbook series, we have also created digital Teacher Guides. These Guides include, amongst other things, ideas for group work, suggestions for organising class discussion using the Explores and detailed, customisable unit plans.

1 Year 3 review

1 Year 3 review

KEY CONCEPT

Relationships

RELATED CONCEPTS

Models, Representation, Systems

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Relationships and systems can be modelled to represent scientific and technological innovations.

Factual

- What determines whether two lines are parallel?
- How do you solve systems of equations graphically?

Conceptual

- How can the solution of an inequality be represented on a number line?
- How do you find the surface area of a 3-dimensional (3D) object?

Debatable

- Is there a best way of solving a system of linear equations?
- Can a statistical representation be misleading?

Do you recall?

1 Perform the following calculations.

a $2\frac{1}{2} \times 3\frac{1}{3}$ **b** $\frac{2}{3} - \frac{1}{2} \times 1\frac{1}{3}$

c $(0.1)^2 + 2.01 - 1.03$ **d** $4.05 \div 0.9 \times 0.01$

2 Simplify $a^4 b^3 \div (a b^2)^2$

3 If Sema earns €1260 in 30 days, how much does she earn per day?

1.1 Numbers

1.1.1 Working with fractions and decimals

Explore 1.1

Can you convert the following decimal to a fraction and fraction to a decimal?

a 21.34 **b** $3\frac{11}{25}$

$\frac{3}{4} = 75\% = 0.75$

Rational numbers or **fractions** are numbers that can be written as $\frac{a}{b}$ where a and b are integers and b is not zero.

Fractions can be expressed as decimals: $\frac{35}{10} = 3.5$

Decimals can be expressed as fractions: $2.15 = \frac{215}{100} = \frac{43}{20} = 2\frac{3}{20}$

Worked example 1.1

Convert the following fraction to a decimal and the decimals to fractions.

a $\frac{12}{9}$ **b** 1.45 **c** $1.\dot{4}\dot{5}$

Challenge Qc

Solution

We need to convert the fractions into decimals and the decimal into a fraction.

To convert a fraction to a decimal we divide the numerator by the denominator. Sometimes it is easier to write the fraction as an equivalent fraction with a power of 10 in the denominator first.

To convert a decimal to a fraction we write the decimal without the decimal point in the numerator and a power of 10 in the denominator. To work out which power of 10, we count the number of decimal places. If the decimal has n decimal places then we need 10^n in the denominator. For a recurring decimal, we need to cancel the recurring part.

a $\frac{12}{9} = 1\frac{3}{9} = 1\frac{1}{3} = 1.333... = 1.\dot{3}$

b $1.45 = \frac{145}{100} = \frac{5 \times 29}{5 \times 20} = \frac{29}{20}$

c Let $x = 1.\dot{4}\dot{5}$

Then $100x = 145.\dot{4}\dot{5}$

We subtract the first equation from the second:

$100x - x = 145.454545\ldots - 1.454545\ldots$

$99x = 144$

$x = \frac{144}{99} = \frac{16}{11}$

We use percentages to calculate discounts.

Percentages can be represented as fractions or decimals, and vice versa.

For example: $15\% = \frac{15}{100} = 0.15$

To calculate a percentage or a fraction of a number, we multiply the number by the percentage or the fraction. For example, to calculate 15% of €200:

$15\% \times 200 = \frac{15}{100} \times 200 = €30$

Worked example 1.2

Express the percentage as a fraction and the fraction as a percentage.

a 12.15% **b** $\frac{3}{7}$

Challenge Qb

Solution

a $12.15\% = \frac{12.15}{100} = \frac{1215}{10\,000} = \frac{243}{2000}$

b $\frac{3}{7} \approx 0.4286 = \frac{42.86}{100} = 42.86\%$

Hint

You could also use your calculator to divide numbers.

Worked example 1.3

Camille uses 25% of her salary for rent, 20% for food, 15% for leisure and 10% for transport. She has €300 left for other things. How much is her salary?

Solution

We are given different percentages of an unknown salary to be found. We are also given the remaining amount. First we find the total percentage she spends, then we find what percentage is left. We know that the remaining percentage represents €300, so we can use this to calculate the monthly salary.

Camille spent 25% + 20% + 15% + 10% = 70% of her salary

The percentage of her salary that is left is 100% − 70% = 30%. We know that this is €300. If 30% is €300, then 1% is €10 and 100% is €1000.

Another way to calculate the salary is to use algebra. Let the unknown salary be s. If 30% of s is €300, then we can represent the situation as an equation:

$$0.3 \times s = 300 \Rightarrow s = \frac{300}{0.3} = 1000$$

Reflect

Can you think of any other ways to approach the problem in Worked example 1.3?

1.1.2 Order of operations

Explore 1.2

Evaluate these two expressions. Are the solutions the same? Why or why not?

$$\frac{3}{4} \div \frac{1}{4} \times \frac{1}{3} \qquad \frac{3}{4} \times \frac{1}{4} \div \frac{1}{3}$$

Fact

The **order of operations** rule helps us to work out expressions correctly when there is more than one operation involved. The rule to solve mixed operations is:

1. (P) Parentheses first: simplify expressions within brackets.
2. (E) Exponents (Indices): squares, cubes, square roots and so on.
3. (DM) Division and Multiplication from left to right.
4. (AS) Addition and Subtraction from left to right.

You can remember this rule as PEDMAS.

Worked example 1.4

Simplify the following expression.

$$\left(\frac{1}{2} - 1\right)^2 \div \left(1 - \frac{1}{3}\right)$$

Fact

The order of operations is also called BIDMAS: Brackets, Indices, Division and Multiplication, Addition and Subtraction.

Solution

First we simplify the expressions in brackets, then we divide.

$$\left(\frac{1}{2}-1\right)^2 \div \left(1-\frac{1}{3}\right) = \left(\frac{1-2}{2}\right)^2 \div \left(\frac{3-1}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^2 \div \left(\frac{2}{3}\right) = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$$

Brackets are done first, then exponents, then multiplication and division from left to right.

Worked example 1.5

Find the value of the variable x in this equation.

$12 \times (x-3) - 3^3 \div 9 \times 2 = 9$

Solution

We need to rearrange the equation to find the value of x.

We can use the PEDMAS rules to simplify the expression on the left side.

$12 \times (x-3) - 3^3 \div 9 \times 2 = 9$	Simplify powers first.
$12 \times (x-3) - 27 \div 9 \times 2 = 9$	From left to right: division then multiplication.
$12 \times (x-3) - 3 \times 2 = 9$	
$12 \times (x-3) - 6 = 9$	Add 6 to both sides of the equation.
$12 \times (x-3) = 15$	Divide both sides of the equation by 12.
$x - 3 = \frac{15}{12}$	Add 3 to both sides.
$x = 4.25$	

We can check our solution by substituting $x = 4.25$ into the original expression.

$$12 \times (4.25-3) - 3^3 \div 9 \times 2 = 12 \times 1.25 - 27 \div 9 \times 2$$

$$= 15 - 3 \times 2 = 15 - 6 = 9$$

Therefore, the value of x is correct.

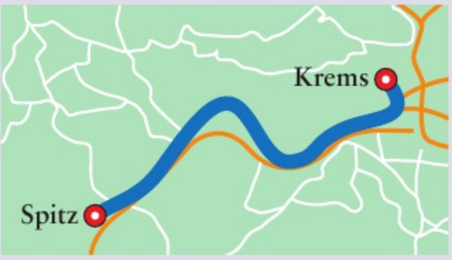
Reflect

Can you think of another method of solving Worked example 1.5?

1.1.3 Ratio, proportion and rate

Explore 1.3

Spitz and Krems are two towns in Lower Austria. The road distance between them is 21 km. The towns are shown on a 1 : 10 000 scale map. What is the distance between the towns on the map along the road?



Fact

When we buy a monitor, we are often interested in the ratio of width to height. A standard monitor usually has a ratio of 4 : 3 whereas a widescreen monitor has a ratio of 16 : 9

The comparison of two quantities in the same units is called a **ratio**. For example, a class has 12 boys and 13 girls. The ratio of boys to girls is 12 : 13 or $\frac{12}{13}$

If two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal, then $\frac{a}{b} = \frac{c}{d}$ and the two ratios are said to be **in proportion**. For example, $\frac{3}{5}$ and $\frac{12}{20}$ are in proportion because $\frac{3}{5} = \frac{12}{20}$
The **cross-product of the diagonals** of the proportion are equal to each other:

$3 \times 20 = 5 \times 12 = 60$

If the units of the numerator and denominator of a ratio are different, the ratio is known as a **rate**. For example, if a car travels 100 km in 2 hours, the rate can be expressed as 100 km : 2 hours or $\frac{100}{2} = \frac{50}{1}$ or 50 km/h

The **unit** of the rate is km/h and we would say that the car travels at 50 km/h

Worked example 1.6

A rectangular swimming pool's dimensions are in the ratio of 2 : 5
Its perimeter is 42 m.

a Find its dimensions. **b** Calculate its area.

Solution

We are given the ratio of length and width as well as the perimeter. We need to find the length, width and area of the pool. We can use the ratio of 2 : 5 and the 42 m perimeter of the rectangle to establish two relationships between the length and the width.

Let a be the width and b be the length.

a From the ratio: $\frac{a}{b} = \frac{2}{5} \Rightarrow a = \frac{2}{5}b$

From the perimeter: $2a + 2b = 42$ or $a + b = 21$

We can substitute the value of a from the first equation into the second equation:

$\frac{2}{5}b + b = 21 \Rightarrow \frac{7}{5}b = 21$

Multiplying both sides by $\frac{5}{7}$ we get:

$b = 21 \times \frac{5}{7} = 15$

Substitute this value for b into the first equation:

$a = \frac{2}{5}b = \frac{2}{5} \times 15 = 6$

Width = 6 m and length = 15 m

b The area of a rectangle is found by multiplying length times width. The area of the rectangle is given by 6×15, so it is $90\,m^2$

Looking back, we see that $\frac{6}{15} = \frac{2}{5}$

Also, the perimeter is given by $2 \times (6 + 15) = 2 \times 21 = 42$, which confirms that the answer is correct.

Reflect

Is there another way of solving Worked example 1.6?

Worked example 1.7

A car travels 150 km from Salzburg to Munich in 2 hours. At the same average speed, how many hours would it take to travel from Munich to Regensburg, 125 km away?

Solution

The average speed is the same in both trips. This means that the rate of distance divided by time is the same in both, so we have a proportion.

We can represent the unknown length of time by x hours.

$$\frac{125}{x} = \frac{150}{2} = 75$$

We can now set up and solve an equation to find x.

$$\frac{125}{x} = 75 \Rightarrow x = \frac{125}{75} = \frac{5}{3} = 1\frac{2}{3} \text{ hours (or 1 hour 40 minutes)}$$

We can check the solution by putting the answers back into the proportion:

$$\frac{150}{2} = \frac{125}{\frac{5}{3}} = 75$$

Worked example 1.8

A map has a scale of 1 : 100 000 (1 cm on the map represents 100 000 cm or 1 km in real life). If the actual road distance between two towns is 12 km, what is the distance between these towns on the map?

Solution

We need to find the distance between two towns on the scaled map. We can use the scale of 1 : 100 000 and the fact that the towns are 12 km apart to construct the proportion and find the map distance between the towns.

Let x cm be the distance between the towns on the map.

Then 12 km = 1 200 000 cm, so

$$\frac{1}{100\,000} = \frac{x}{1\,200\,000}$$

We can solve this equation to find the distance between the towns.

$x = \frac{1\,200\,000}{100\,000} = 12$, and so the distance is 12 cm

1.1.4 Indices (exponents)

Explore 1.4

Did you know that technically one kilobyte is not 1000 bytes but 1024 bytes? Some systems use the word kibibyte for 1024 bytes and kilobyte for 1000 bytes. Can you find out where 1024 came from? Can you tell how many bytes are technically in one megabyte? How many kilobytes are in one megabyte?

Indices or **exponents** are used to show how many times the same number appears in a product, such as $a \times a \times a \times a \times a = a^5$. For example, $4^3 = 4 \times 4 \times 4 = 64$, where 4 is the **base** and 3 is the **index** or the **exponent**.

To write a number in **index form**, we write it as a base and an index: 4^3

To write a number in expanded form, we write out the full multiplication: $4 \times 4 \times 4$

Worked example 1.9

Convert each expression that is in index form into expanded form and each expression that is in expanded form into index form.

a x^5y^2

b $3 \times 3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 2 \times 2$

c $(-3)^2$

d $5 \times 5 \times 5 \times 5$

Fact

A floppy disk used to be a standard data storage device. They could usually hold 1.44 megabytes of data. These days, a tiny USB stick can hold many times that amount.

Fact

Indices is the plural form of index.

Index form is also called **exponential form**.

The combined base and index is called a **power** of the base, so 64 is a power of 4.

Solution

a This expression has two bases: x and y. The index for x is 5, so x is present 5 times in the product. The index for y is 2, so y is present twice in the product.

$x \times x \times x \times x \times x \times y \times y$

b In this expression, 3 is present 5 times in the product and 2 is present 4 times.

$3^5 \times 2^4$

c $(-3) \times (-3)$

d 5^4

There are six **laws of indices** that help you to solve problems.

Law 1: The zeroth power of any non-zero number is 1.

$a^0 = 1, a \neq 0$

For example, $100^0 = 1$

Law 2: The first power of any number is itself.

$a^1 = a$

For example, $7^1 = 7$

Law 3: When two numbers with the same base are multiplied together, their indices are added.

$a^m \times a^n = a^{m+n}$

For example, $2^3 \times 2^2 = 2 \times 2 \times 2 \times 2 \times 2 = 2^{3+2} = 2^5$

Law 4: When two numbers with the same base are divided, the index of the divisor is subtracted from the index of the dividend.

$\frac{a^m}{a^n} = a^{m-n}$

For example, $\frac{5^4}{5^3} = \frac{5 \times 5 \times 5 \times 5}{5 \times 5 \times 5} = 5^{4-3} = 5^1$ or 5

and $\frac{3^2}{3^7} = 3^{2-7} = 3^{-5} = \frac{1}{3^5}$

Law 5: If there are indices of indices, then the indices are multiplied.

$(a^m)^n = a^{m \times n}$

For example, $(2^3)^2 = 2^{3\times 2} = 2^6$

This law can be justified by using Law 3: $(2^3)^2 = 2^3 \times 2^3 = 2^{3+3} = 2^6$

Connections

These laws include examples of an extension of a definition. Initially saying that 3^5 means the product of five 3s, i.e. $3 \times 3 \times 3 \times 3 \times 3$, is not of help in understanding the meaning of 3^{-5}, but the rules which apply to the original definition lead by pattern to an obvious extension, requiring a broader definition.

You will see the same type of extension in trigonometry when definitions initially applied to right-angled triangles are extended to apply to angles of any size, including those far too big to fit in a triangle.

Law 6: If a number has a negative index, it can be written as the reciprocal of the number with a positive index.

$$a^{-m} = \frac{1}{a^m}$$

For example, $3^{-1} = \frac{1}{3^1} = \frac{1}{3}$

Worked example 1.10

Simplify these expressions using the laws of indices.

a $6^2 \times 6^3 \times 6^{-2}$ **b** $\frac{2^3 \times 2^4}{2^5}$

c $(10^2)^3$ **d** $9^1 \times 9^5 \div 9^4$

Solution

a $6^2 \times 6^3 \times 6^{-2} = 6^{2+3+(-2)}$
$= 6^3$
$= 6 \times 6 \times 6$
$= 216$

Law 3: When two numbers with the same base are multiplied together, their indices are added.

b $\frac{2^3 \times 2^4}{2^5} = \frac{2^{3+4}}{2^5} = \frac{2^7}{2^5}$ Law 3

$= 2^{7-5} = 2^2 = 4$

Law 4: When two numbers with the same base are divided, their indices are subtracted.

c $(10^2)^3 = 10^{2\times3} = 10^6 = 1\,000\,000$ Law 5: If there are indices of indices, then the indices are multiplied.

d $9^1 \times 9^5 \div 9^4 = 9^{1+5-4} = 9^2 = 81$ Laws 3 and 4

Challenge

Worked example 1.11

Simplify these expressions.

a $(a^{12} \times a^8 \div a^{-5})^0$, $a \neq 0$ **b** $3^{3^2} \times 3$ **c** $(5 \times 2^2)^3$

Solution

a The zeroth power of any non-zero number is 1, so the value of the calculation inside the parentheses does not matter.

$(a^{12} \times a^8 \div a^{-5})^0 = 1$

b $3^{3^2} \times 3 = 3^{3\times3} \times 3^1$ Law 2

$= 3^{9+1} = 3^{10} = 59\,049$ Law 3

c $(5 \times 2^2)^3 = (5 \times 4)^3$

$= 20^3 = 20 \times 20 \times 20 = 8000$

Hint Qb

When there are several levels of powers with no brackets, start from the top and work down.

Reflect

Can we find different combinations of the indices laws to simplify each expression in Worked example 1.11? Look at the method shown below as an alternative for part c, then try to come up with a different approach for parts a and b.

$$(5 \times 2^2)^3 = (5 \times 2^2) \times (5 \times 2^2) \times (5 \times 2^2)$$

$$= (5 \times 4) \times (5 \times 4) \times (5 \times 4)$$

$$= 8000$$

Practice questions 1.1

1 Convert the following into decimals.

a $\frac{11}{40}$ **b** $\frac{12}{18}$ **c** $2\frac{33}{50}$ **d** $\frac{27}{20}$

2 Write these as simple fractions.

a 2.3 **b** 0.75 **c** 1.555… **d** 0.66

3 Complete the following table. The first row has been done for you.

Fraction	Decimal	Percentage
$\frac{35}{40}$	0.875	87.5%
	$3.\dot{1}$	
		58%
$2\frac{3}{8}$		
	0.63	

4 Vigo spends 10% of his money on books. If he spent €12.50 on books, how much money does he have left?

5 Simplify these expressions.

a $10\frac{2}{4} \times \frac{8}{12} \div 1\frac{1}{2}$

b $1.21 + 12.1 \div 10$

c $2\frac{1}{3} \div 3\frac{1}{2} \times \frac{2}{3}$

d $(1.2)^2 \div 12$

6 Simplify these ratios.

a 12 : 26

b 14 cm to 35 cm

c 21 boys : 30 girls

d $\frac{120\text{ m}}{88\text{ m}}$

7 Find the missing variable(s) in these proportional expressions.

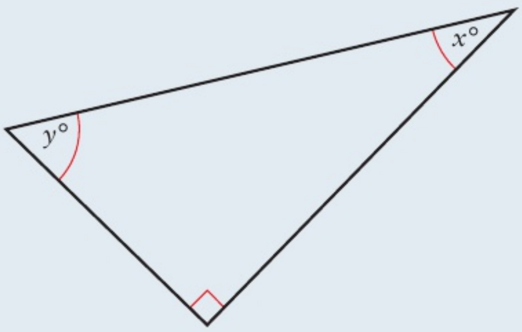
a $\frac{18}{24} = \frac{x}{36}$

b $\frac{2}{7} = \frac{36}{y}$

c $\frac{33}{z} = \frac{55}{75}$

d $3:2:1 = 12:w:t$

8 Find the angles $x°$ and $y°$ in this triangle if the ratio $x:y = 4:5$



9 Complete the following to convert these rates to their equivalents in the given units.

a 8 km/h = ☐ m/min

b 12 km/L = ☐ m/mL

c 1200 g/h = ☐ kg/min

d 12 500 cm³/g = ☐ m³/kg

10 Simplify these expressions.

a $10a^2b^3 \times 3ab^2$

b $(4xy^3)^2$

c $48a^3b^2 \div 12a^2b$

d $120x^5 \div 24x^3y$

1.2 Algebra

1.2.1 Algebraic operations

Explore 1.5

Can you simplify these expressions?

a $11a - 9 - 22x + 14a + 11 + 25x$ **b** $\dfrac{3xy}{6x^2} \times \dfrac{5x^2}{10y}$

We can summarise common algebraic operations in three groups.

1. Add or subtract like terms in an expression. Like terms are the terms with exactly the same variables.

   To simplify the expression $3a - 12 + 4b - 5a + 3b + 7$ we add or subtract the like terms: $(3a - 5a) + (4b + 3b) + (-12 + 7) = -2a + 7b - 5$

2. Multiply or divide algebraic expressions even if they have unlike terms. Multiply or divide the coefficients first, then the variables.

   $12xy \times -3axy^2 = (12 \times -3) \times (xy \times axy^2) = -36ax^2y^3$

   $36x^2y^3z \div 12xy^2z = (36 \div 12)(x^2y^3z \div xy^2z) = 3xy$

3. When you simplify mixed algebraic expressions, apply the order of operations (PEDMAS).

   $$\begin{aligned} 2x^2 - 2 \times (3x)^2 &= 2x^2 - 2 \times 9x^2 \\ &= 2x^2 - 18x^2 \\ &= -16x^2 \end{aligned}$$

In this example, we expanded the parentheses by squaring the $3x$ term, then we did the multiplication and finally we subtracted like terms to reduce the expression to its simplest form.

Worked example 1.12

Simplify each expression.

a $2(x - 2y) + 3(y - 2x)$ **b** $\dfrac{2x}{3} - \dfrac{x}{2} \times \dfrac{3}{5}$ **c** $\dfrac{xy^2}{6} \div \dfrac{2x^2y}{12}$

Solution

a We start by expanding the brackets, then collect like terms.

$2(x - 2y) + 3(y - 2x) = 2x - 4y + 3y - 6x = -4x - y$

b $\frac{2x}{3} - \frac{x}{2} \times \frac{3}{5} = \frac{2x}{3} - \frac{3x}{10}$ Do the multiplication first.

$= \frac{2x \times 10 - 3x \times 3}{3 \times 10}$ Write the fractions with a common denominator.

$= \frac{20x - 9x}{30}$

$= \frac{11x}{30}$ Collect like terms in the numerator.

c $\frac{xy^2}{6} \div \frac{2x^2y}{12} = \frac{xy^2}{6} \times \frac{12}{2x^2y}$

$= \frac{12xy^2}{12x^2y}$

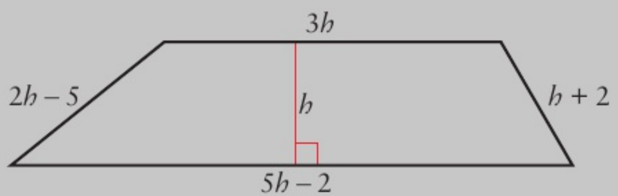
$= \frac{y}{x}$

Worked example 1.13

A trapezium has the measures shown in the diagram below.

a Find the perimeter and the area of the trapezium.

The measures of a similar trapezium are given in terms of h as shown:



b Find the perimeter and the area of this trapezium and compare with the answers in part a.

Solution

a The perimeter is the sum of the sides:

$45 + 17 + 73 + 25 = 160$

area is (average of parallel sides) × (distance between them):

$\frac{45 + 73}{2} \times 15 = 885$

b The perimeter is the sum of the sides:

perimeter: $3h + h + 2 + 5h - 2 + 2h - 5 = 11h - 5$

area is (average of parallel sides) × (distance between them):

$$\frac{3h + 5h - 2}{2} \times h = \frac{8h - 2}{2} \times h = h(4h - 1)$$

Looking back at the perimeter and area, we can check the answers for congruent trapezia, since taking the value $h = 15$ produces corresponding equal dimensions.

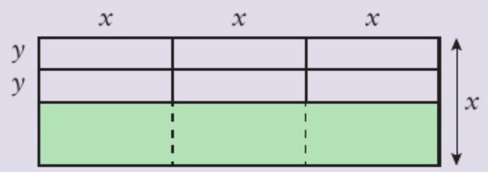
perimeter: $11h - 5 = 11 \times 15 - 5 = 160$

area: $h(4h - 1) = 15 \times (60 - 1) = 885$

1.2.2 Expansion and factorisation

Explore 1.6

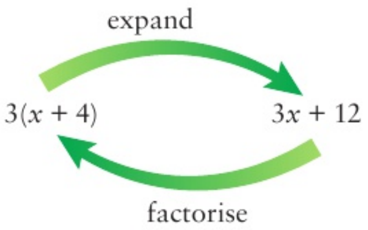
Here is a diagram with segments as shown. Can you write algebraic expressions, in at least two ways, to represent the area of the shaded region?



Expansion and factorisation of algebraic expressions are inverse operations.

$3(x + 4) = 3x + 12$ is expansion.

$3x + 12 = 3(x + 4)$ is factorisation.



Worked example 1.14

Expand the following expressions.

a $4(2a + 5b - 3c)$ **b** $(a - 2b)(a + 2b)$

Solution

a Multiply each term inside the brackets by 4.

$4(2a + 5b - 3c) = 8a + 20b - 12c$

Reminder

Remember that a reliable method for multiplying out binomial brackets is called FOIL, standing for First, Outside, Inside, Last. For example:

Outside

First

$$(2x - 3)(x + 4) = 2x^2 + 8x - 3x - 12$$

Inside

Last

b Multiply each term inside the first brackets by each term inside the second brackets.

$$\begin{aligned}(a - 2b)(a + 2b) &= a(a + 2b) - 2b(a + 2b)\\ &= a^2 + 2ab - 2ab - 4b^2\\ &= a^2 - 4b^2\end{aligned}$$

Worked example 1.15

Factorise each expression.

a $xy + yz + yw$ **b** $a^2 + ab + ac + bc$ **c** $a^2 - b^2$

Solution

a Look for a common factor. In this case, each term has a factor of y.

$$xy + yz + yw = y(x + z + w)$$

b The first two terms have a common factor of a, and the second two terms have a common factor of c.

$$\begin{aligned}a^2 + ab + ac + bc &= a(a + b) + c(a + b)\\ &= (a + b)(a + c)\end{aligned}$$

Now we have two terms with a common factor of $(a + b)$.

c We either recognise this as a difference of two squares:

$$a^2 - b^2 = (a - b)(a + b)$$

or we can add and subtract the term ab:

$$\begin{aligned}a^2 - b^2 = a^2 - ab + ab - b^2 &= a(a - b) + b(a - b)\\ &= (a - b)(a + b)\end{aligned}$$

Note that $(a - b)$ is a common factor.

Worked example 1.16

Factorise the quadratic expression $x^2 - 6x + 5$

Solution

To factorise $x^2 + bx + c$, we look for the factors of the constant term c that add up to the linear coefficient b. The only two factors of 5 that add up to -6 are -1 and -5, so:

$$x^2 - 6x + 5 = (x - 5)(x - 1)$$

$-5x$

$-x$

The sum of these two products gives us the $-6x$ term.

Reflect

The method in Worked example 1.15 is called **factorising by grouping**. Is it possible to factorise all trinomial quadratics (three-term quadratics) using this method? Can you use the same method to factorise the expression $x^2 + 12x + 11$? What about $x^2 + 5x + 11$?

1.2.3 Equations and inequalities

Explore 1.7

The time, T, taken by a pendulum for one swing is given by the formula $T = 2\pi\sqrt{\frac{l}{g}}$ where $g = 9.8\ \text{m/s}^2$ and l is the length of the pendulum in metres.

If the time for one swing is 5 s, what is the length of the pendulum, to the nearest cm?

An equation is an algebraic expression that involves the equal sign, =
An inequality involves one or two of the following signs:

- less than $<$
- greater than $>$
- less than or equal to $\leqslant$
- greater than or equal to $\geqslant$

Algebraic expression	Equation	Inequality
$2x + 5$	$2x + 5 = 17$	$2x + 5 > 17$

To solve an equation or an inequality, we apply the same inverse operations on both sides of the equation or the inequality until the variable is isolated.

Fact

Foucault's pendulum was created as an experiment to demonstrate the Earth's rotation.

Hint

An equation is like an old-fashioned scale. Both sides have to be the same for the scale to balance.

Worked example 1.17

Solve the two-step equation and inequality. Show the solutions on a number line.

a $3y - 7 = 25$ **b** $3y - 7 < 25$

Solution

We need to solve the equation and inequality to find the value of the unknown variable and graph the solution on a number line.

We apply the same inverse operations on both sides of the equation or inequality. For both parts, y is multiplied by 3 and then 7 is subtracted from the result. The inverse operations are adding 7 to both sides and then dividing by 3.

a $3y - 7 = 25$

$3y - 7 + 7 = 25 + 7$ Add 7 to both sides and simplify.

$3y = 32$

$\frac{3y}{3} = \frac{32}{3}$ Divide both sides by 3 and simplify.

$y = 10.\dot{6}$

b $3y - 7 < 25$

$3y - 7 + 7 < 25 + 7$

$3y < 32$ Follow similar steps to part a.

$\frac{3y}{3} < \frac{32}{3}$

$y < 10.\dot{6}$

We can substitute each answer into the original equation or inequality to verify that the solution is correct.

a $y = \frac{32}{3}$, so $3y - 7 = 3\frac{32}{3} - 7 = 32 - 7 = 25$ which is true.

b Choose any value less than $10.\dot{6}$. Choose $y = 10$ for example,

$3y - 7 = 3 \times 10 - 7 = 30 - 7 = 23 < 25$

Reminder

The inequality was strict (no equals sign) so is a limit, but not included in the solution set, therefore an open circle is used on the number line.

Practice questions 1.2

1 Simplify each of these algebraic expressions.

a $3x + 5a - 5x + 2a$

b $3(x + y) - 2(3x - y)$

c $2ab - 5bc + 3ab + 7cb$

d $-2x \times 4y + 5xy$

e $\dfrac{45x^2w^2}{15xw^3}$

f $169ab^2c^3 \div 13a^3b^2c$

2 Simplify each of these algebraic fractions.

a $\dfrac{x}{3} - \dfrac{x}{2} + x$

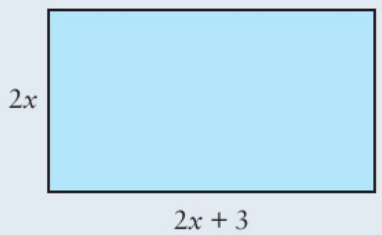
b $\dfrac{2y}{3} + \dfrac{y}{4} + \dfrac{y}{12}$

c $\dfrac{2x^2}{5} \times \dfrac{4x^3}{3}$

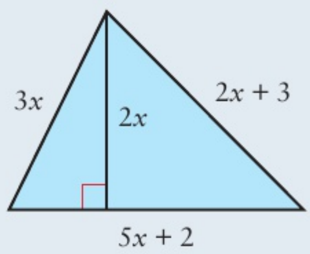
d $\dfrac{3xy^2}{5} \div \dfrac{12x^3}{10}$

3 Find the perimeter and area of the following shapes.

a



b



4 Solve each of these equations.

a $3x + 7 = 43$

b $\dfrac{2x - 6}{3} = 4$

c $3(y + 1) - (y - 1) = 32$

d $2(3x - 1) = 5(x + 2)$

Hint Q4d

Expand both sides first and then subtract a sufficient amount of x from both sides to leave a term in x only on one side.

5 Solve each inequality and graph the solution on a number line.

a $3x - 2 \leqslant 7$

b $\dfrac{2x - 1}{5} \geqslant 1$

c $12x - 13 > 8x + 11$

d $\dfrac{a}{3} - \dfrac{a}{2} < 3$

6 Expand each expression.

a $3x(2x + 3)$

b $(a - 2)(a + 2)$

7 Factorise fully the given expressions.

a $3x^2 - 6x$

b $y(y - 1) + 6y$

c $-6x^2y - 12xy^2$

d $x^2 - 12x - 13$

Challenge Q7d

Challenge Q8a

Hint Q8b

The volume of a cube is calculated by taking the cube of a side length.

Challenge Q8c

Challenge Q8e

8 Write an equation for each of these problems and solve them.

a The perimeter of a rectangle is 20 cm. Its area is 16 cm^2. Find the maximum possible length of the rectangle.

b The volume of a cube is 3375 cm^3. Find the side length of the cube.

c The perimeter of a rectangle is 80 cm and its area is 375 cm^2. Find its dimensions.

d The sum of three consecutive even numbers is 72. Find the numbers.

e The sum of two numbers is 22 and their difference is 8. Find the product of the numbers.

1.3 Geometry

1.3.1 Coordinate geometry

Fact

All points in the Cartesian plane have x and y coordinates and can be written as (x, y). An equation involving x and y puts a restriction on the coordinates, limiting them to a certain locus – in these cases, a straight line.

Explore 1.8

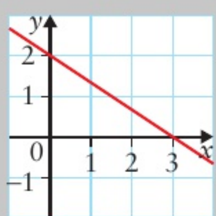
1 Do the points $A(-12, 10)$, $B(8, 0)$, $C(4, -6)$ and $D(-16, 4)$ form a parallelogram?

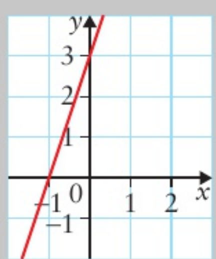
2 Can you sketch the graph of the straight line described by $y = -3x + 1$?

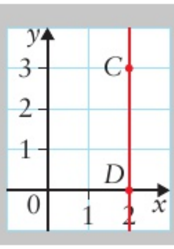
Equations in the form $y = mx + c$ (**gradient–intercept form**) or $ax + by + c = 0$ (**standard form**) describe linear relationships whose graphs are **straight lines** on a coordinate plane. To graph a straight line, we need a minimum of two points on the line.

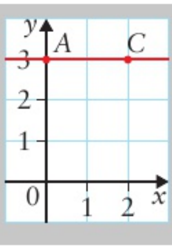
Worked example 1.18

Find an equation of each line by using the gradient and the y-intercept.

a


b


c 

d 

Solution

We need to find an equation of each straight line. For parts a and b we need to find the gradient, m, and the y-intercept, c. For part c we need to find the x-intercept. For part d we need to find the y-intercept.

a y-intercept: $c = 2$

gradient: $m = \frac{-2}{3}$ $\qquad \frac{\text{rise}}{\text{run}}$

The equation of the line is $y = \frac{-2}{3}x + 2$

b y-intercept: $c = 3$

gradient: $m = \frac{3}{1} = 3$

The equation of the line is $y = 3x + 3$

c The gradient is not defined for vertical lines. The line passing through points $D(2,0)$ and $C(2,3)$ is the vertical line with equation $x = 2$, since all points on the line have an x-coordinate of 2.

d The gradient is zero for horizontal lines. The line passing through points $A(0,3)$ and $C(2,3)$ is the horizontal line with equation $y = 0x + 3$ or $y = 3$, since all points on the line have a y-coordinate of 3.

Worked example 1.19

Draw the graphs of $y = 2x + 1$ and $y = x + 3$ on the same coordinate plane and find their point of intersection.

Solution

We need to draw the graphs of both straight lines and find the point common to both lines.

We can draw the graphs of the lines by identifying their x- and y-intercepts and drawing the line passing through the points. When we have drawn the lines, we can identify their intersection point graphically.

For $y = 2x + 1$:

When $x = 0$, $y = 2(0) + 1 = 1$, so the y-intercept is $A(0, 1)$

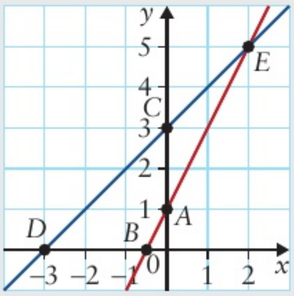
When $y = 0$, $0 = 2x + 1 \Rightarrow -1 = 2x \Rightarrow x = -\frac{1}{2}$, so the x-intercept is $B\left(-\frac{1}{2}, 0\right)$

For $y = x + 3$:

When $x = 0$, $y = 0 + 3 = 3$, so the y-intercept is $C(0, 3)$

When $y = 0$, $0 = x + 3 \Rightarrow x = -3$, so the x-intercept is $D(-3, 0)$

If we draw the graphs of both lines we can find their intersection point.



We can see from the graph that the point of intersection is $E(2, 5)$

We can check that point $E(2, 5)$ is on both lines by seeing if the point E satisfies both equations. We have $(x, y) = (2, 5)$.

Substituting $x = 2$, $2x + 1$ gives:

$2(2) + 1$

$= 5$, so the point $E(2, 5)$ is on the line $y = 2x + 1$, we check that it gives the value of y

Similarly, for $x + 3$:

$(2) + 3$

$= 5$, so the point $E(2, 5)$ is on $y = x + 3$

Therefore, $E(2, 5)$ is the point of intersection of the two lines.

Connections

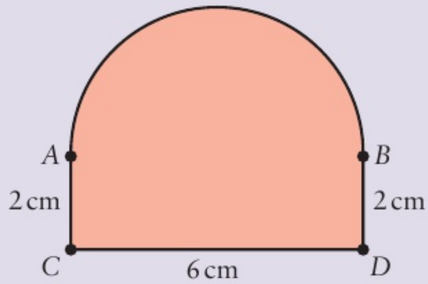
The intersection of two lines can be found algebraically as well. You will investigate how to solve two equations simultaneously in Chapter 5.

1.3.2 Perimeter, area and surface area

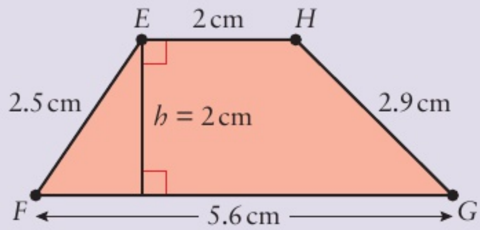
Explore 1.9

Can you find the perimeter and the area of these shapes?

a



b



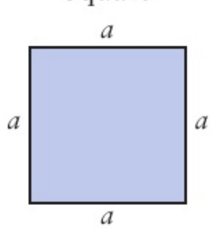
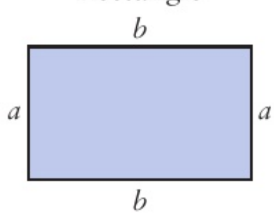
> **Connections**
>
> 
>
> When designing a stained glass window, it's important to know the area of each panel so that you know how much glass to order.

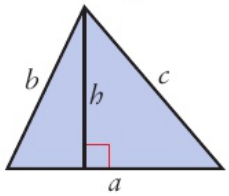
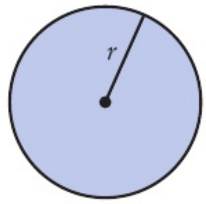
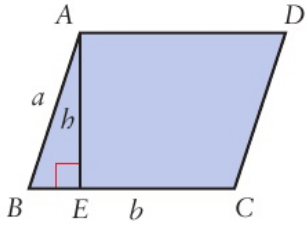
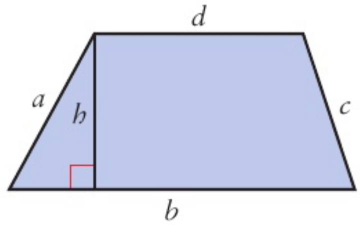
To find the **perimeter**, P, of an object, we add together the lengths of all the sides. Perimeter is measured in metres (m), centimetres (cm) and a variety of other units that measure length.

To find the **area**, A, of an object, we calculate the 2-dimensional space it covers. Area is measured in units such as square metres (m^2), square centimetres (cm^2) and so on.

The perimeter of a circle has a special name, the **circumference**, C.

The perimeter and the area of various shapes can be calculated using the relevant formulae as summarised in the following table.

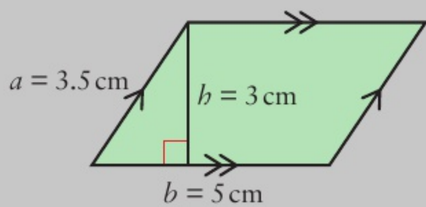
Square	Rectangle
	
$A = a \times a = a^2$	$A = a \times b$
$P = 4 \times a$	$P = 2 \times (a + b)$

Triangle	Circle
 $A = \frac{1}{2} \times a \times h$ $P = a + b + c$	 $A = \pi \times r^2$ $C = 2 \times \pi \times r$
Parallelogram	**Trapezium**
 $A = b \times h$ $P = 2 \times (a + b)$	 $A = \frac{h \times (b + d)}{2}$ $P = a + b + c + d$

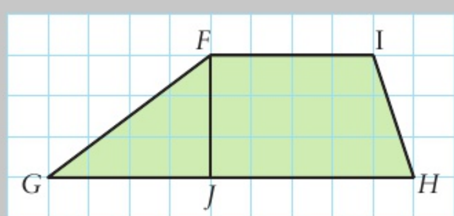
Worked example 1.20

Find the perimeter and the area of the following shapes.

a



b



Solution

We can apply the perimeter and area formulae for each shape. For the trapezium, we can find the lengths of FG and IH using Pythagoras' theorem.

a $P = 2 \times (3.5 + 5) = 17$, so the perimeter is 17 cm

$A = 3 \times 5 = 15$, so the area is 15 cm^2

b Using Pythagoras' theorem:

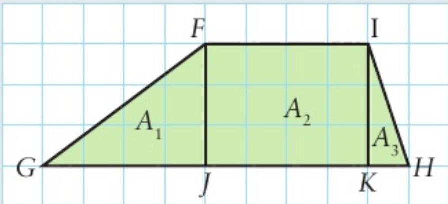
$FG^2 = 3^2 + 4^2 = 25 = 5^2$, so $FG = 5$

$IH^2 = 3^2 + 1^2 = 10$, so $IH = \sqrt{10}$

$P = 4 + 5 + 9 + \sqrt{10} = 18 + \sqrt{10}$, so the perimeter is $(18 + \sqrt{10})$ units

$A = \frac{3 \times (4 + 9)}{2} = \frac{39}{2} = 19.5$, so the area is 19.5 units^2

We can check our value for the area of the trapezium by dividing it into a rectangle and two triangles and calculating the area of each part separately.



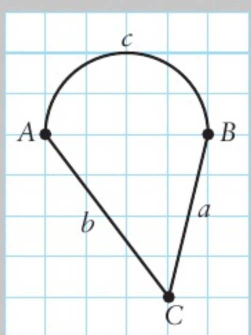
$A = A_1 + A_2 + A_3$

$A = \frac{4 \times 3}{2} + 3 \times 4 + \frac{3 \times 1}{2} = 6 + 12 + 1.5 = 19.5$

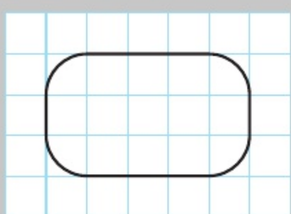
Worked example 1.21

Find the perimeter and area of the following composite shapes. The shapes are drawn on grid paper.

a



b



Solution

a The composite shape is made from a semicircle and a triangle. For the perimeter, the sides a and b can be found using Pythagoras' theorem. We can find c using the formula for the circumference. For the area, we know that the radius of the semicircle is 2 units, and the height and base of the triangle are 4 units and 4 units.

$P = a + b + c$

The circumference of a circle is $2\pi r$, so the circumference of a semicircle is πr

Therefore, $c = \pi \times 2 = 2\pi$

$b^2 = 3^2 + 4^2 = 25 = 5^2$, so $b = 5$

$a^2 = 1^2 + 4^2 = 17$, so $a = \sqrt{17}$

$P = 5 + 2\pi + \sqrt{17}$, so the perimeter is $(5 + 2\pi + \sqrt{17})$ units

$$A = \text{area of semicircle} + \text{area of triangle} = \frac{1}{2} \times \pi \times 2^2 + \frac{4 \times 4}{2}$$
$$= 2\pi + 8$$

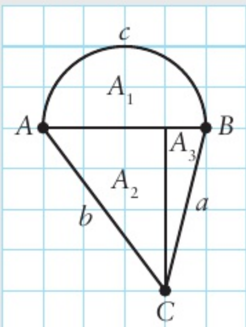
so the area is $(2\pi + 8)$ units^2

b The perimeter of the shape is the combination of four straight sides and four quarter circles. The four quarter circles make a full circle. The area is the combination of four quarter circles and 16 unit squares.

$P = 3 + 3 + 2 + 2 + 4 \times \left(\frac{1}{4} \times 2\pi \times 1\right) = 10 + 2\pi$, so the perimeter is $(10 + 2\pi)$ units

$A = 16 \times 1^2 + 4 \times \left(\frac{1}{4} \times \pi \times 1^2\right) = 16 + \pi$, so the area is $(16 + \pi)$ units^2

We can check the area of the composite shape using a different approach. Divide the area into three parts: $A = A_1 + A_2 + A_3$ as shown in this diagram.



So $A = \frac{1}{2} \times \pi \times 2^2 + \frac{3 \times 4}{2} + \frac{1 \times 4}{2} = 2\pi + 6 + 2 = 2\pi + 8$

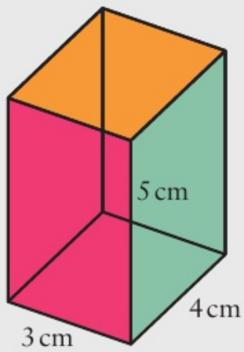
The surface area, SA, of a solid 3D object is the sum of the areas of its faces.

Worked example 1.22

Find the surface area of a rectangular prism with the dimensions of 3 cm × 4 cm × 5 cm

Solution

The diagram shows a rectangular prism with the dimensions 3 cm × 4 cm × 5 cm



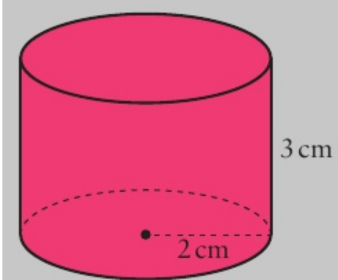
There are six rectangular faces, with opposite faces having the same area.

The surface area of the shape is:

$$SA = (2 \times (3 \times 4)) + (2 \times (3 \times 5)) + (2 \times (4 \times 5))$$
$$= 2 \times 12 + 2 \times 15 + 2 \times 20$$
$$= 24 + 30 + 40$$
$$= 94, \text{ so the surface area is } 94\,\text{cm}^2$$

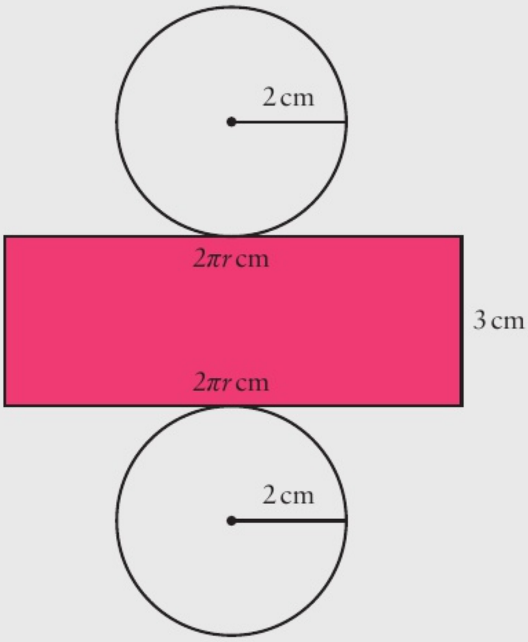
Worked example 1.23

Find the surface area of a cylinder with radius 2 cm and height 3 cm.



Solution

When we unravel the cylinder, the surface area is made of two identical circles and a rectangle. The length of the rectangle is equal to the circumference of the circles.



To find the area of the cylinder, we need to find the area of a rectangle and two identical circles.

We know that $r = 2$ and $h = 3$

We substitute these values into the formulae for the area of a circle and a rectangle.

area of two circles is given by $2 \times \pi \times r^2 = 8\pi$

area of the rectangle is given by $2\pi r \times h = 2\pi \times 2 \times 3 = 12\pi$

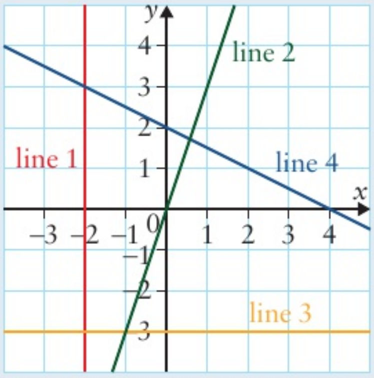
The surface area of the cylinder is given by $8\pi + 12\pi = 20\pi$, so the surface area is 20π cm^2

Practice questions 1.3

1 Draw the graph of each of these straight lines on a coordinate plane.

a $y = 2x + 3$ b $y = -x - 3$ c $y = 3x + 1$

2 Write down an equation for each line shown.



3 Write down an equation of the line that:

a has a gradient of 5 and a y-intercept of 2

b has a gradient of -2 and a y-intercept of 1

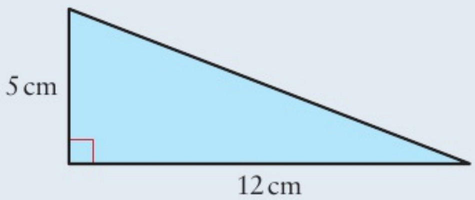
c has a gradient of 1.5 and a y-intercept of -3

d goes through points $A(1, 1)$ and $B(-2, 4)$.

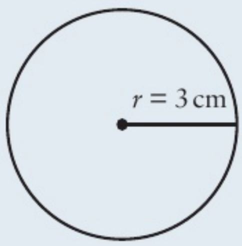
4 Is the point $A(3, 4)$ on the line $y = 3x - 2$? Explain your answer.

5 Find the perimeter and area of these shapes.

a

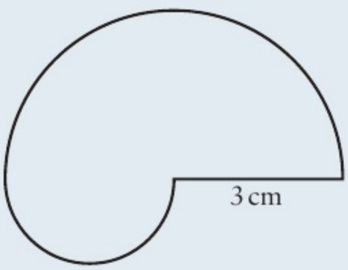


b

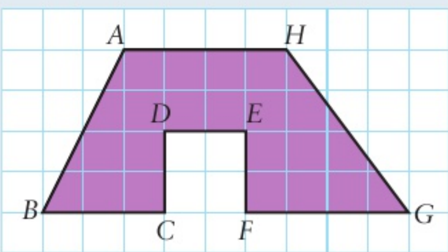


6 Find the area of each of these composite shapes.

a



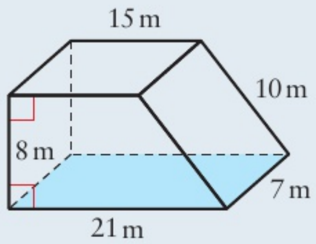
b



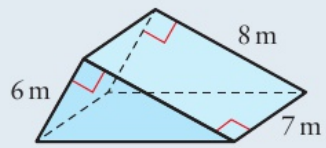
Challenge Q7

7 Find the surface area of each prism.

a



b



1.4 Probability and statistics

1.4.1 Probability

Explore 1.10

You throw a die and flip a coin. Can you find the probability of the outcome being a multiple of 3 and a head?

The probability, P, of an event, E, can be calculated using the formula

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ represents the number of ways the event can happen and $n(S)$ represents the number of possible outcomes, which is known as the sample space. All probabilities range from 0 (impossible) to 1 (certain).

Probabilities can be represented as a fraction, a percentage or a decimal number. For example, $25\% = 0.25 = \frac{1}{4}$

Worked example 1.24

Pune and Melissa throw two regular unbiased 6-sided dice and look at the sum of the numbers on the top faces. Find the probability of throwing a sum:

a of 8

b less than 10

c larger than 7.

Solution

We need to calculate the probability of each event.

The sample space (S) of throwing two regular 6-sided dice and adding the numbers on the top faces is shown in the table. For each part, the answer will be the number of outcomes that match the criteria divided by the total number of possible outcomes. The total number of possible outcomes is 36.

S		First die					
		1	2	3	4	5	6
Second die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

a There are five outcomes with a sum of 8 (2 and 6, 6 and 2, 5 and 3, 3 and 5, 4 and 4).

P(sum of 8) $= \frac{5}{36}$

b There are six outcomes with a sum of 10 or more (6 and 4, 4 and 6, 5 and 5, 6 and 5, 5 and 6, 6 and 6). Since we are looking for a sum of less than 10, the number of outcomes is $36 - 6 = 30$

P(sum less than 10) $= \frac{30}{36} = \frac{5}{6}$

c There are 15 outcomes with a sum of 8 or more.

P(sum larger than 7) $= \frac{15}{36} = \frac{5}{12}$

Fact

If the probability of an event A is P(A), then the probability of not A, P(A'), is $1 - $ P(A). A' is called the complement of A.

1.4.2 Statistics

Explore 1.11

Can you show how you find the mean, mode and median of this data set?

2 2 2 3 3 3 3 4 4 4 4 4 5 5 5 6 6 7 8 8 9 9 9 10 10 10 10

When working with a data set, it is helpful to present the data in a **frequency table** where it is easy to see repeated data. We can find the **cumulative frequency** for each row by adding the frequency of that row to the frequencies of all the previous rows.

Fact

It is easier to analyse a data set when the data is organised in a frequency table or when the data is ordered. For example, from smallest to largest.

The main measures of data analysis are shown below.

The **range** shows the spread of a data set. It is calculated by taking the difference between the highest and lowest data values.

- The **mean** is the ratio of the sum of the data to the number of data values. It is often shown as $\bar{x}$. If there is a frequency chart, then the mean can be calculated by the formula
  $\bar{x} = \frac{\sum fx}{\sum f}$, where x represents the data values, f represents the frequencies and $\sum$ (sigma) represents the sum.
- The **mode** is the data value that occurs most often in the data set.
- The **median** is the middle data value when the data set is arranged in order. If there are two middle numbers, the median is the mean of the two middle values.

The mean, mode and median are **measures of central tendency**.

Worked example 1.25

Calculate the measures of central tendency and the range of this data set.

Outcome (x)	Frequency (f)	Cumulative frequency
1	5	5
2	9	14
3	3	17
4	5	22
5	5	27

Solution

range = 5 − 1 = 4

The mode is the value with the highest frequency.

mode = 2

The median is the middle data value. There are 27 values, so the median is the $\frac{27+1}{2} = 14$th value.

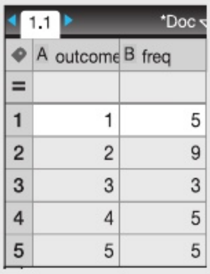
median = 2

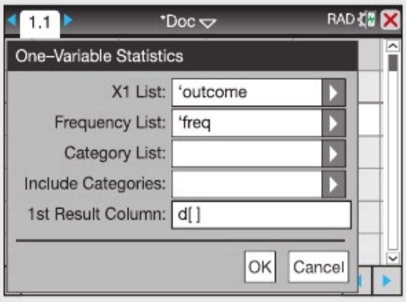
$$\text{mean} = \bar{x} = \frac{1 \times 5 + 2 \times 9 + 3 \times 3 + 4 \times 5 + 5 \times 5}{27} = \frac{77}{27} = 2.85 \text{ (3 s.f.)}$$

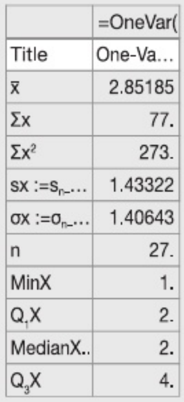
Connections

In a spreadsheet the command SUMPRODUCT applied to the x and f columns will produce $\sum fx$ in one step.

The central tendency can be calculated with a GDC. The images show extracts from a TI-Nspire calculator application for the same data.







Outliers are data values that are not consistent with the majority of the data, either because they are too small or too large. For example, in the data set 2, 2, 3, 4, 5, 5, 7, 7, 8, 1, 9, 8, 10, 10, 35, the value of 35 is an outlier as it is much larger than all the other values.

Worked example 1.26

On a biology field trip, an area is divided into square metres and the number of plants found in each section is counted. The results are given in the following data set.

10	10	10	11	11	11	11	12	12	13	13	13	14	15	15
15	15	16	16	16	16	16	18	19	19	20	20	20	88	100

a Create a frequency table for the data set.

b Calculate the measures of central tendency and the range.

c It was discovered that the two numbers 88 and 100 were included by mistake. Calculate the measures of central tendency and range with these values removed from the data set. Which one is affected the most?

Solution

a We need to count how many times each outcome appears in the data set.

Outcome (x)	Frequency (f)	fx
10	3	30
11	4	44
12	2	24
13	3	39
14	1	14
15	4	60
16	5	80
18	1	18
19	2	38
20	3	60
88	1	88
100	1	100
Totals	30	595

b mode = 16

There are 30 data values so the median is the average of the 15th and 16th data values, both of which are 15. You can find this by mentally calculating the cumulative frequency.

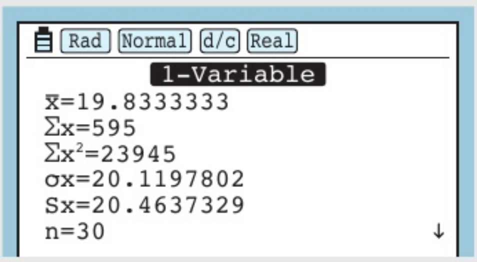
median = 15

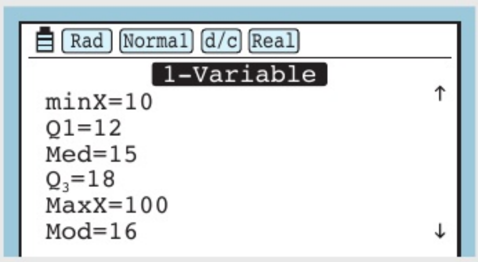
$$\text{mean} = \frac{\sum fx}{\sum f}$$

$$= \frac{595}{30} = 19.83 \text{ (2 d.p.)}$$

range = 100 – 10 = 90

These results are confirmed in the GDC output.





c The mode is unchanged: mode = 16

There are now 28 data values so the median is the average of the 14th and 15th data values.

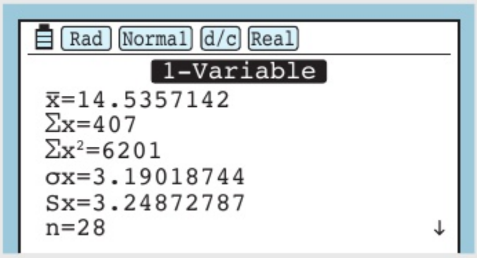
median = 15

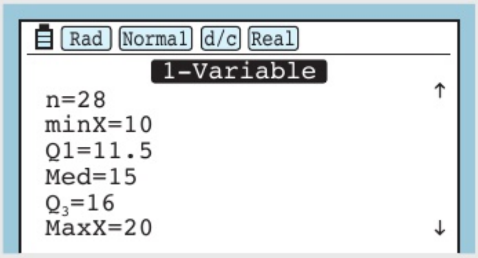
$$\text{mean} = \frac{\sum fx}{\sum f}$$

$$= \frac{407}{28} = 14.54 \text{ (2 d.p.)}$$

range = 20 – 10 = 10

The GDC result confirms these calculations.





In this case, deleting the outliers had no effect on the mode or median.

It had a small effect on the mean, and a large effect on the range.

Practice questions 1.4

1 List the sample space for the given events.

- **a** Days of the week starting with the letter S
- **b** The sum of the numbers when two dice are rolled
- **c** Months starting with the letter J
- **d** A coin being tossed

2 A letter is chosen at random from the letters in the word MATHEMATICS. Find the probability of choosing:

- **a** the letter A
- **b** a vowel
- **c** the letter C
- **d** a letter that is not W.

3 Find the probability of the complementary event in each case.

a In a group of people, the probability that someone's birthday is in the first half of the year is $\frac{11}{20}$

b The probability of getting an even number when you roll a fair dice.

c The probability of an airline's planes taking off on time is 29%.

d The probability of winning a game is 0.95

4 The points that a basketball team scored in their last 25 games are summarised in the following table.

Score (S)	30–39	40–49	50–59	60–69	70–79
Frequency (f)	3	7	8	4	3

Based on this data, what is the probability of the team scoring:

a more than 49 points in a game

b fewer than 70 points in a game

c 60 or more points in a game?

5 The data below shows the number of 'assists' a football player made per game for 22 football matches.

1 2 0 0 2 1 3 1 2 1 4 4 5 5 2 0 0 2 1 5 5 2

Find:

a the mode

b the mean

c the range

d the median.

6

Outcome (x)	Frequency (f)	Cumulative frequency
0.5	3	3
1.5		8
2.5	7	
3.5	9	24
4.5		26
5.5	4	

a Copy the cumulative frequency table and fill in the missing values.

b Find the range of the data.

c Calculate the measures of central tendency.

Self assessment

- I can convert rational numbers to decimals and vice versa.
- I can convert decimals to percentages and vice versa.
- I can convert recurring decimals into fractions.
- I can find a percentage of a number.
- I can apply the order of operations.
- I can simplify ratios.
- I can solve proportions.
- I can simplify rates.
- I know the laws of indices.
- I can factorise and expand algebraic expressions.
- I know what an inverse operation is.
- I can solve one-step and two-step equations.
- I can describe inequality signs.
- I can solve inequalities and show the solution on a number line.
- I can factorise simple algebraic expressions.
- I can expand parentheses of algebraic expressions.
- I can find an equation of a line.
- I can graph a line on a coordinate plane.
- I know how to represent horizontal and vertical lines as equations.
- I can find the gradient of a line.
- I know the gradient–intercept form of a line.
- I can find the circumference of a circle.
- I can find the areas of a triangle, square, rectangle and circle.
- I can find the area of a composite shape.
- I can find the surface area of a solid.
- I can find the probability of an event.
- I know that probability is between 0 and 1.
- I know what complementary events are.
- I can describe the range, mean, mode and median.
- I can create a frequency table.

Check your knowledge questions

1 Copy and complete the table with equivalent fractions, decimals and percentages. The first row has been done for you.

Fraction	Decimal	Percentage
$\frac{12}{25}$	0.48	48%
	$1.\dot{1}$	
$\frac{9}{20}$		
		140%
	0.89	

2 Use the order of operations to simplify these expressions.

a $1.03 + 12.3 \div 1.23$

b $5^3 \times 2^2 \div 10$

c $(6 + 15 \div 3)^2 - 4^3$

d $(0.1)^2 + \frac{1}{10}$

3 Write down these ratios and rates in their simplest form.

a $36:108$

b 120 km : 3 hours

c $\frac{116\text{ cm}}{132\text{ cm}}$

d $25:40:60$

4 Calculate the value of the missing variable(s) in these proportions.

a $15:25:40 = 3:x:y$

b $\frac{11}{17} = \frac{33}{x}$

c $\frac{42}{36} = \frac{y}{6}$

d $\frac{11}{z} = \frac{33}{120}$

5 If Elias's car travels 270 km in 3 hours, calculate how far he would travel in 5 hours at the same speed.

Challenge Q6

6 Simplify this expression as much as possible.

$$(12x^2y)^2 \div 18x^3y \times \frac{5}{8xy}$$

Challenge Q7b

7 Write these algebraic expressions in their simplest form.

a $x - \frac{x}{2} - \frac{x}{3}$

b $(12ab^2c)^2 \div 9abc \times 15a^2b^2c$

8 Solve these equations.

a $5(2x - 1) + 3(x - 4) = 9$

b $\frac{2x + 3}{5} - \frac{x - 2}{3} = 12$

9 Solve these inequalities and graph each solution on a number line.

a $2x + 3 < 9$

b $3(x + 1) - 2(x - 1) \leqslant 6$

c $\frac{x}{3} - \frac{x}{2} \geqslant 2$

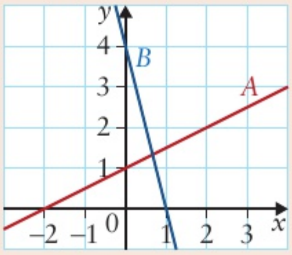
d $\frac{2x - 1}{3} - \frac{3x - 2}{2} - 1 > 0$

Challenge Q10

10 Find the circumference of a circle, if its area is $81\pi\text{ cm}^2$.

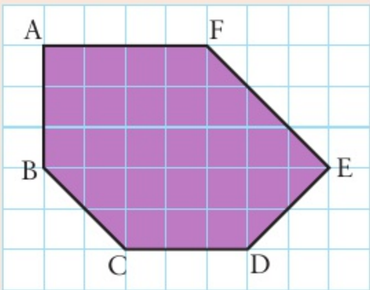
11 Draw graphs of the lines $y = 2x + 3$ and $y = -2x - 1$ on the same coordinate plane and identify their point of intersection.

12 Write down equations of the lines *A* and *B* in this diagram.

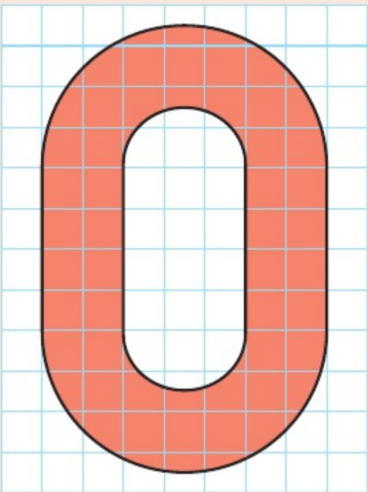


13 Calculate the areas of each of these shapes.

a



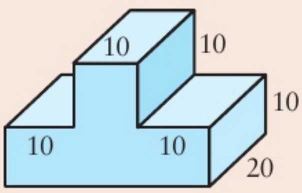
b



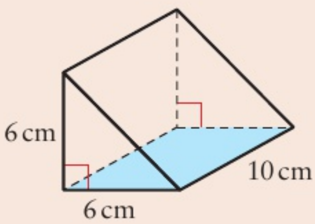
Challenge Q13b

14 Calculate the surface area of each of these solids. Apart from the two right-angled triangles, all faces are rectangles.

a



b



15 A letter is chosen at random from the phrase PYTHAGORAS WAS NOT IRRATIONAL. Write down the probability of choosing:

a a letter P

b a letter R.

16 In a mail sorting office, the number of damaged boxes per day is recorded on six consecutive days. The results are shown in the following table.

Number of damaged boxes (x)	Frequency (f)	Cumulative frequency
5	5	5
6		11
7	6	
8	4	21
9		28
10	2	

a Copy the cumulative frequency table and fill in the missing values.

b Find the range of the data.

c Calculate the measures of central tendency.

Year 3 extension

2

2 Year 3 extension

KEY CONCEPT

Form

RELATED CONCEPTS

Models, Representation, Systems

GLOBAL CONTEXT

Identities and relationships

Statement of inquiry

Representing mathematical concepts in different forms can identify complex relationships between variables.

Factual

- Can a surd be represented by a point on the number line?
- What is the difference between a rational number and an irrational number?

Conceptual

- What is a polynomial?
- Are surds always rational?

Debatable

- Does a visual proof achieve the same standard as an analytical one?

Do you recall?

1 Write $\frac{8}{28}$ in its simplest form.

2 Express $0.\dot{7}\dot{2}$ as a fraction.

3 Expand and simplify $(2k + 1)^2$

4 Factorise $6a^2 - 11a + 4$

5 Sketch $y = 3x + 4$ on a coordinate plane.

2.1 Classifying numbers and properties of numbers

2.1.1 Rational and irrational numbers

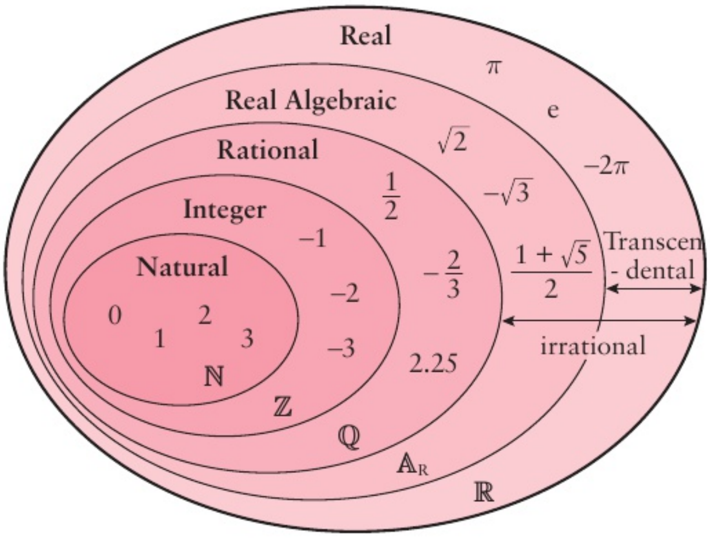
Explore 2.1

1 Can you classify each of these numbers as rational or irrational?

$\frac{7}{8}$ $\quad \sqrt[3]{64}$ $\quad 1.2\dot{3}$ $\quad \sqrt{20}$ $\quad \left(\frac{-3}{5}\right)^4$ $\quad 5\pi$ $\quad \sqrt{144}$ $\quad 3^{\frac{1}{2}}$

2 Can you plot $\sqrt{5}$ on a number line?

Any number that can be written as a fraction $\frac{a}{b}$, where a and b are integers and b is not zero, is a rational number. Rational numbers include natural numbers and integers, terminating and recurring decimals. For example, $\frac{7}{9}$, $\sqrt{49}$, -35, 10^5, $3.1\dot{2}$, $-\frac{101}{87}$ are all rational numbers. The set of rational numbers is denoted $\mathbb{Q}$.



Numbers that cannot be written as a fraction $\frac{a}{b}$ are called **irrational numbers**. Irrational numbers are non-terminating and non-recurring decimals. $\sqrt{2}$, $\sqrt[3]{2}$, $10\sqrt{7}$, $\sqrt{3}$ are a few irrational numbers.

The **real numbers** are all the rational and irrational numbers. The set of real numbers is denoted $\mathbb{R}$.

Fact

There are other more accurate but less popular approximations of π such as $\frac{355}{113}$

Connections

Some artists and architects believe the golden ratio makes the most pleasing and beautiful shape. Many buildings and artworks exhibit the golden ratio, such as the Greek Temple in Agrigento, Sicily.

Fact

The numbers π, e (Euler's number) and φ (the golden ratio) are three well-known irrational numbers with many applications. These numbers have all been calculated to over a trillion decimal places with no pattern found. The first 30 digits of each number are:

$\pi = 3.14159265358979323846264338327\ldots$

$e = 2.71828182845904523536028747135\ldots$

$\varphi = 1.61803398874989484820458683436\ldots$

The popular approximation of $\pi \approx \frac{22}{7} = 3.1416\ldots$ is close but **not exact**.

The golden ratio can be expressed in terms of another irrational number: $\varphi = \frac{1+\sqrt{5}}{2}$

Worked example 2.1

Classify each number below as rational or irrational. Represent the rational numbers in the form $\frac{a}{b}$

$\frac{3\pi}{2}$ 0.14 $\sqrt[3]{16}$ $\sqrt[3]{27}$ $(-3)^3$ $\sqrt{36}$ $(-10)^2$ $0.888\ldots$ $\sqrt{10}$

Solution

We need to classify each number as rational or irrational. In some cases the answer is not immediately obvious. A rational number multiplied by an irrational number will always give an irrational number.

We can summarise the solution in a table. The definition of a rational number in the form $\frac{p}{q}$ does not mean that rational numbers will always appear in that form. Note particularly the integers that can be rewritten with a denominator of 1 to conform to the definition, e.g. $12 = \frac{12}{1}$

Number	$\frac{3\pi}{2}$	0.14	$\sqrt[3]{16} = 2\sqrt[3]{2}$	$\sqrt[3]{27} = 3$	$(-3)^3$	$\sqrt{36} = 6$	$0.888\ldots$	$(-10)^2$	$\sqrt{10}$
Rational?	No	Yes	No	Yes	Yes	Yes	Yes	Yes	No
Fraction		$\frac{7}{50}$		$\frac{3}{1}$	$\frac{-27}{1}$	$\frac{6}{1}$	$\frac{8}{9}$	$\frac{100}{1}$	

Challenge

Worked example 2.2

Show that $\sqrt{2}$ is an irrational number.

Solution

We need to use a sequence of logical steps to show that $\sqrt{2}$ is irrational. We need to show that $\sqrt{2}$ cannot be expressed as a fraction in the form $\frac{a}{b}$ where a and b are coprime (without common factors) integers and $b \neq 0$.

The task of writing $\sqrt{2}$ as a fraction does not appear to be possible directly, so we will try an indirect method. Since a number can be either rational or irrational, we start by assuming that $\sqrt{2}$ is rational. Then we show that this leads to a contradiction and so is not possible. We then conclude that $\sqrt{2}$ must be irrational.

Assume that $\sqrt{2}$ is a rational number and can be written as $\frac{a}{b}$ in its simplest form. That is, a and b are integers that have no common factor and b is not zero.

Then,

$$\sqrt{2} = \frac{a}{b} \Rightarrow (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2 \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \qquad \textbf{(1)}$$

This means that a^2 is an even number. Thus, a must be an even number because the square of an odd number cannot be even.

If a is even, then it can be expressed as $a = 2k$, where k is an integer. Now, substitute this value into equation **1**:

$$a^2 = 2b^2 \Rightarrow (2k)^2 = 2b^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow 2k^2 = b^2$$

This, in turn, implies that b is also an even number. This means that a and b have a common factor, namely 2. But this contradicts the initial condition that a and b have no common factor. Thus, our assumption that $\sqrt{2}$ is rational cannot be true, and we conclude that it must be irrational.

Reminder

$\Rightarrow$ is called "the implies sign".

$A \Rightarrow B$ can be read as "A implies B", or "if A, then B".

For example, $a + 7 = 11 \Rightarrow a = 4$ could be read: "if 7 more than a is 11, then a is 4".

$A \Rightarrow B$ is equivalent to saying that either B is true or A must be false.

For example:

47° and x° are angles forming a straight line

$\Rightarrow 47 + x = 180 \Rightarrow x = 133$

If it turns out that $x \neq 133$, then we must be wrong to believe that 47° and x° form a straight line.

Reflect

The method used in Worked example 2.2 is called **proof by contradiction**. Is it possible to use a similar method to show that $\sqrt{3}$ and $\sqrt{5}$ are irrational numbers? Can you see where a similar method would fail to prove that $\sqrt{4}$ is irrational?

Challenge

Worked example 2.3

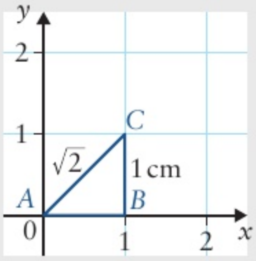
Plot $\sqrt{2}$ on a number line.

Solution

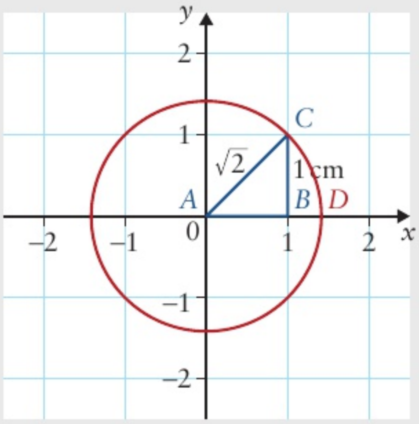
We know that $\sqrt{2}$ is an irrational number, so we cannot measure its position on a number line directly.

We can use Pythagoras' theorem to construct a length equal to $\sqrt{2}$. Using a coordinate grid and circle geometry, we can then copy that length onto a number line.

Start by constructing a right-angled triangle ABC on a coordinate grid, with $AB = BC = 1\,\text{cm}$
Then, by Pythagoras' theorem, $1^2 + 1^2 = AC^2 = 2$ and $AC = \sqrt{2}$



To mark this distance on a number line, draw a circle on the coordinate grid with centre A and radius $AC = \sqrt{2}$. This circle intersects the x-axis (the number line) at D.



Since $AD = AC = \sqrt{2}$, we can see that D represents $\sqrt{2}$ on the number line.

We can use the method shown to construct $\sqrt{2}$ on a number line. But because $\sqrt{2}$ is an infinite non-recurring decimal number ($\sqrt{2} = 1.414\,213\,562\ldots$), it is not possible to plot it as a decimal number.

Reflect

Can the method in Worked example 2.3 be used to represent other irrational numbers, such as $\sqrt{3}$ and $\sqrt{5}$, on a number line?

Worked example 2.4

Between which two consecutive integers does $\sqrt{45}$ lie?

Solution

We need to look for the two square numbers on each side of 45. Then $\sqrt{45}$ will lie between the square roots of these numbers.

36 and 49 are the nearest square numbers to 45, so $\sqrt{45}$ lies between $\sqrt{36}$ and $\sqrt{49}$

That is, $\sqrt{36} < \sqrt{45} < \sqrt{49}$, or $6 < \sqrt{45} < 7$

We can use a calculator to check our answer.

$\sqrt{45} = 6.708...$, which is between 6 and 7.

Practice questions 2.1.1

1 Classify each number as a rational number or an irrational number.

a $\frac{12}{99}$ b $\sqrt{20}$ c $\frac{22}{7}$ d $12.1\dot{2}$

e $\frac{\pi}{6}$ f $\sqrt{5} \times \sqrt{5}$ g $\sqrt{3} + \sqrt{3}$ h $2 + \sqrt{16}$

i $\frac{1}{\sqrt{3}}$ j $\frac{\sqrt{128}}{\sqrt{2}}$

2 Use a calculator to find approximations to three decimal places of each irrational number.

a $\sqrt{2}$ b $\sqrt{3}$ c $\sqrt{5}$ d $\sqrt{10}$ e $\sqrt{15}$

3 Between which two consecutive integers does each irrational number lie?

a $\sqrt{59}$ b $\sqrt{87}$ c $\sqrt{125}$ d $\sqrt{180}$ e $\sqrt{288}$

4 Use a calculator to evaluate each number. Then write the numbers in ascending order.

$\sqrt{5}$ 2^3 $\sqrt[3]{2}$ 3.12 $\frac{5}{\sqrt{2}}$ $\frac{\pi}{2}$

Challenge

Connections

Surds are expressions that involve irrational numbers. If a surd can be simplified completely, such as $\sqrt{25} = 5, \sqrt[3]{64} = 4, \sqrt[4]{16} = 2$, then it is a rational number. But if it cannot be simplified fully, such as $\sqrt{50} = 5\sqrt{2}$, $\sqrt[3]{16} = 2\sqrt[3]{2}$, then it is an irrational number. You will explore surds further in Chapter 16.

Hint Q4

Use the form given in the question to order the numbers from smallest to greatest.

Challenge Q5

Challenge Q6

Hint Q6

A counter-example is an example that opposes or contradicts a theory.

5 Plot $\sqrt{6}$ on a number line.

6 Are the following statements true or false? Give an example or counter example for each answer.

a The product of two irrational numbers is always an irrational number.

b The quotient of two irrational numbers is always an irrational number.

c The sum of two irrational numbers is always an irrational number.

d The difference of two irrational numbers is always an irrational number.

2.1.2 Properties of real numbers

Explore 2.2

Can you work out which of these equations are true? If you are unsure, try substituting a different number for each letter.

a $a + b = b + a$

b $a \times (b \times c) = (a \times b) \times c$

c $a \div (b \div c) = (a \div b) \div c$

d $a - (b + c) = (a - b) + (a - c)$

- $\mathbb{R}$ has the property of **closure** under multiplication, addition and subtraction. This means that performing any of these operations on two numbers that are in $\mathbb{R}$ will result in a number that is also in $\mathbb{R}$. For $\mathbb{R}$ with zero removed, there is also closure for division. We can put this as follows.

  For any $a, b \in \mathbb{R}$:

  - $a + b \in \mathbb{R}$
  - $a \times b \in \mathbb{R}$
  - $a - b \in \mathbb{R}$
  - $a \div b \in \mathbb{R}$ (for $b \neq 0$)

- Addition and multiplication are **commutative**. This means that for any two real numbers a and b, $a + b = b + a$ and $a \times b = b \times a$. For example, $2 + 3 = 3 + 2 = 5$, and $2 \times 3 = 3 \times 2 = 6$. Subtraction and division are *not* commutative. For example, $4 - 2 \neq 2 - 4$ and $4 \div 2 \neq 2 \div 4$

- Addition and multiplication are also **associative**. This means that for any real numbers a, b and c, $a + (b + c) = (a + b) + c$ and $(a \times b) \times c = a \times (b \times c)$. For example:

  $3 + (2 + 4) = 3 + 6 = 9$
  $(3 + 2) + 4 = 5 + 4 = 9$

  $3 \times (2 \times 4) = 3 \times 8 = 24$
  $(3 \times 2) \times 4 = 6 \times 4 = 24$

Fact

The symbol $\in$ means 'is an element of', 'is a member of' or 'is in'. So $a, b \in \mathbb{R}$ means that both a and b are elements of $\mathbb{R}$.

- Subtraction and division are *not* associative.

  If subtraction were associative then $24 - (8 - 4)$ would be the same as $(24 - 8) - 4$

  First expression: $24 - (8 - 4) = 24 - 4 = 20$

  Second expression: $(24 - 8) - 4 = 16 - 4 = 12$

  So $24 - (8 - 4) \neq (24 - 8) - 4$

  Similarly, if division were associative then $24 \div (8 \div 4)$ would be the same as $(24 \div 8) \div 4$

  First expression: $24 \div (8 \div 4) = 24 \div 2 = 12$

  Second expression: $(24 \div 8) \div 4 = 3 \div 4 = 0.75$

  So $24 \div (8 \div 4) \neq (24 \div 8) \div 4$

- 0 and 1 are the **identity** elements of addition and multiplication respectively. For any real number a, $0 + a = a + 0 = a$ and $1 \times a = a \times 1 = a$. For example, $0 + 5 = 5 + 0 = 5$ and $1 \times (-5) = (-5) \times 1 = -5$

- Subtraction and division do not have an identity element.

- Multiplication is **distributive** over addition and subtraction. This means that for any real numbers a, b and c:

  $a \times (b + c) = a \times b + a \times c$ and $a \times (b - c) = a \times b - a \times c$

  For example:

  $5 \times (6 - 2) = 5 \times 4 = 20$
  $5 \times 6 - 5 \times 2 = 30 - 10 = 20$
  so $5 \times (6 - 2) = 5 \times 6 - 5 \times 2$

- For any given real number a, the **additive inverse** is $-a$, because $a + (-a) = (-a) + a = 0$. For example, the additive inverse of 5 is -5

- For any real number a ($a \neq 0$), the **multiplicative inverse** is $\frac{1}{a}$, because $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$. For example, the multiplicative inverse of $\frac{2}{3}$ is $\frac{1}{\frac{2}{3}} = \frac{3}{2}$, because $\frac{2}{3} \times \frac{3}{2} = \frac{3}{2} \times \frac{2}{3} = 1$

Fact

The identity element of an operation is the element that has no effect when that operation is performed.

Connections

The properties of closure, associativity, identity, inverse, commutativity, and distributivity can be applied to many structures in mathematics. They are important generalisations in that once a newly defined operation and set are known to have some of the properties, then a number of well known results can be applied. You are likely to revisit these properties in matrices, vectors and functions.

Fact

The inverse of an element under an operation is the element that results in the identity element when the operation is performed.

Worked example 2.5

Write down the additive and multiplicative inverses of these real numbers.

$\sqrt{2}$ $\quad -3.2 \quad \frac{11}{8} \quad 8$

Solution

Number	Additive inverse	Multiplicative inverse
$\sqrt{2}$	$-\sqrt{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
-3.2	3.2	$\frac{1}{-3.2} = -0.3125$
$\frac{11}{8}$	$-\frac{11}{8}$	$\frac{1}{\frac{11}{8}} = \frac{8}{11}$
8	-8	$\frac{1}{8}$

We can check our answers by adding or multiplying:

$\sqrt{2} + (-\sqrt{2}) = 0$, so $-\sqrt{2}$ is the additive inverse of $\sqrt{2}$

$\sqrt{2} \times \frac{1}{\sqrt{2}} = 1$, so $\frac{1}{\sqrt{2}}$ is the multiplicative inverse of $\sqrt{2}$

Reflect

Thinking skills

Exploring the closure property on $\mathbb{Z}$, the set of integers

Answer the following questions to investigate whether or not the four operations (addition, subtraction, multiplication and division) have the closure property on integers. If not, give a counter example.

1 For any integers a and b, is $(a + b)$ always an integer?

2 If a and b are integers, is $(a - b)$ always an integer?

3 If a and b are integers, is $(a \times b)$ always an integer?

4 If a and b are integers, is $(a \div b)$ always an integer?

5 Which of the four operations have the closure property on integers?

Worked example 2.6

Simplify the following expressions.

a $\sqrt{6} \times (\sqrt{6} + \sqrt{3})$ **b** $xy^2 \times (ax - by)$

Solution

The distributive property of multiplication over addition and subtraction can be used to simplify these expressions.

a $\sqrt{6} \times (\sqrt{6} + \sqrt{3}) = \sqrt{6} \times \sqrt{6} + \sqrt{6} \times \sqrt{3}$

$= \sqrt{36} + \sqrt{18}$

$= 6 + 3\sqrt{2}$

b $xy^2 \times (ax - by) = xy^2 \times ax - xy^2 \times by$

$= ax^2y^2 - bxy^3$

Practice questions 2.1.2

1 Use the associative property to rewrite these expressions.

a $3 \times (4 \times 5)$ b $(0.2 + 0.5) + 1.2$

c $x(yz)$ d $\frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$

2 For each of these expressions, use the distributive property to expand and then simplify where possible.

a $9 \times (12 - 3)$ b $1.2 \times (0.2 + 0.3)$

c $(w + z) \times x$ d $6 \times (x + 2) - 5 \times (x - 3)$

3 Find the additive inverse of each number.

a 123 b 12.35 c $\sqrt{5}$ d $\pi + 2$

4 Find the multiplicative inverse of each number.

a -32 b 0.3 c $\frac{\pi}{4}$ d $\sqrt{15}$

5 Simplify each expression.

Challenge Q5

a $3y - 15 + (-3y)$ b $\left(\frac{11}{12} + \frac{3}{5}\right) + \frac{2}{5}$

c $34x - 22y + 12x + 38y$

Challenge Q6

6 Do the natural numbers have the closure property for subtraction?

7 Is there an identity element for the division operation? Explain.

2.2 Algebra extension

2.2.1 Properties of equations and inequalities

Explore 2.3

1 If $x = 3y$ and $y = 5z$, what can you say about x and z?

2 Lukas is older than his sister Sophia. Their cousins are Marco and Rob, and Marco is older than Rob. Rob is older than Lukas. What can you say about Marco and Sophia?

Fundamental properties of equations and inequalities:

- The **reflexive** property indicates that for every real number x, $x = x$
  Inequalities are not reflexive: $x \ngtr x$
- The **symmetry** property indicates that for all real numbers x and y, if $x = y$, then $y = x$. This is not true for inequalities. If $x < y$ then $y \ngtr x$; in fact, $y > x$. Similarly if, $x > y$ then $y \nless x$; in fact $y < x$
- The **transitive** property indicates that for all real numbers x, y and z, if $x = y$ and $y = z$, then $x = z$. This property applies to inequalities too. If $x < y$ and $y < z$, then $x < z$, and similarly for the other types of inequality.
- The **equivalency** property indicates that if the same operations are applied to both sides of an equation or inequality, the equation or inequality remains true. Equivalency does not apply to an inequality when both sides are multiplied or divided by a negative number. Also note that in all cases division by zero is impossible and therefore division by an unknown or variable should include consideration of whether the unknown or variable could be zero.

Hint

The same equivalencies apply to all types of inequality: $x < y$, $x > y$, $x \leqslant y$, $x \geqslant y$

Note that if we multiply or divide by a negative number we need to reverse the inequality sign.

For x, y, a real numbers, if $x = y$ then:

$x + a = y + a \qquad x - a = y - a \qquad x \times a = y \times a \qquad x \div a = y \div a$

For x, y, a, b real numbers, $b > 0$, if $x < y$ then:

$x + a < y + a \qquad x - a < y - a \qquad x \times b < y \times b \qquad x \div b < y \div b$

For x, y, c real numbers, $c < 0$, if $x < y$ then:

$x \times c > y \times c \qquad x \div c > y \div c$

Worked example 2.7

Solve:

a $4x + 25 = 3x + 36$

b $\frac{x}{2} + 2x - 1 \geqslant 2x - \frac{x}{2} + 5$

Solution

To solve the equation and the inequality, we need to apply operations to both sides until the variable is isolated on one side and numbers on the other side.

We will do this by using equivalency rules.

a $4x + 25 = 3x + 36$

$4x + 25 - 25 = 3x + 36 - 25$ Subtract 25 from both sides.

$4x = 3x + 11$ Simplify.

$4x - 3x = 3x + 11 - 3x$ Subtract $3x$ from both sides.

$x = 11$

b $\frac{x}{2} + 2x - 1 \geqslant 2x - \frac{x}{2} + 5$

$\frac{x}{2} + 2x - 1 + 1 \geqslant 2x - \frac{x}{2} + 5 + 1$ Add 1 to both sides.

$\frac{x}{2} + 2x \geqslant 2x - \frac{x}{2} + 6$ Simplify.

$\frac{x}{2} + 2x - 2x \geqslant 2x - \frac{x}{2} + 6 - 2x$ Subtract $2x$ from both sides.

$\frac{x}{2} \geqslant -\frac{x}{2} + 6$ Simplify.

$\frac{x}{2} + \frac{x}{2} \geqslant -\frac{x}{2} + 6 + \frac{x}{2}$ Add $\frac{x}{2}$ to both sides.

$x \geqslant 6$

We can check our answers by substituting them into the original equation and inequality.

a Substituting $x = 11$ into the original equation gives

$4x + 25 = 4 \times 11 + 25 = 69$

$3x + 36 = 3 \times 11 + 36 = 69$

so $x = 11$ satisfies $4x + 25 = 3x + 36$

b Choose any number larger than or equal to 6, say 8.

$x = 8 \Rightarrow \frac{x}{2} + 2x - 1 = \frac{8}{2} + 2 \times 8 - 1 = 19$

$x = 8 \Rightarrow 2x - \frac{x}{2} + 5 = 2 \times 8 - \frac{8}{2} + 5 = 17$

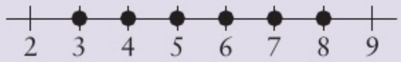
so $x = 8$ satisfies $\frac{x}{2} + 2x - 1 \leqslant 2x - \frac{x}{2} + 5$

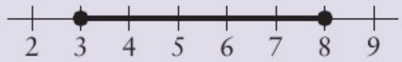
2.2.2 Set and interval notation

Explore 2.4

How can you represent all the real numbers between 2 and 11 on a number line? If you include 2 and 11 in this set, how would you represent the new set?

Can you describe the sets below in words and using mathematical notation?

a 

b 

We use **set notation** and **interval notation** to represent a group of numbers with symbols. We use curly brackets, { } for sets, and the inequality signs $<$, $>$, $\leqslant$, $\geqslant$ to show the interval of numbers.

For example, we can represent all the real numbers between 3 and 5, including 3 and 5, using set notation as $\{x \mid 3 \leqslant x \leqslant 5\}$

Hint

This is read 'The set of numbers x, such that x is between 3 and 5 inclusive'.

The same set of numbers represented using interval notation is $[3, 5]$.
The square brackets, are used to show that the endpoints of the interval are included.

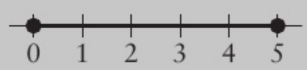
If 3 and 5 are not included, we write this in set notation as $\{x \mid 3 < x < 5\}$.
To represent this in interval notation, use round brackets: (3, 5)

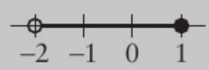
Fact

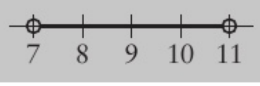
The interval notation for this is sometimes written as]3, 5[instead of (3, 5).

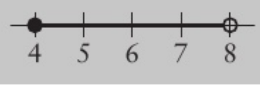
Worked example 2.8

Write interval and set notation to represent each number set.

a 

b 

c 

d 

Solution

The circles show the endpoints of the number sets. A filled circle means that the endpoint is included in the set, so we use a square bracket for the interval notation and a $\leqslant$ or $\geqslant$ symbol for the set notation. An unfilled circle means that the endpoint is not included in the set, so we use a round bracket for the interval notation and a $<$ or $>$ symbol for the set notation.

a $[0, 5]$ or $\{x \mid 0 \leqslant x \leqslant 5\}$

b $(-2, 1]$or $\{x \mid -2 < x \leqslant 1\}$

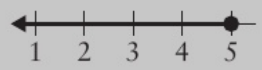
c $(7, 11)$ or $\{x \mid 7 < x < 11\}$

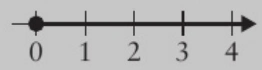
d $[4, 8)$ or $\{x \mid 4 \leqslant x < 8\}$

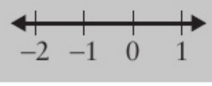
If one side of the interval has no definite limit or end, then we use ∞, the infinity sign. We can specify with a + (optional) or − sign whether it continues to positive or negative infinity. Since ∞ is not a number, it cannot be included in the set, so we use open interval notation. For example $x \geqslant 2$ is written as $[2, +\infty)$

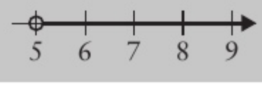
Worked example 2.9

Write interval and set notation for these infinite number sets.

a 

b 

c 

d 

Solution

An arrow pointing to the left means that the set continues to $-\infty$.
An arrow pointing to the right means that the set continues to $+\infty$

a $(-\infty,5]$ or $\{x \mid \leqslant 5\}$

b $[0,\infty)$ or $\{x \mid x \geqslant 0\}$

c $(-\infty,\infty)$ or $\mathbb{R}$

d $(5,\infty)$ or $\{x \mid 5 < x\}$

Hint

In Worked example 2.9, equivalent forms of the set notation are:

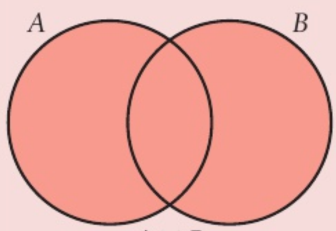
a It is enough to write $\{x \mid x \leqslant 5\}$

c It is enough to write the symbol for real numbers, $\mathbb{R}$.

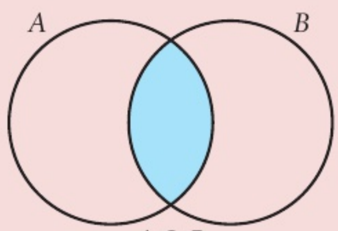
d It is enough to write $x > 5$.

Fact

The symbol $\cup$ is used to show the **union** of two sets. The symbol $\cap$ is used to show the **intersection** of two sets. For example, for two sets A and B, $A \cup B$ means all the elements that are in A **or** B, and $A \cap B$ means all the elements common to both A **and** B.



$A \cup B$



$A \cap B$

Worked example 2.10

For the sets $A = \{5, 7, 11, 13\}$ and $B = \{1, 2, 3, 4, 5\}$, find the sets $A \cup B$ and $A \cap B$.

Solution

$A \cup B$ contains all the elements that are in A or B.

$A \cup B = \{1, 2, 3, 4, 5, 7, 11, 13\}$

$A \cap B$ contains all the elements that are in both A and B. The only element in both A and B is 5.

$A \cap B = \{5\}$

Fact

An element is either a member of a set or it is not. Therefore we do not repeat elements when listing members of a set.

Worked example 2.11

Sketch the following number sets on a number line.

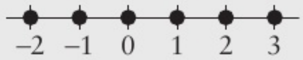
a $A = \{x \mid x \in \mathbb{Z}, -2 \leqslant x < 4\}$ **b** $B = \{x \mid x \in \mathbb{R}, -3 < x < 3\}$

c $A \cup B$ **d** $A \cap B$

Solution

We need to plot the elements of A, B, $A \cup B$ and $A \cap B$ on a number line.

a Set A includes all the integers between -2 and 4, including -2 but not including 4. So the elements of set A are $A = \{-2, -1, 0, 1, 2, 3\}$



b Set B includes all the real numbers between -3 and 3, not including the end points. Since there are infinitely many numbers, we can only show them on the number line as a line segment. We use unfilled circles for the endpoints, to show that they are not included in the set.

c $A \cup B$ includes all the real numbers between -3 and 3 and also includes 3 (because A includes 3).

d $A \cap B$ includes the elements of A that are between -3 and 3.

$A \cap B = \{-2, -1, 0, 1, 2\}$

Practice questions 2.2.2

1 Describe each set with interval notation.

a $\{x \mid -19 \leqslant x < 2\}$

b $\left\{x \mid \frac{2}{3} \leqslant x \leqslant 12\right\}$

c $\{x \mid 0 < x < 19\}$

d $\left\{x \mid \frac{\pi}{2} < x \leqslant 4\pi\right\}$

2 Describe each interval with set notation.

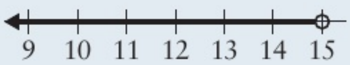
a $\left(-\frac{1}{2}, 5\right)$

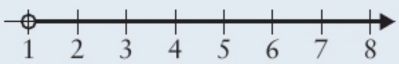
b $[2.3, 5.9]$

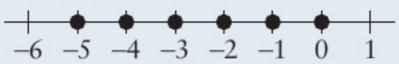
c $[-1, 1)$

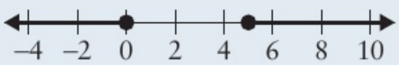
d $\left(-5, \frac{11}{2}\right]$

3 Write the set shown on each number line using set notation and interval notation.

a 

b 

c 

d 

4 Sketch each set, defined on the set of real numbers, on a number line.

a $(1, 11)$

b $\left[-\frac{1}{2}, \frac{13}{2}\right]$

c $\{x \mid -2 \leqslant x < 7.5\}$

d $(2, 6) \cup [4, 8]$

5 Show the intersection of the set of prime numbers less than 20 and the set of odd numbers between 4 and 12 using set notation.

6 Use interval notation to show the solution of the inequality of $2x + 5 > 12$.

7 What is the intersection of the set of odd integers and the set of even integers? Why?

Hint Q3d

The number line shows the union of the two number sets.

Challenge Q4d

Challenge Q7

2.2.3 Absolute values and their graphical representation

Explore 2.5

If $|x - 2| = 4$, what are the possible values of x?

$|a|$ is the **absolute value** of a real number a. $|a|$ represents the distance between the number a and zero. The distance cannot be negative, so the absolute value is always non-negative.

$|a| = a$ if $a \geqslant 0$ and $|a| = -a$ if $a < 0$

This can also be written as:

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Worked example 2.12

Find the value of each expression.

a $|-5|$ **b** $|12 - 34|$ **c** $|2^4 - 3 \times 4|$

Solution

We need to simplify the term inside the absolute value sign and then apply the definition to make the number a positive number.

a $-5 < 0$ so $|-5| = 5$

b Simplify the expression within the absolute value first.

$|12 - 34| = |-22| = 22$

c Simplify the expression within the absolute value first.

$|2^4 - 3 \times 4| = |16 - 12| = |4| = 4$

Worked example 2.13

Challenge

Sketch the graph of the equation $y = |x - 2|$ on a coordinate plane.

Solution

We need to sketch the graph of $y = |x - 2|$ on a coordinate plane, making sure that all possible points are considered.

When $x \geqslant 2$ then $x - 2 \geqslant 0$, so $|x - 2| = x - 2$. For values of x greater than or equal to 2 we need to sketch the graph of $y = x - 2$

When $x < 2$, then $x - 2 < 0$, so $|x - 2| = -(x - 2) = 2 - x$
For values of x less than 2 we need to sketch the graph of $y = 2 - x$

Since both equations describe straight lines, we need a minimum of two points to sketch each part of the graph.

For the first part we choose $x = 2$ and $x = 3$

When $x = 2$, $y = 2 - 2 = 0$, so $A(2, 0)$ is the x-intercept of $y = x - 2$

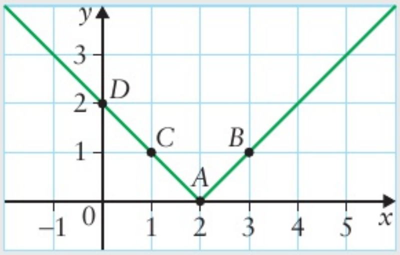
When $x = 3$, $y = 3 - 2 = 1$, so $B(3, 1)$ is a point on the line $y = x - 2$

For the second part we choose $x = 1$ and $x = 0$

When $x = 1$, $y = 2 - 1 = 1$, so $C(1, 1)$ is on the line $y = 2 - x$

When $x = 0$, $y = 2 - 0 = 2$, so $D(0, 2)$ is the y-intercept of the line $y = 2 - x$

Plot and connect the points to sketch the graph of $y = |x - 2|$



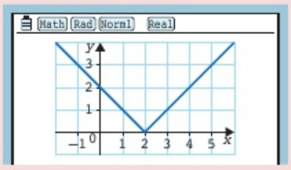
The graph of $y = |x - 2|$ is made of two straight lines with equation $y = x - 2$ when $x \geqslant 2$ and $y = 2 - x$ when $x < 2$

Hint

The choice of values of x can be any number within the domain; i.e. $x \geqslant 2$ for the first part and $x < 2$ for the second part. The chosen values are picked for the convenience of calculations.

Fact

This graph can be constructed with graphing software or using a GDC.



Worked example 2.14

Solve each inequality and show each solution on a number line.

a $|x + 1| < 2$ **b** $\left|\dfrac{x}{2}\right| \geqslant 3$

Challenge

Solution

To solve the inequalities, we need to split each one into two cases: first where the expression inside the absolute value is greater than or equal to zero, then where the expression is less than zero.

a For $x \geqslant -1$ we have $x + 1 \geqslant 0$, so $|x + 1| = x + 1$

So for $x \geqslant -1$ the inequality becomes $x + 1 < 2$, and so $x < 1$

Therefore, we have $x \geqslant -1$ and $x < 1$ which implies that $-1 \leqslant x < 1$

For $x < -1$ we have $x + 1 < 0$ so $|x + 1| = -(x + 1) = -x - 1$

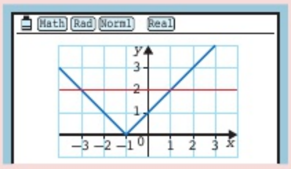
So for $x < -1$ the inequality becomes $-x - 1 < 2$, and so $x > -3$

Therefore $-3 < x < -1$

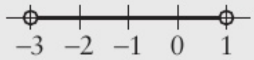
The solution is the union of $-1 \leqslant x < 1$ and $-3 < x < -1$, which gives $-3 < x < 1$

Fact

You can use your GDC or available software to solve the inequality by graphing $y = |x + 1|$ and $y = 2$ and looking for the values where the first graph is below the second.



Sketching the solution on a number line:



To check our solution, we can substitute a few values for x into the original inequality.

If $x = 0$, which is in the solution interval, then $|x + 1| = |0 + 1| = 1 < 2$, and so satisfies the inequality.

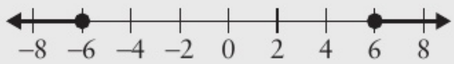
If $x = 2$, which is outside the solution interval, then $|x + 1| = |2 + 1| = 3$, which is not less than 2 and so does not satisfy the inequality.

b If $\frac{x}{2} \geqslant 0$, then the inequality becomes $\frac{x}{2} \geqslant 3$ and so $x \geqslant 6$

If $\frac{x}{2} < 0$, then the inequality becomes $-\frac{x}{2} \geqslant 3$ and so $x \leqslant -6$

The solution is the union of the two: $x \geqslant 6$ or $x \leqslant -6$

Sketching the solution on a number line:



Practice questions 2.2.3

1 Find the value for each number.

a $|\pi|$ **b** $|-2|$ **c** $\left|\frac{3}{2}\right|$ **d** $|-\sqrt{2}|$

2 Simplify and work out the absolute value expressions.

a $|121 - 139|$ **b** $|3^2 - \sqrt{16}|$

c $\left|\frac{3 \times 4 - 2^2}{16 - 5^2}\right|$ **d** $|(3 - 12) + (25 - 34)|$

3 Solve each equation.

a $|x - 6| = 5$ **b** $|2x - 5| = 11$ **c** $\left|\frac{2x}{3}\right| = 4$ **d** $\left|\frac{x - 5}{6}\right| = 2$

4 Solve each inequality.

a $|x - 11| < 3$ **b** $|x + 3| \geqslant 3$

c $\left|\frac{x}{3}\right| \leqslant 5$ **d** $\left|\frac{x + 2}{4}\right| > 3$

Challenge Q5

5 Solve the inequality and show the solution on a number line.

$|(2x + 3) - (3x - 2^3)| \leqslant 12$

6 Draw the graph of the equation $y = |3x - 2|$

2.2.4 Polynomials

Explore 2.6

1 Find the value of $P(3)$ if $P(x) = 3x^3 - 12x^2 + 5x + 4$

2 Is $x - 1$ a factor of the polynomial $P(x) = 2x^3 + 3x^2 - 4x - 1$? Explain.

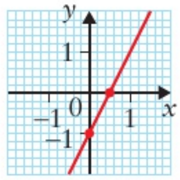
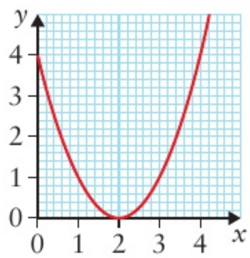
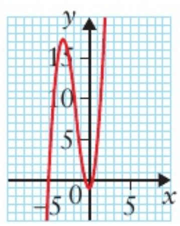
A **polynomial** in a single variable, x, is an algebraic expression that is the sum of **terms** in the form $a_n x^n$. The general form of a polynomial is:

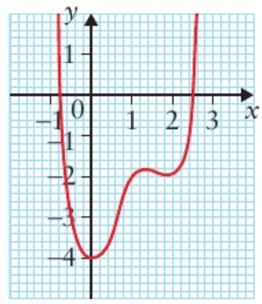
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ \dots \ + a_2 x^2 + a_1 x^1 + a_0 x^0$$

The values a_n are called **coefficients**, and are real numbers. x is the **variable** and n is a positive integer or zero. The value of n is the **degree** of the polynomial. The term with no variable (or a variable with a power of zero (a^0)) is the **constant term**.

The coefficient of x^n is called the **leading coefficient**.

Polynomials can be named according to their degree. The table below gives some examples.

Name and degree (n)	Example	Graph
Linear: $n = 1$	$P(x) = 2x - 1$	 Straight line
Quadratic: $n = 2$	$P(x) = x^2 - 4x + 4$	 Parabola
Cubic: $n = 3$	$P(x) = x^3 + 5x^2 - 1$	

Quartic: $n = 4$	$P(x) = x^4 - 4x^3 + 4x^2 + x - 4$	

Worked example 2.15

State whether each expression is a polynomial. Give a reason for your answer.

a $3 - \dfrac{2}{x}$ **b** $x^2\sqrt{x} - 3x + 5$

c $5x^3 - \dfrac{4}{3}x^2 + \sqrt{5}\,x - \pi$ **d** $3^x + 5$

Solution

An algebraic expression is a polynomial if all the powers of the variables are positive integers or zero. Coefficients can be any real number.

a $3 - \dfrac{2}{x}$ is not a polynomial. The term $\dfrac{-2}{x}$ can be written as $-2x^{-1}$, so the power of x is a negative integer.

b $x^2\sqrt{x} - 3x + 5$ is not a polynomial. The term $x^2\sqrt{x}$ can be written as $x^2x^{\frac{1}{2}} = x^{\frac{5}{2}}$, so the power of x is not a positive integer.

c $5x^3 - \dfrac{4}{3}x^2 + \sqrt{5}\,x - \pi$ is a polynomial. The powers of the variables are all positive integers or zero.

d $3^x + 5$ is not a polynomial because 3^x is not a variable with an integer power; it is an exponential expression.

Worked example 2.16

For each polynomial, state the degree, the leading coefficient, and the constant term.

a $5x^4 - \dfrac{1}{2}x^3 + 5x$ **b** $10 - 2x + 3x^2 - 5x^3$

c $\sqrt{2}\,x^3 + 5x^2$ **d** $4x^2 + 0.5x^3 - 7.5$

Fact Qb

If all the coefficients in a polynomial are non-zero numbers, it's called a **complete** polynomial.

Solution

The degree is the highest power of the variable.

The leading coefficient is the coefficient of the highest degree term.

The constant term is the term that does not contain a variable.

	Polynomial	Degree	Leading coefficient	Constant term
a	$5x^4 - \frac{1}{2}x^3 + 5x$	4	5	0
b	$10 - 2x + 3x^2 - 5x^3$	3	-5	10
c	$\sqrt{2}x^3 + 5x^2$	3	$\sqrt{2}$	0
d	$4x^2 + 0.5x^3 - 7.5$	3	0.5	-7.5

Adding and subtracting polynomials

To add or subtract polynomials, we collect like terms.

If $P(x) = x^2 + 4x + 4$ and $Q(x) = 3x^2 - 2x + 1$ then:

$P(x) + Q(x) = (x^2 + 3x^2) + (4x - 2x) + (4 + 1) = 4x^2 + 2x + 5$

$P(x) - Q(x) = (x^2 - 3x^2) + (4x - (-2x)) + (4 - 1) = -2x^2 + 6x + 3$

Multiplying polynomials

To multiply polynomials, we use the distributive property.

If $A(x) = 3x$, $B(x) = x + 3$, $C(x) = 2x - 1$, then:

$A(x) \times B(x) = 3x \times (x + 3) = 3x \times x + 3x \times 3 = 3x^2 + 9x$

$B(x) \times C(x) = (x + 3) \times (2x - 1) = x \times 2x - x \times 1 + 3 \times 2x - 3 \times 1$

$= 2x^2 - x + 6x - 3 = 2x^2 + 5x - 3$

Fact

The sum, difference and product of two polynomials are always polynomials themselves. What about the quotient of two polynomials?

Worked example 2.17

$P(x) = 3x^2 - 5x$, $Q(x) = 2x - x^2$ and $R(x) = 2 - x$

Write each expression in simplified form.

a $P(x) + Q(x)$ **b** $Q(x) - R(x)$ **c** $Q(x) \times R(x)$

Solution

a $P(x) + Q(x) = 3x^2 - 5x + (2x - x^2)$

$= (3x^2 - x^2) + (2x - 5x)$

$= 2x^2 - 3x$

b $Q(x) - R(x) = 2x - x^2 - (2 - x)$

$= -x^2 + (2x - (-x)) + (0 - 2)$

$= -x^2 + 3x - 2$

c $Q(x) \times R(x) = (2x - x^2) \times (2 - x)$

$= 2x \times 2 - 2x \times x - x^2 \times 2 + -x^2 \times -x$

$= 4x - 2x^2 - 2x^2 + x^3$

$= x^3 - 4x^2 + 4x$

Investigation 2.1

Exploring the product of binomials

A **binomial** is a polynomial with two terms. Answer the following questions to investigate whether there is a pattern for the product of two binomials.

1 **a** Make a copy of the table below and fill in the expansion of each binomial product.

Binomial	Binomial product
$(x + 1) \times (x + 1)$	
$(x + 2) \times (x + 2)$	
$(x + 3) \times (x + 3)$	
$(x + 4) \times (x + 4)$	

b Is there a pattern for the product? Explain your finding.

c Can you summarise the binomial product of $(x + a) \times (x + a)$ as a formula? Show that your formula works for other examples.

2 **a** Work out each binomial product.

Binomial	Binomial product
$(x - 1) \times (x - 1)$	
$(x - 2) \times (x - 2)$	
$(x - 3) \times (x - 3)$	
$(x - 4) \times (x - 4)$	

b Can you summarise the binomial product of $(x - a) \times (x - a)$ as a formula? Justify your formula with examples

3 Use similar examples for the product of $(x + a) \times (x - a)$. What is the formula of this binomial product? Show that it works for other examples too.

Explore 2.7

Look at the table below and write the factorised and expanded form for each product. The first one has been done for you.

Product	Factorised form	Expanded form
$(x + a) \times (x + a)$	$(x + a)^2$	$x^2 + 2ax + a^2$
$(x - a) \times (x - a)$		
$(x + a) \times (x - a)$		
$(x + a) \times (x + a) \times (x + a)$		
$(x - a) \times (x - a) \times (x - a)$		

Dividing polynomials

When we divide 13 by 5, the quotient is 2 and the remainder is 3. So, we can write $13 = 5 \times 2 + 3$. That is, dividend = divisor × quotient + remainder

$$\begin{array}{rl} & 2 \quad \leftarrow \text{quotient} \\ \text{Divisor} \rightarrow 5 & \overline{)13} \quad \leftarrow \text{dividend} \\ & \underline{-10} \\ & 3 \quad \leftarrow \text{remainder} \end{array}$$

The same division rule can be applied to polynomials.

> **Connections**
>
> There are many different ways of setting out long division of numbers and algebraic expressions.
> If you know a different method, compare it with this one and show that the same two have the same effect.

Worked example 2.18

Divide $P(x) = x^2 + 4x + 5$ by $D(x) = x - 2$

Solution

First make sure that each polynomial is ordered, with its highest degree term first, followed by the other terms in descending order of their exponents.

> **Hint**
>
> The exponent is the power of the variable, such as 2 in x^2.

The dividend is $x^2 + 4x + 5$ and the divisor is $x - 2$. We write them in the same way as for numerical division.

$$x - 2\overline{)x^2 + 4x + 5}$$

We divide the first term of the dividend by the first term of the divisor, and put the result in the answer. In this case, we divide x^2 by x and put the quotient, x, in the answer.

$$\begin{array}{r} x \qquad\quad \\ x - 2\overline{)x^2 + 4x + 5} \end{array}$$

We multiply the divisor by that answer and put the result in the line below the dividend, making sure to align the terms in columns.

$$\begin{array}{rl} & x \\ x-2 & \overline{)\,x^2+4x+5} \\ & \underline{x^2-2x} \end{array}$$

Then we subtract to create a new polynomial.

$$\begin{array}{rl} & x \\ x-2 & \overline{)\,x^2+4x+5} \\ & \underline{x^2-2x} \\ & \qquad +6x+5 \end{array}$$

Now, with the new polynomial, we repeat the same steps above until we get a new polynomial with a smaller degree than the divisor.

$$\begin{array}{rll} & x+6 & \text{quotient} \\ x-2 & \overline{)\,x^2+4x+5} & \\ & \underline{x^2-2x} & \\ & \qquad +6x+5 & \\ & \qquad \underline{+6x-12} & \\ & \qquad\qquad +17 & \text{remainder} \end{array}$$

Finally, we can write it in the form:

dividend = divisor × quotient + remainder

$$x^2 + 4x + 5 = (x - 2)(x + 6) + 17$$

That is, $P(x) = D(x) \times Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.

Factor and remainder theorem

For any polynomial $P(x)$, when we divide $P(x)$ by $(x - a)$ we can find a quotient $Q(x)$ and remainder R. Thus we can write $P(x) = (x - a) \times Q(x) + R$. Two theorems can be deduced from this result.

Remainder theorem

If $P(x) = (x - a) \times Q(x) + R$ then the remainder of the division is $P(a)$.

This is because $P(a) = (a - a) \times Q(a) + R = 0 + R = R$

Factor theorem

$(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$

We can verify this using the remainder theorem. The remainder when we divide $P(x)$ by $(x - a)$ is $P(a)$. If $(x - a)$ is a factor, the remainder must be zero. Therefore, $P(a) = 0$. Also if $P(a) = 0$, then $R = 0$, so $P(x) = (x - a)Q(x)$, showing that $(x - a)$ is a factor of $P(x)$.

For example, if we divide $x^2 - 3x + 2$ by $x - 1$, the remainder is $P(1) = 1^2 - 3(1) + 2 = 0$. So $x^2 - 3x + 2$ is divisible by $x - 1$. If we use long division we can find the quotient, $x - 2$, and remainder 0, so $x^2 - 3x + 2 = (x - 1) \times (x - 2)$

Worked example 2.19

Determine whether $x - 2$ is a factor of this polynomial:
$P(x) = x^3 - x^2 - 4x + 4$

Solution

We need to see if $x - 2$ divides $P(x) = x^3 - x^2 - 4x + 4$

Using the factor theorem, if $P(2) = 0$, then $x - 2$ divides $P(x)$ exactly.

$P(2) = 2^3 - 2^2 - 4 \times 2 + 4 = 8 - 4 - 8 + 4 = 0$

So $x - 2$ is a factor of $P(x) = x^3 - x^2 - 4x + 4$

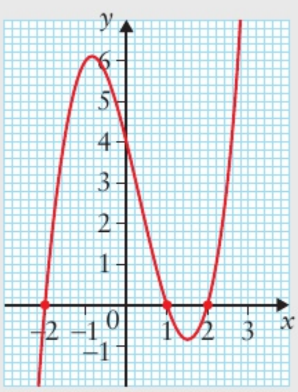
We can then use long division.

$$
\begin{array}{r}
x^2 + x - 2 \\
x - 2 \overline{) x^3 - x^2 - 4x + 4} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
x^2 - 4x \qquad \\
\underline{x^2 - 2x} \qquad \\
-2x + 4 \\
\underline{-2x + 4} \\
0
\end{array}
$$

So $x^3 - x^2 - 4x + 4 = (x - 2) \times (x^2 + x - 2)$

In this case we can also factorise the quotient: $x^2 + x - 2 = (x + 2) \times (x - 1)$

So $x^3 - x^2 - 4x + 4 = (x - 2) \times (x + 2) \times (x - 1)$



Connections

We can use a graphical calculator or graphical software program to generate the graph of $P(x)$ from Worked example 2.19.

Practice questions 2.2.4

1 Find the degree of each polynomial.

a $3x^2 - 2x + 5$

b $5x^4 - x^3 + 5x^2 - 1$

c $\frac{1}{\sqrt{2}}x - \frac{x^2}{3} + 12x^3$

d $1 - 2x + 3x^2 - 4x^5$

2 Write the leading term and constant of each polynomial.

a $x^2 - 3x + 2$

b $-x^4 - 2x^3 + 3x^2 - 1$

c $\frac{1}{2}x^3 + 2x^4 - 5x$

d $4 - 3x + 2x^2 - x^3$

3 State whether or not each expression is a polynomial.

a $\frac{1}{x} + \frac{x^2}{2}$

b $1 - 5x + 3x^2$

c $5^x - x^5$

d $x^3 - 2x^2 + x^{\frac{1}{2}}$

Questions 4–7 refer to the following polynomials:

$P(x) = x - 1 \qquad Q(x) = 2x - 4 \qquad R(x) = x^3 - x^2 + x - 1$

4 Simplify each of these operations.

a $P(x) + Q(x)$

b $R(x) - Q(x) - P(x)$

c $R(x) - P(x) \times Q(x)$

5 Simplify each of these products.

a $P(x) \times Q(x)$

b $P(x) \times R(x)$

c $Q(x) \times R(x)$

6 Is $P(x)$ a factor of $R(x)$? Give a reason for your answer.

7 Find the remainder when $R(x)$ is divided by $Q(x)$.

Self assessment

- I can identify rational and irrational numbers.
- I can plot irrational numbers on a number line.
- I can prove that $\sqrt{2}$ is an irrational number.
- I can work out which consecutive integers an irrational number lies between.
- I can do operations on irrational numbers.
- I know what is meant by the property of closure.
- I can use the commutative property and the associative property.
- I know the identity elements of multiplication and addition.
- I can find the additive and multiplicative inverses of a number.
- I know what is meant by the property of distribution.
- I can use set and interval notation.
- I can represent a number set on a number line.
- I can use union $\cup$ and intersection $\cap$ of sets.
- I know what is meant by the reflexive property.
- I know what is meant by the symmetry property.
- I can maintain equivalence to solve equations and inequalities.
- I can find the absolute value of a number.
- I can solve equations and inequalities with absolute value.
- I know what a polynomial is.
- I can find the degree, the leading coefficient and the constant term of a polynomial.
- I can find the sum and difference of polynomials.
- I can multiply two polynomials.
- I can divide two polynomials.
- I can work out if a given polynomial is a factor of another given polynomial.
- I can find the remainder of a polynomial when it is divided by $x - a$
- I can find the product of binomials.

Check your knowledge questions

1 Classify each number as rational or irrational.

a 2^{-3} b $3^{\frac{2}{3}}$ c $\sqrt{25}$ d $\pi + \frac{\pi}{6}$ e $3 - \sqrt{2}$

2 Find the two consecutive integers each irrational number lies between.

a $\sqrt{19}$ b $\sqrt{44}$ c $\sqrt{59}$

d 3π e $\sqrt{15} + \sqrt{23}$

Challenge Q2e

3 Find the multiplicative inverse of each number.

a $\frac{-11}{13}$ b 2^3 c $\sqrt{5}$ d $\frac{1}{\frac{\pi}{2}}$ e $2 - x$

Challenge Q3e

4 Find the additive inverse of each number.

a 2^4 b -0.5 c $\pi - 2$ d $\sqrt{2}$ e $\frac{\pi}{2} - x$

5 Simplify each expression using the commutative and associative properties.

a $12y - 15 + (-9y) + 8$

b $\left(\frac{1}{2} + \frac{2}{3}\right) + \frac{1}{3}$

c $14x - 2y + 2x + 18y$

6 Describe each set using interval notation.

a $\{x \mid -1 \leqslant x < 5\}$

b $\left\{x \mid \frac{12}{5} \leqslant x \leqslant 14\right\}$

c $\{x \mid -4 < x < 9\}$

d $\left\{x \mid \frac{3\pi}{2} < x \leqslant 2\pi\right\}$

7 Write each interval using set notation.

a $\left(-\frac{2}{3}, 1.5\right)$

b $[2, 13]$

c $[-1, 1) \cup (1, 5]$

d $\left(-\frac{\pi}{2}, \pi\right]$

8 Use interval notation to show the solution to $2x + 5 > 21$

9 Solve $3x + 5(2x + 5) = 6(x + 12)$ by using the equivalency rules.

Challenge Q10d

10 Find the absolute value of each number.

a $\left|\pi - \frac{3\pi}{2}\right|$

b $|11 - 9|$

c $\left|\frac{3}{2} - \frac{4}{3}\right|$

d $|-\sqrt{2^3}|$

11 Solve each of these equations.

a $|x - 3| = 11$

b $|3x - 2| = 16$

c $\left|\frac{x}{5}\right| = 11$

d $\left|\frac{x-1}{5}\right| = 3$

Challenge Q12

12 Solve the inequality and show the solution on a number line.

$\left|\frac{2x-5}{3}\right| \leqslant 3$

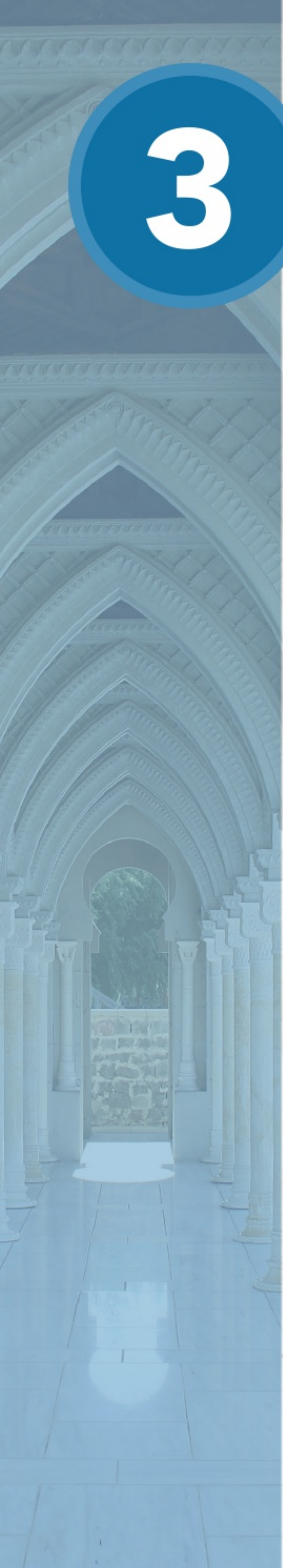
13 For the polynomials $P(x) = x^2 + 3x + 2$ and $Q(x) = x + 1$, find:

a $P(x) - Q(x)$

b $P(x) \times Q(x)$

c the remainder when $P(x)$ is divided by $Q(x)$

3 Relations and functions



3 Relations and functions

KEY CONCEPT

Form

RELATED CONCEPTS

Models, Representation, Systems

GLOBAL CONTEXT

Identities and relationships

Statement of inquiry

Decision-making can be improved by using models to represent relationships in different forms.

Factual

- How does mathematics describe the relations between objects?
- What representation can be used to describe relations?

Conceptual

- In what contexts are unique answers important?
- Are mathematical operations always reversible?

Debatable

- Does a visual representation of mathematics carry sufficient information and precision?
- Do mathematical relations have to follow an obvious rule?

Do you recall?

1 Find an equation of a line with y-intercept (0, 3) and gradient 2.

2 Find an equation of a line through the points (0, 4) and (4, 0).

3 Find the tenth term in the sequence 1, 4, 7, 10, 13, … using the obvious pattern.

4 Is 16 the only number that is 12 more than 4?

5 Is 4 the only number that can be squared to give 16?

6 If you are asked how many numbers there are between 1 and 10 inclusive, what needs to be said about the type of number to be sure of the answer?

3.1 Relations

3.1.1 Making a connection

Explore 3.1

Make a list of the countries you have visited.

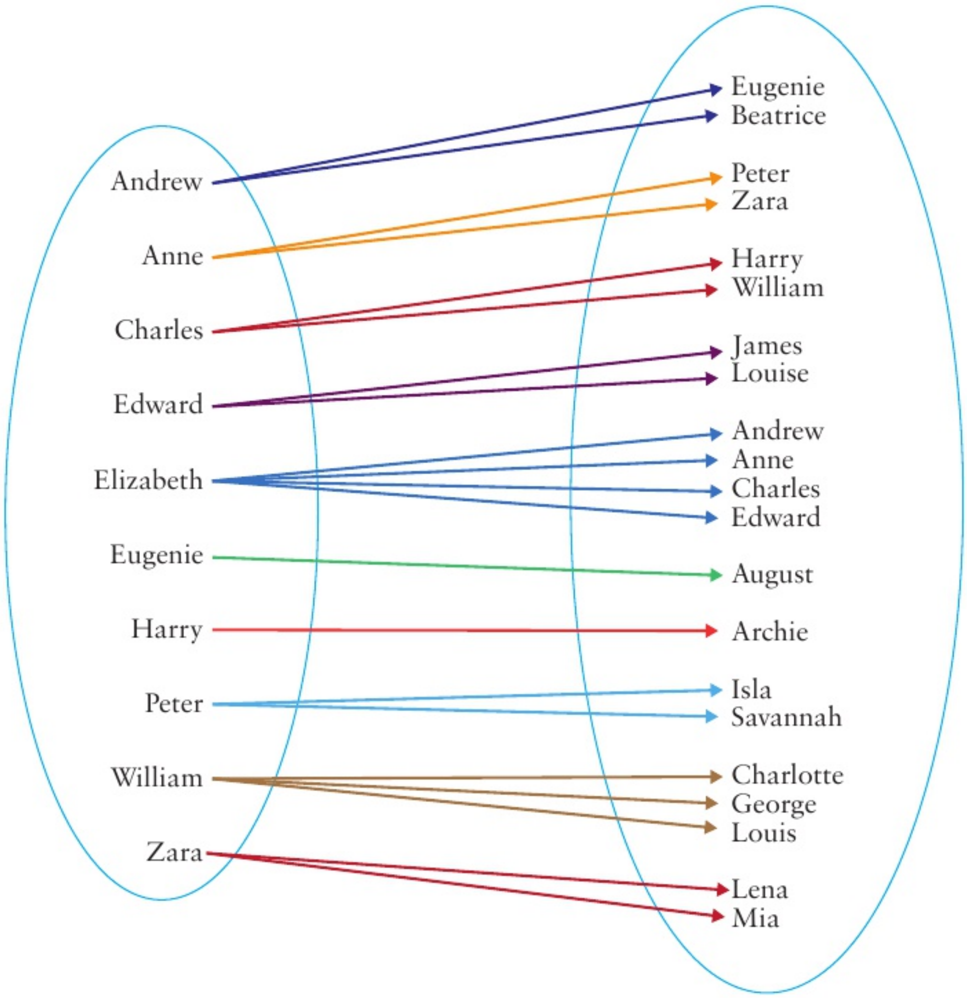
List all the members of your class on the left side of a page, all the countries any of you have visited on the right side, and then make connections between each person and their countries. Which country has the most links? Are there some countries with just one link?

How many members of your class can be uniquely identified from just the list of countries they have visited?

The mathematical idea of a **relation** means a link between two sets. It can be applied in many contexts and represented in a number of different ways.

is the parent of ↑ →	Andrew	Anne	Archie	August	Beatrice	Charles	Charlotte	Edward	Eugenie	George	Harry	Isla	James	Lena	Louis	Louise	Mia	Peter	Savannah	William	Zara
Andrew					×				×												
Anne																		×			×
Charles											×									×	
Edward													×			×					
Elizabeth	×	×				×		×													
Eugenie				×																	
Harry			×																		
Peter												×							×		
William							×			×					×						
Zara														×			×				

The grid on the previous page shows the parental relations between Queen Elizabeth II and her direct descendants in the British Royal family. For example, Zara is the parent of Lena and Mia.



This diagram is another representation of the same information. It is called a mapping diagram. The parents, on the left, form a set called the **domain**. The children, on the right, form another set called the **range**.

A relation links the elements of the domain and those of the range. Expressed in words, the relation can be stated as 'is the parent of'.

Because this concerns the direct descendants, one parent is shown in each family. However, one parent may have many children. This is an example of a **one-to-many** relation.

There are some elements of the domain that also appear in the range (for example, William). You will find that this is quite common for numerical relations.

Worked example 3.1

Snails keep eating Eva's tomato plant, so she decided to watch them for an hour one evening and find how far they travelled towards the plant. Her results are shown in the table. The initial distance from the tomato plant is x cm, and the distance after one hour is y cm.

x	321	540	282	236	82	123	211	211
y	274	523	198	187	30	156	145	209

Represent this information by a set of ordered pairs.

Solution

The set of initial distances, x cm, is the domain. The set of distances after one hour, y cm, is the range.

We have used ordered pairs before to define the x- and y-coordinates of points in a Cartesian graph. The order of the pair of numbers, say (3, 5), is important. The first number, 3, is in the domain, whereas the second, 5, is in the range.

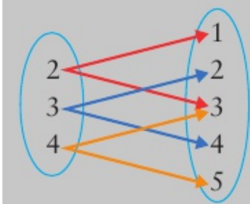
The set of ordered pairs for Eva's data is:

{(321, 274), (540, 523), (282, 198), (236, 187), (82, 30), (123, 156), (211, 145), (211, 209)}

Looking back, we see that for the snails there is no obvious rule connecting x to y, except that $y < x$ in most cases.

Worked example 3.2

Look at the mapping diagram.



a Identify the domain and range.

b Translate the information into a set of ordered pairs.

c Find a rule that could connect elements in the domain to those in the range.

d Follow the rule to suggest four additions to the domain and range in the form of ordered pairs.

e Use the ordered pairs as points and plot them on Cartesian axes.

Solution

Understand the problem

The arrows in the mapping diagram point from the elements in the domain to those in the range. Each arrow represents a connection, so there should be as many ordered pairs as there are arrows.

Make a plan

Identify the domain and range. To find a possible rule, pick one point in the domain and explain its connection to the points in the range. Then check the rule to see if it applies to the other points in the domain.

Carry out the plan

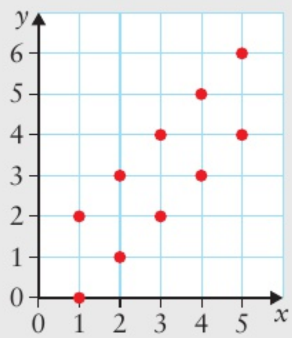
a The domain is the set $\{2, 3, 4\}$. The range is the set $\{1, 2, 3, 4, 5\}$

b The set of ordered pairs is $\{(2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5)\}$

c 2 connects to 1 and 3, which are its nearest integer neighbours.
This works also for the connections from 3 and 4 in the domain.
So, the rule is that $x \rightarrow x \pm 1$

d If we include 1 and 5 in the domain and add 0 and 6 to the range then we obtain the additional pairs: $(1, 0)$, $(1, 2)$, $(5, 4)$, and $(5, 6)$

e



Reflect

Is there enough evidence in the mapping diagram to be sure that you have the correct rule?

Would a graph of just the original pairs help you find the rule?

Might you have a relation in which there is no obvious rule?

Communication skills

Hint Q1a

You may need to look up some of these books.

Practice questions 3.1.1

1 Use the relation 'is the author of' to link the elements of the domain:

{Louisa May Alcott, Jane Austin, Agatha Christie, Charles Dickens, Victor Hugo, Jack London, J.K. Rowling}

with the range:

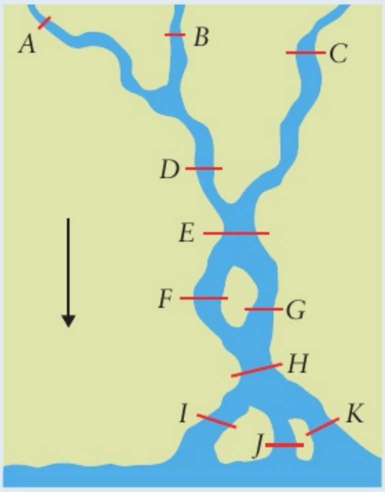
{A Christmas Carol, A Tale of Two Cities, Emma, Harry Potter and the Goblet of Fire, Les Misérables, Little Women, Murder on the Orient Express, Pride and Prejudice, The Call of the Wild, The Cuckoo's Calling, White Fang}

a Fill in a copy of the table.

b Draw a mapping diagram.

is the author of	A Christmas Carol	A Tale of Two Cities	Emma	Harry Potter and the Goblet of Fire	Les Misérables	Little Women	Murder on the Orient Express	Pride and Prejudice	The Call of the Wild	The Cuckoo's Calling	White Fang
Louisa May Alcott											
Jane Austin											
Agatha Christie											
Charles Dickens											
Victor Hugo											
Jack London											
J.K. Rowling											

2 The picture shows a river estuary with various observation points labelled A to K. The arrow shows the general direction of flow of the river.



Logs float unimpeded down the river towards the sea and are observed passing the various observation points.

a If a log is observed at a particular point, which other points does it definitely pass, either before or after the observation point?

b Create a table to show the connections.

3 For each of the following mapping diagrams:

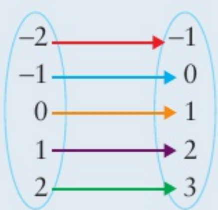
i identify the domain and range

ii write all the connections in the relation as a set of ordered pairs

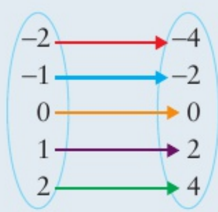
iii where possible, suggest a rule for the relation

iv if you find a rule, suggest four further ordered pairs that would extend the domain and range while respecting the rule.

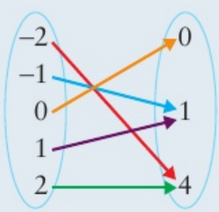
a



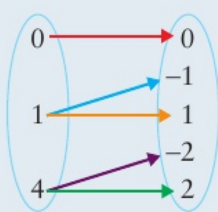
b



c



d



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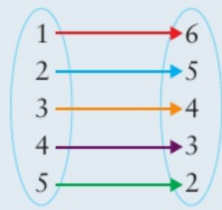

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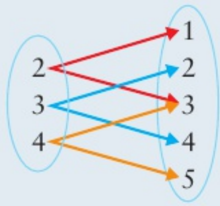


4 Draw a mapping diagram for each of the following sets of ordered pairs. In each case, plot the points corresponding to the ordered pairs on Cartesian axes and suggest a rule for the relation from the domain to the range. Add two more points, to each graph, that satisfy the same rule.

- **a** $\{(3, 1), (6, 2), (9, 3), (12, 4)\}$
- **b** $\{(1, 1), (2, 8), (3, 27), (4, 64)\}$
- **c** $\{(0, 0), (1, -1), (2, -2), (3, -3)\}$
- **d** $\{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
- **e** $\{(0, 9), (1, 4), (2, 1), (3, 0), (4, 1), (5, 4), (6, 9)\}$

3.1.2 One-to-one relations

Explore 3.2

A computer has been set up to assign you an ID code made up of the first three letters of your family name followed by the first two letters of your given name. For example, the code for the name **AR**IANA **GRA**NDE would be **GRAAR**.

Would your ID code be unique in your class; in your extended family; in your school; in the entire country?

What could be added to the code to make sure it is unique?

You might need to remember a list for a test. There are some well-known techniques and here are two examples.

A memory aid

The common French verbs that conjugate with être in the passé composé (past perfect tense) are

Monter, **R**etourner, **S**ortir, **V**enir, **A**ller, **N**aître, **D**escendre, **E**ntrer, **R**ester, **T**omber, **R**entrer, **A**rriver, **M**ourir, **P**artir

In this order, the initial letters form 'Mrs Vandertramp', which is an invented name, but one that is quite easy to remember when it has been repeated a few times.

Thinking skills

A memory palace

The noble gases are helium (He), neon (Ne), argon (Ar), krypton (Kr), xenon (Xe) and radon (Rn).

To remember them, you could attach post-it notes to various objects in your room (it does not have to be a palace) and get used to seeing them. When you want to remember them, close your eyes for a moment and think of your room, looking at each object to recall the names of the gases.

The first of these examples uses a **one-to-one relation**. Each verb has a single letter in Mrs Vandertramp and each letter in Mrs Vandertramp has a single verb.

In the memory palace, each noble gas has a single object in the room to identify it, but there are objects in the room that do not have labels. To make the second example one-to-one you would have to restrict the objects in the room to just those with labels.

Look at the picture on the opening page of this chapter. Each column on the right is connected to one column on the left. As you walk through, you do not need to count the columns to know that there are as many on the right as there are on the left. The columns are in a one-to-one relation. This is one of the most fundamental concepts of mathematics, and is the basis of our ability to count. If you have ever counted on your fingers, you have used a one-to-one relation between your fingers and the objects you are counting.

For a one-to-one relation there must be the same number of elements in the domain as in the range. This sounds easy to check, but there is the possibility that both sets are infinite. There must also be a rule that links an element in the domain to an element in the range but does not link either to more than one element.

Worked example 3.3

Determine whether or not there is a one-to-one relation:

a between the people in your class and their birthdays

b between a domain of the first ten positive whole numbers and a range of how many factors each number has

c between the number of sheep in a field and the total number of legs of the sheep in the field.

Solution

a Everyone in the class has a birthday. For this to be a one-to-one relation we have to apply the condition that no two people in the class can have the same birthday.

If that condition does not apply then it is not a one-to-one relation.

b All prime numbers have exactly two factors, so 2, 3, 5 and 7 in the domain will all map onto 2 in the range. Therefore, it cannot be a one-to-one relation.

c In this case there is a one-to-one relation. For example, with 10 sheep in the field there would be 40 legs, and if there were 40 legs there would be 10 sheep.

Connections

Factors: can you find two other counter-examples in addition to the prime numbers?

Practice questions 3.1.2

1 Identify which of the following situations involve a one-to-one relation.

  a Everyone in a room is seated and there are no spare chairs.

  b It is difficult to count animals in a field if they keep moving, so every day a farmer makes sure that no cows have wandered off by checking that every milking bay is occupied at milking time.

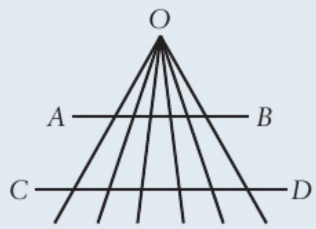
  c The players in each of two football teams (before anyone is sent off).

d The letters A, B, C and the different arrangements that can be made from the letters A, B, C.

e Authors and the books written by the authors.

f The elements of the periodic table and their symbols.

g The points where the segment AB is cut by the rays coming from O and those where CD is cut by the same rays.



2 Which of the mapping diagrams in question 3 of Practice questions 3.1.1 represent one-to-one relations?

3 Which of the sets of ordered pairs in question 4 of Practice questions 3.1.1 represent one-to-one relations?

3.2 Functions

3.2.1 Definition of a function

Explore 3.3

$(-2)^2 = 4$, so would it be correct to say that -2 is the square root of 4?

$(-2)^3 = -8$, so is -2 the cube root of -8?

Ask similar questions for other integer powers, $(-2)^n$.

How would you define the nth root?

The distinction between **one** and **many** is important when describing functions. 'Many' in this context means more than one.

A function can be thought of as a process with an input and an output. Two different inputs could give the same output, but with a function you cannot obtain different outputs from the same input.

As it is a type of relation, we can use an arrow to show the link:
input → output

For example:

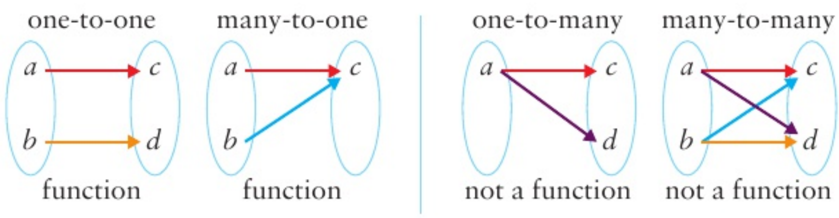
- **Add 5** is a function: $3 \rightarrow 8$ and $-3 \rightarrow 2$. There just one output for each input.
- **Square** is a function: $3 \rightarrow 9$ and $-3 \rightarrow 9$. In this case two different inputs give the same output, but there is still just one output to each input.

Functions are so common in mathematics that a relation that is not a function usually needs a longer description. A simple example would be **'is a factor of'**. This definition would take an input of 12 and give six numbers in the output: $12 \rightarrow 1, 2, 3, 4, 6, 12$

With six possible outputs from one input, this cannot be a function.

Remember that a function gives a unique output for each input.

We define a function as a **one-to-one** or **many-to-one** relation.



In a mapping diagram, if there is only one arrow leaving each element of the domain then it fulfils the one-to-one or many-to-one condition. It represents a function because there is no doubt about where that element is mapped to in the range.

If there is at least one member of the domain that has two or more arrows leaving it, then the relation is not a function.

Worked example 3.4

Consider the following sets of ordered pairs defining a relation from the first number in the ordered pair to the second number.

$$A = \{(0, 2), (1, 3), (2, 3), (3, 3), (4, -3)\}$$
$$B = \{(0, -3), (0, 2), (2, 1), (2, 3), (3, 1)\}$$

Do A and B represent functions?

Solution

Understand the problem

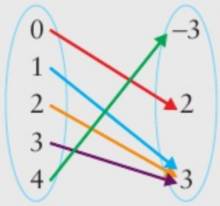
In each of the ordered pairs, the first number is in the domain and the second in the range. The number of connections between numbers is what determines whether or not this relation is a function.

Make a plan

We can draw a mapping diagram based on the ordered pairs to show the connections clearly.

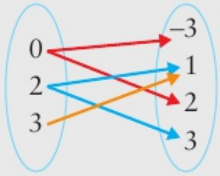
Carry out the plan

A has the domain $\{0, 1, 2, 3, 4\}$ and the range $\{-3, 2, 3\}$.



For A, each number in the domain has a single arrow leaving it, so relation A is a function. Because 3 in the range has many arrows arriving at it, it is a many-to-one function.

B has the domain $\{0, 2, 3\}$ and the range $\{-3, 1, 2, 3\}$.



Both 0 and 2 in the domain have many (more than one) arrows coming from them, so this is not a function. In the range, 1 has many arrows arriving, so it is a many-to-many relation.

Look back

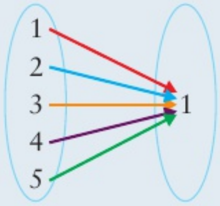
The key to identifying a function is knowing that a number in the domain should have only one corresponding number in the range. In practical terms, this usually involves a rule for which there is only one output for each input. For example, the rule might be to square each element in the domain.

Practice questions 3.2.1

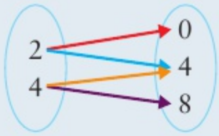
1 Which mapping diagram in question 3 of Practice questions 3.1.1 represents a many-to-one function?

2 For each of the following relations, decide if it is a one-to-one function, a many-to-one function, or not a function at all.

a



b



c The set of ordered pairs $\{(1, 5), (2, 9), (3, 13), (4, 17), (5, 21)\}$

d Any integer mapped to a number twice its size

e Any integer mapped to its square

f Any integer mapped to its cube

g Any square mapped to the integers that could be squared to obtain it

h The example of the British royal family from the start of the chapter, with the domain and range swapped and using the relation 'is the child of'

i The planets of our solar system mapped to their moons

j Any positive integer mapped to its approximate value when rounded to the nearest multiple of 10

k A number written to the nearest 10 mapped to the original number

3 32 chocolates are shared equally between a number of people, with some chocolates remaining. Is the relation connecting the number remaining to the number of people a function?

4 Consider a relation that maps x onto a number y that is greater than x.

a Write a list of five numbers greater than 10 and give some possible values of y when $x = 10$

b If you knew that $y = 13$, would you know that $x = 10$?

c Is x a function of y?

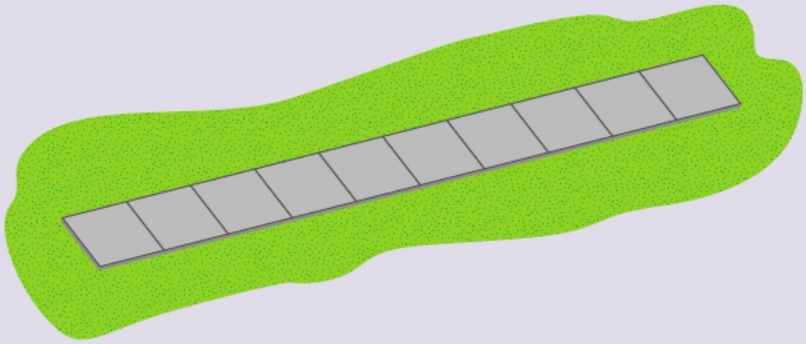
d Is y a function of x?

e Would your answers to parts c and d be different if x mapped onto a number y that is the smallest integer greater than x?

3.2.2 Beyond integer domains

Explore 3.4

To finish a path of 80 cm width there is a choice between pouring concrete or laying paving slabs. The slabs are squares measuring 80 cm by 80 cm. If they do not fit exactly into the length of the path, then a whole slab will have to be cut to length. The concrete can be mixed to fill a path of any length.



Pouring the concrete will cost €12 per metre length of path. For example, pouring an 8-m path would cost €96.

Paving slabs cost €9.25 each. For an 8-m path, the cost would be €92.50, with 10 slabs fitting exactly and no waste from cutting a slab.

Calculate the costs of each method for a 9-m path.

On a single set of axes, draw graphs to show the costs for each method for whole number path lengths between 5 m and 12 m.

What is the difference between the two graphs?

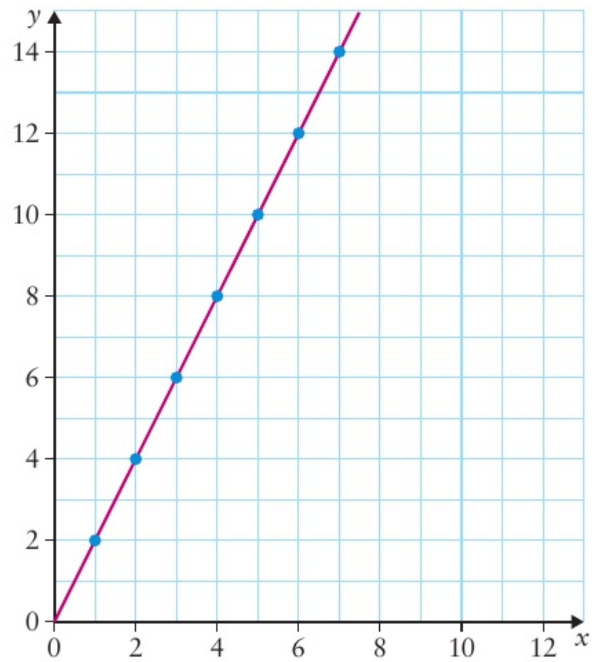
Reflect

Look back at your solutions to the one-to-one relations in questions 3 and 4 of Practice questions 3.1.1, particularly the values you added when there was a clear rule. Also look at the graphs you drew. Was it tempting to join the dots? Would it have been justified?

In the numerical examples so far in this chapter, the domain and range have been elements from the set of integers.

Consider the set of ordered pairs $\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}$.

The rule for the relation is apparently 'double the number'. So, we could enlarge the domain with $(6, 12)$ and $(7, 14)$. But we could also include points between those already given, for example $\left(\frac{1}{2}, 1\right)$ or $\left(\frac{5}{4}, \frac{5}{2}\right)$ which still obey the same rule. Thus the domain is no longer restricted to integers.



When all the points are plotted, a straight line can be drawn through them with the equation $y = 2x$. Any point on this line could be added to the domain and range without altering the rule.

In this way, we can extend the idea of relations between sets of integers to relations between sets of real numbers. The idea of one-to-one starts to get more complicated as there are an infinite number of mappings even within a small interval.

Reflect

Find the relations from questions 3 and 4 of Practice questions 3.1.1 that produce points lying on straight lines when graphed. Look for a pattern that applies to their rules.

Self-management skills

Worked example 3.5

During the coming month, a student will be commuting frequently to study at a distant university. The journey is by train and these are the ticket price options:

Present this information as a graph.

What recommendations would you give the student?

Solution

Understand the problem

The decision facing the student depends on the number of return journeys in the month. To make a recommendation we need to show how the cost of each option varies as the number of journeys changes. The number of journeys is the independent variable and belongs to the domain. The total cost of one-day returns depends directly on the number of return journeys. The season ticket is a fixed cost that does not vary with the number of journeys. The railcard is in part a fixed cost, and in part dependent on the number of journeys. The three different costs rely on three different functions and each belongs to the range of its own function.

A 40% reduction leaves 60% to pay.

Make a plan

We can draw a graph for each function to show which is cheapest for different numbers of journeys.

Typically, a graph uses x as the independent variable. There are three different formulae, so each has its own graph and range. To plot them on a graph, we calculate the cost €y in terms of the number of journeys, x.

We can see, apart from the season ticket, that the cost will rise in direct proportion to the number of journeys, so it is reasonable to assume a linear relationship. Only two points are needed to position a straight line, so two examples from each type of ticket will be enough for the graph.

Carry out the plan

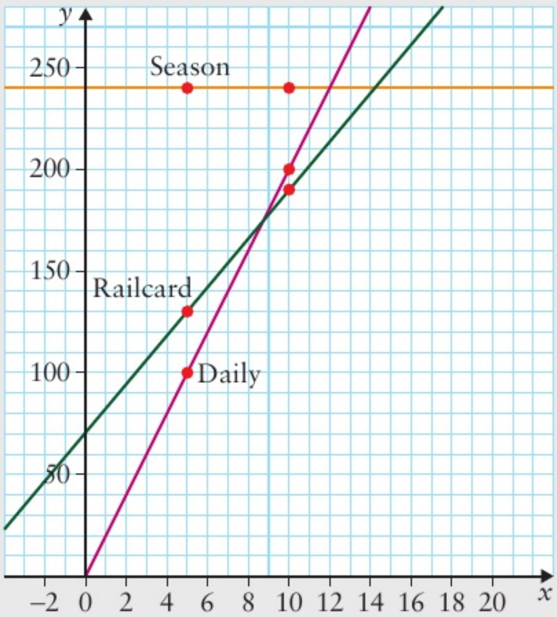
For five return journeys ($x = 5$) the cost in euros will be:

- One-day return: $y = 20 \times 5 =$ €100, giving the ordered pair (5, 100)
- Monthly season ticket: $y =$ €240, giving the ordered pair (5, 240)
- Monthly railcard: $y = 70 + 0.6 \times 20 \times 5 =$ €130, giving the ordered pair (5, 130)

For 10 return journeys ($x = 10$) the cost in euros will be:

- One-day return: $y = 20 \times 10 = €200$, giving the ordered pair (10, 200)
- Monthly season ticket: $y = €240$, giving the ordered pair (10, 240)
- Monthly railcard: $y = 70 + 0.6 \times 20 \times 10 = €190$, giving the ordered pair (10, 190)

Plotting the points for each ticket and drawing the lines gives us a way to compare the costs.



From the graph, up to 8 return journeys in the month would be cheapest with one-day returns, between 9 and 14 would be cheaper with the railcard and 15 and above would be cheaper with the season ticket.

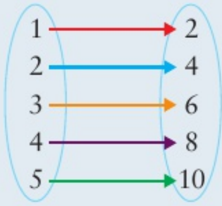
Look back

While it is convenient to plot the lines continuously, the domain is restricted to positive whole values as it is not possible to make fractional or negative return journeys.

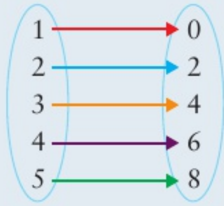
Practice questions 3.2.2

1 For each of the mapping diagrams, identify the rule and plot the ordered pairs on a coordinate graph. Then join the points to extend the domain to include all real numbers within the limits of the original domain.

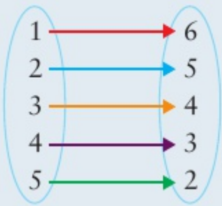
a



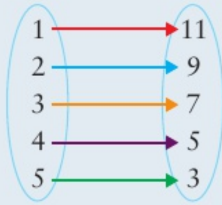
b



c



d



2 For each part in question 1, find the value in the range that corresponds to 1.1 in the domain.

3 You work in a shop which has a wide range of items for sale between €49.99 and €499.99.

The owner decides to clear a lot of stock to reorganise the layout of the shop. She therefore decides to have a sale to encourage people to purchase the stock. She is undecided on what offer to make and asks you to create a single sheet giving a visual comparison of the sale price against the original price for each of the following options:

A: €30 off every price

B: a 15% reduction in all prices

C: a 10% reduction and then €10 off the resulting price

Represent the three offers on a single set of axes showing graphs for the reduced costs against the original costs for any price between €49.99 and €499.99.

4 At the airport, you can change your money from dollars into euros. The service costs \$5, and for every additional dollar you get €0.85.

a Is the relation between the number of dollars and the number of euros a function?

b How many euros would you get for \$100?

c How many euros would you get for \$200?

d Use the results to plot points on a graph and draw a line through them to show the relation.

e Use your graph to determine how many dollars it would cost to obtain €50.

3.2.3 Function notation

Explore 3.5

The value of a function is defined as the original number subtracted from its cube.

Consider, but don't necessarily answer, the following questions:

a What is the value of the function when the original number is 0?

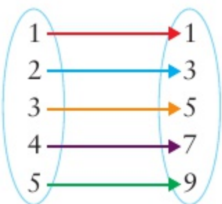
b What is the value of the function when the original number is −2?

c Which original number could give 24 as the value of the function?

d Which original numbers would be trebled by the function?

Devise a way of asking these questions so that they are much shorter, using symbols rather than words wherever possible. You might like to create your own notation.

A function converts the positive integers into the odd numbers. The first few terms are shown in the mapping diagram, but there are infinitely many terms, so a mapping diagram cannot show the full domain and range. A set of ordered pairs would have the same limitation.



What is needed is a way to express the rule: doubling the number in the domain and subtracting 1.

If n is any number in the domain (the set of positive integers) then the rule would turn it into $2n - 1$. This is a function, as it is not possible to obtain two different answers in the range for one value of n in the domain.

We will call the function f.

We can define the function as:

$f(n) = 2n - 1$

We read this as 'f of n equals two n minus one'.

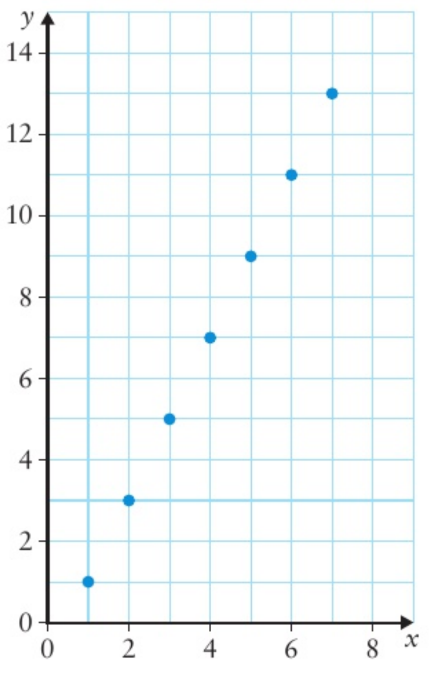
We can replace n with positive integers, so $f(2) = 3$ and $f(100) = 199$, but we can also replace n with another letter: $f(x) = 2x - 1$

$f(n) = 2n - 1$ and $f(x) = 2x - 1$ each describe the function f completely.

To draw a Cartesian graph, we need a relation between x- and y-coordinates to decide whether a point in the plane is part of the graph.

We put $y = f(x)$

The domain for this example could be that x is a positive integer, $x \in \mathbb{Z}^+$



Unlike the function notation, the graph is limited in the same way as the mapping diagram. It can only show some relations. The positive integers are infinite, but our graph is finite.

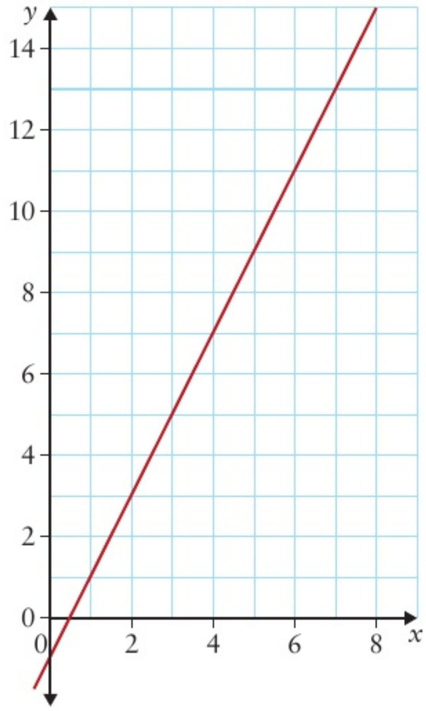
Fact

There is another form of notation that is often used to define functions, which imitates a mapping diagram.

$f: n \rightarrow 2n - 1$

We read this as 'f maps n onto two n minus one'.

If we change the domain to all real numbers, the graph has an advantage as it represents an infinite number of values along the line in a way that the mapping diagram cannot.



We describe this as the graph of $y = f(x)$, where $f(x) = 2x - 1$, defined on the domain $x \in \mathbb{R}$.

This is a long way of writing what we would call the graph of $y = 2x - 1$, but the notation allows much greater flexibility in use.

Communication skills

f is the most common symbol used to name a function, but there are many other names used. We often use $f(x)$ and $g(x)$ when comparing two functions and you will encounter common mathematical functions with abbreviated names, such as $\sin(\theta)$ and $\log(x)$.

Worked example 3.6

A function is defined on the set of real numbers by $f(x) = x^3 - x$.

Find the value of:

a $f(1)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(0)$ **e** $f(10)$

Solution

a $f(1) = 1^3 - 1 = 1 - 1 = 0$

b $f(2) = 2^3 - 2 = 8 - 2 = 6$

c $f(-1) = (-1)^3 - (-1) = -1 + 1 = 0$

d $f(0) = 0^3 - 0 = 0 - 0 = 0$

e $f(10) = 10^3 - 10 = 1000 - 10 = 990$

Worked example 3.7

Maddy is comparing two deals from rival electricity suppliers.

Deal A: A standing charge of €30 for every two-month period and a price per unit of €0.10

Deal B: A standing charge of €10 for every two-month period and a price per unit of €0.20

Which would be the better deal for a household that uses 300 units of electricity in a two-month period? Describe both deals in function notation. Illustrate your answer with a graph.

Solution

Understand the problem

The standing charge is paid even if no electricity is used.

The rate of increase in price for Deal B is greater than that for Deal A, so it will give a steeper graph.

Which of the two deals is likely to be cheaper if you use very few units, or if you use a lot?

Make a plan

To put these deals into function notation, a variable is needed and each function has to be named. The variable is the number of units used and combines with the cost per unit, but has no effect on the standing charge.

For a graph, it is a good idea to name the variable x. The functions must produce numbers. They will describe the number of euros in each case.

The question about 300 units gives an indication of the size of the domain and therefore of the x axis.

Carry out the plan

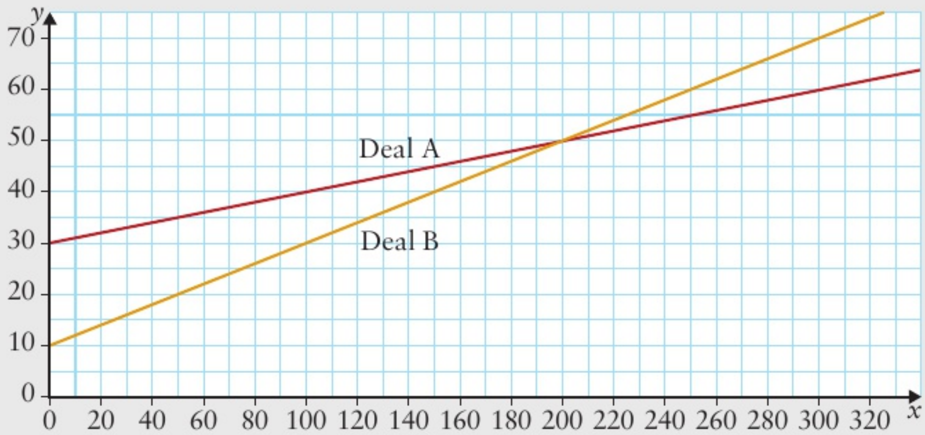
Let x be the number of units used over two months.

Let $A(x)$ be the cost of x units under Deal A in euros.

Let $B(x)$ be the cost of x units under Deal B in euros.

Then $A(x) = 30 + 0.1x$ and $B(x) = 10 + 0.2x$

The graphs of $y = A(x)$ and $y = B(x)$ are shown.



The lines cross at $x = 200$, so any consumption above 200 units per two-month period would cost less with Deal A than Deal B.

Evaluating at $x = 300$:

$A(300) = 30 + 0.1 \times 300 = 60$

$B(300) = 10 + 0.2 \times 300 = 70$

Practice questions 3.2.3

1 Evaluate each of the following functions at the given values from the domain.

a $f(x) = 4x$, find $f(3)$ and $f(-1)$

b $f(x) = x - 2$, find $f(4)$ and $f(2)$

c $f(x) = 4 + x$, find $f(10)$ and $f(-10)$

d $f(x) = x$ rounded to the nearest integer, find $f(2.2)$, $f(1.9)$, $f(-1.3)$

e $f(x) = \frac{12}{x}$, find $f(4)$, $f(2)$, $f(12)$

f $f(x) = 4x^2$, find $f(0)$, $f(2)$, $f(-2)$

2 For each function in question 1, find a possible value of x such that $f(x) = 4$

3 Which functions in question 1 are many-to-one?

Challenge Q4

4 Which of the functions in question 1 cannot have the complete domain of all integers?

5 The cooking time for a joint of beef depends on taste preferences and on the size of the joint. The oven is pre-heated to 180°C and then:

- for rare (lightly cooked) meat, allow 20 minutes for every 450 grams plus an additional 20 minutes
- for medium cooked meat, allow 25 minutes for every 450 grams plus an additional 25 minutes
- for well-done meat, allow 30 minutes for every 450 grams plus an additional 30 minutes.

a Calculate the time taken to cook an 800 gram joint for each of rare, medium and well-done. Keep a note of your calculations as a model for part b.

b Find three functions, $r(x)$, $m(x)$ and $w(x)$, for the cooking time of rare, medium and well done joints respectively, given that the weight of the joint is x grams.

c Use technology to plot the three functions for a sensible domain.

d Two beef joints are put in a large pre-heated oven and taken out at the same time. One joint weighs 1 kg and is to be served rare. The other is to be served well-done. What is the weight of the well-done joint?

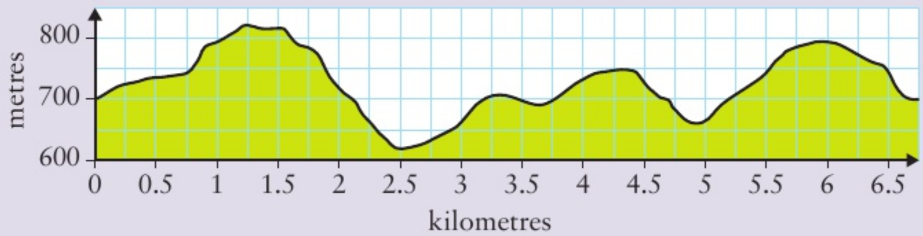
Hint Q5c

Would you cook 5 grams of meat in this way?

3.2.4 Which graphs represent functions?

Explore 3.6

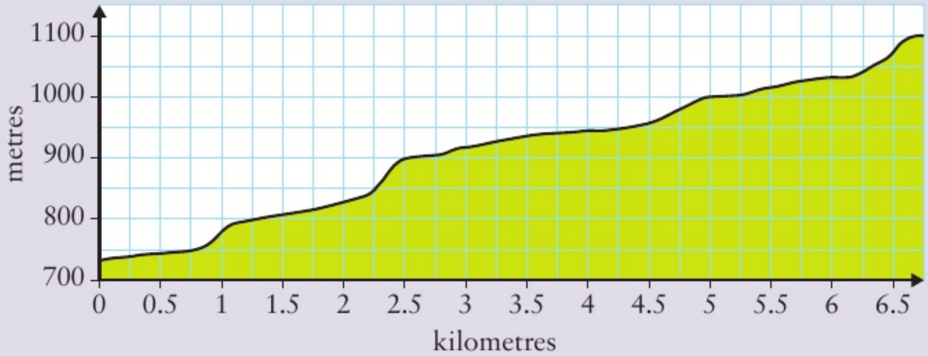
You are planning a walk in a mountainous area. Before you set off, you print an outline map of the route showing the variation in height above sea level.



If you had been walking for 3 km, could you use the map to work out your height above sea level? Would you be able to work out the height at any distance along the walk?

If you knew you were 750 m above sea level, could you use the map to work out how far you had walked? Are there any heights above sea level for which you would know how far you had walked?

Would your answers be different if you were walking to the top of a hill with this map?



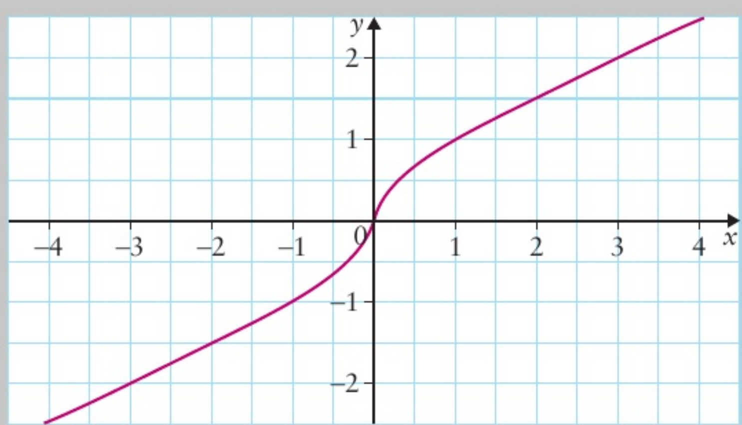
What is it about the nature of the maps that makes a difference?

So far, we have looked at linear functions and their graphs. Unless the line is vertical (with equation $x = c$) it is fairly obvious that a straight-line graph represents y as a function of x. But you will use functions to describe many different shaped graphs. It is possible to identify whether a relation is a function from its graph.

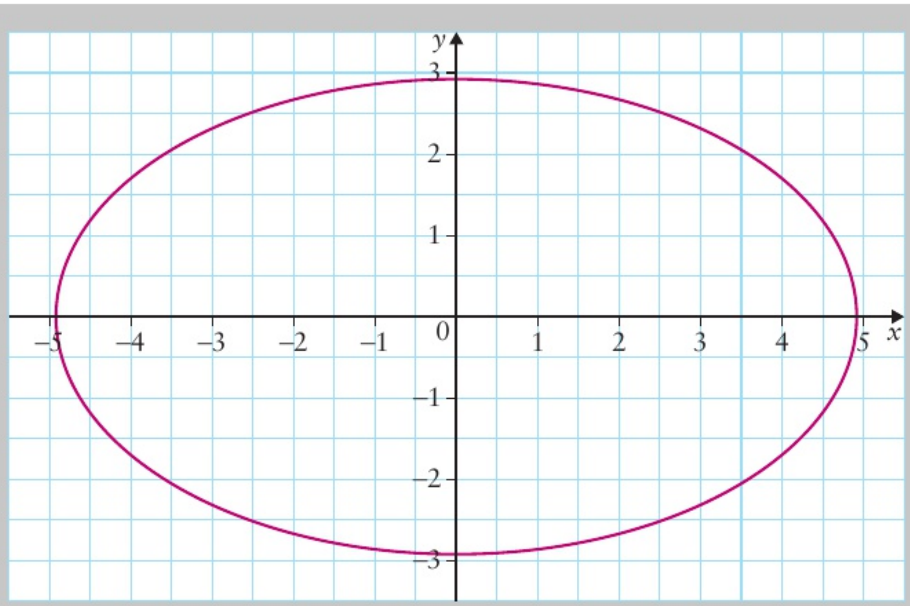
Worked example 3.8

Which of these graphs represents y as a function of x?

a



b



Solution

Understand the problem

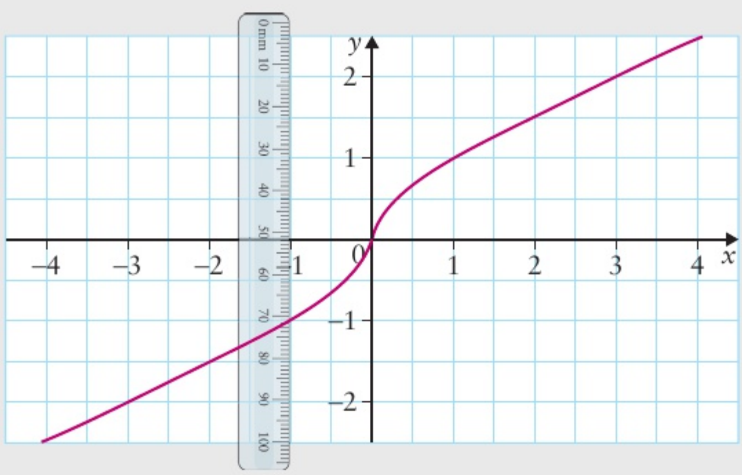
If a graph represents a function, it should be possible to choose any value of x and use the graph to find a unique corresponding value of $y = f(x)$

Make a plan

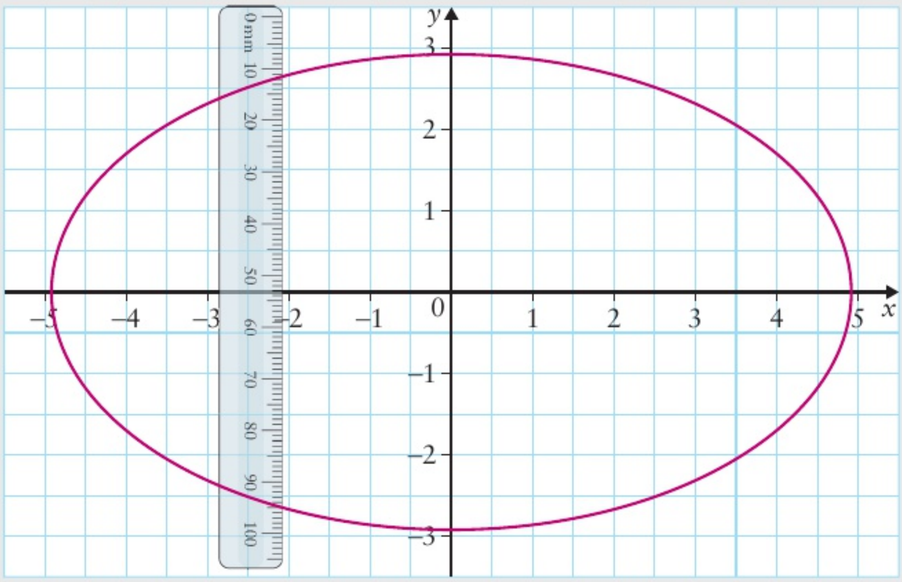
To test for a unique value, we can represent possible values of x (those in the domain and represented on the graph) by vertical lines. We should check at how many points the vertical lines meet the graph. A ruler can be a good instrument for such a check.

Carry out the plan

a As a ruler in a vertical position is moved from left to right across the graph, we can see that it meets the graph in at most one point at a time. Therefore this graph represents a function



b A ruler in a vertical position meets the graph at two points almost everywhere in the domain that is shown. Therefore this graph does not represent a function.



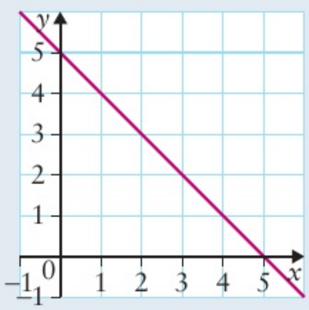
Look back

It seems clear that the relation represented in graph b is not a function, but the relation in graph a is a function, as it does meet the condition of a unique value each time. This method is known as the **vertical line test**.

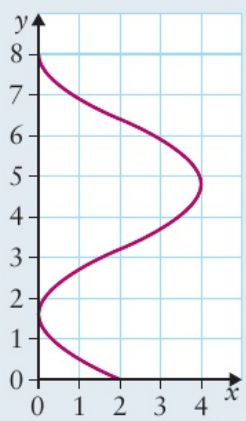
Practice questions 3.2.4

1 Use the vertical line test to determine which of the following graphs represent y as a function of x over the domain that is shown.

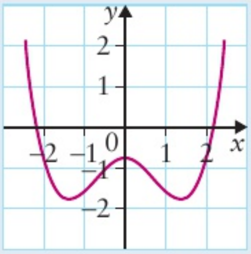
a



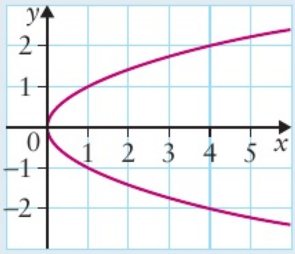
b



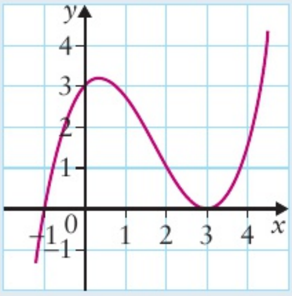
c



d



e



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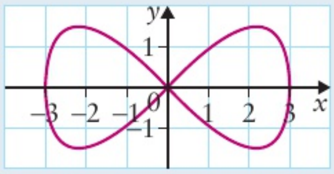


2 Despite the change to the metric system in 1971, the UK still makes regular use of imperial measures. One example is the fuel consumption of a car, which is usually given in miles per gallon (mpg) in the UK. A far more common measure worldwide is litres per 100 km.

- 1 gallon = 4.546 litres
- 1 mile = 1.609 km

a Apart from the size of the units, what is the fundamental difference between the two measures?

b Convert 50 mpg to litres per 100 km.

c Use your calculation from part b as a model to find the function $L(x)$ that will express x mpg in litres per 100 km. Give your answer in its simplest form.

d Find the function $M(x)$ that will express x litres per 100 km in mpg.

e Check that your functions work with 50 mpg and its equivalent in litres per 100 km

f How do you explain the similarity of the two functions?

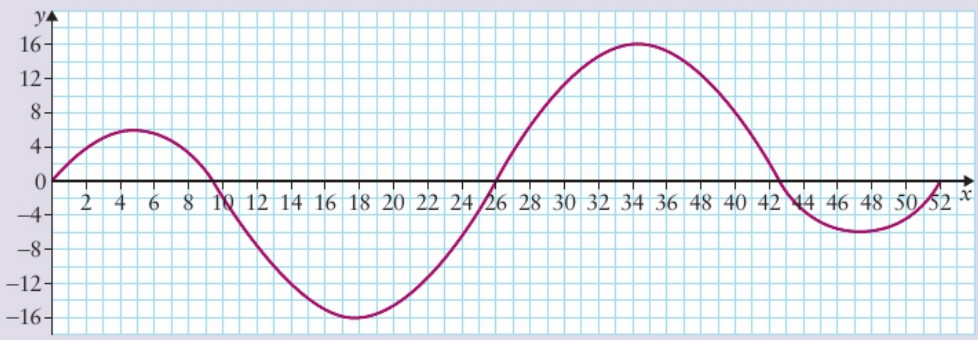
g Use technology to draw the graph of $y = L(x)$ for a suitable domain and range. Compare it with the graph of $y = M(x)$

h A car that is parked with the engine switched off is consuming fuel at a rate of 0 litres per 100 km. Can you express that in mpg?

3.2.5 Inverse functions

Explore 3.7

You might expect the Sun to be at its highest in the sky at the same time every day, but it is not quite that simple. For that to happen, the Earth would have to orbit the Sun in a circle and not tilt on its axis. But in fact, the Earth's orbit is an ellipse and the tilt is about 23.5°. The result is that time measured by a sundial is sometimes ahead of the clock and sometimes behind by up to 16 minutes during the course of a year. The graph (called the Equation of Time) shows the variation in minutes by week of the year. For example, the sundial and clock are aligned 9 weeks and four days into the year (8th March, if not a leap year).



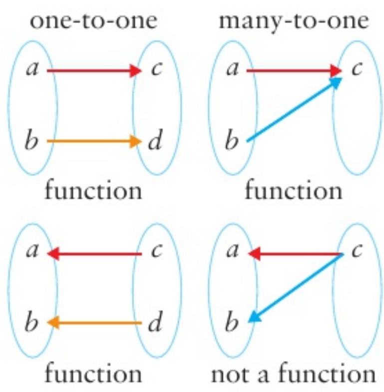
Provided you knew the date, could you use the graph to determine when, either side of a clock's midday, the Sun would be highest in the sky?

If you recorded the time between sundial midday and clock midday, could you use the graph to determine what day of the year it was? Might you need some additional information?

One important question about functions is whether or not the process can be reversed.

Consider the periodic table. We need to decode the abbreviations. If H were used to represent both hydrogen and helium, there would be a problem, which is why He is used for helium. You have seen earlier that the symbols for the elements are in a one-to-one relation with the names of the elements. That becomes important when the relation between sets must work both ways.

To reverse the process of a function, the range and domain swap places and the arrows in a mapping diagram point in the opposite direction. If the rule to achieve this is also a function, it is called the **inverse function**. We can try this with the functions we saw earlier.



Only the one-to-one function has an inverse.

> **Fact**
>
> Inverse functions can be found only for functions that are one-to-one.
>
> We use the notation f^{-1} to represent the inverse of f.

Worked example 3.9

The following rules define functions on the domain of integers. Find the inverse functions of each, if they exist.

a Multiply by 3 **b** Square

c Add 50 **d** Subtract from 100

Solution

In each case, the existence of an inverse function depends on whether the original function is one-to-one. Solving simple equations can give the impression that any step can be undone, but that is based on assumptions about the domain that should be examined in more detail.

As we are told that these are functions, we need to work out whether many (more than one) elements of the domain map to the same element of the range.

a Multiplication is one-to-one (there is only one number that is three times as big as another) and is undone by division, so the inverse function has the rule 'divide by 3'.

b Squaring when the domain includes negative numbers is many-to-one. For example, $4^2 = 16$ and $(-4)^2 = 16$. So, the function is not one-to-one and there is no inverse.

c Addition is one-to-one (there is only one number that is 50 more than another). The inverse function has the rule 'subtract 50'.

d Consider some examples:
$100 - 45 = 55$, $100 - 55 = 45$,
$100 - (-1) = 101$, $100 - 101 = -1$
It becomes clear that subtracting from 100 is reversed by the same process. This is called a **self-inverse**.

> **Fact**
>
> A function is a self-inverse if the inverse function has exactly the same definition as the original function.

Reflect

Reflections are examples of self-inverses. From a mathematical point of view, you are your reflection's reflection.

Consider the reciprocal function: $f(x) = \frac{1}{x}$

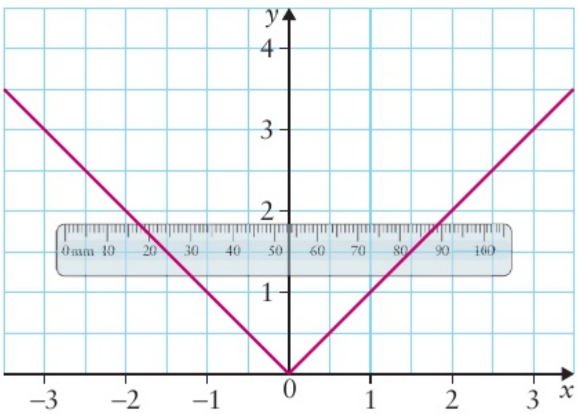
$f\left(\frac{2}{3}\right) = \frac{3}{2}$ and $f\left(\frac{3}{2}\right) = \frac{2}{3}$

f is a self-inverse: $f^{-1}(x) = \frac{1}{x} = f(x)$

Explore these examples: $g(x) = -x$, $h(x) = 1 - x$

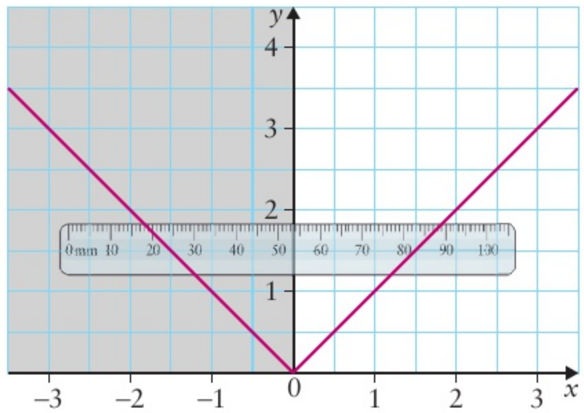
A graph can show y as a function of x. The inverse would be x as a function of y. Again, this is only possible if the original function is one-to-one.

Testing whether x is a function of y on a graph can be done in a very similar way to the test for y as function of x, this time using a **horizontal line test**.



The graph shows a relation that would pass the vertical line test, but not the horizontal line test. The ruler touches the graph at two points, so choosing a value of y would give two values of x and therefore x is not a function of y.

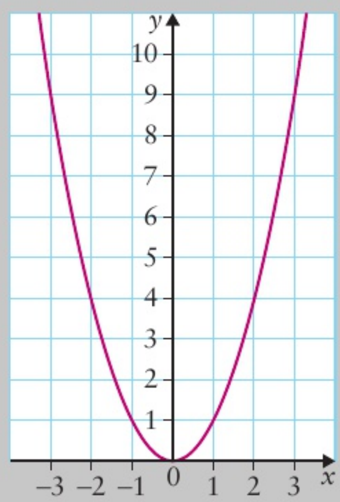
Sometimes an inverse function can be established by limiting the domain and range.



In this case only positive values of x and y are included. This means that x is a function of y.

Worked example 3.10

The graph, $y = f(x)$, of the function $f(x) = x^2$ is defined on the domain of all real numbers. Explain why f has no inverse function. What limitation could be made to be able to define the inverse function f^{-1}?



Solution

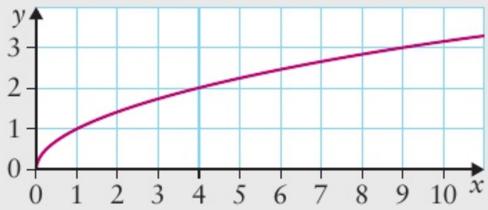
First we need to check that f is a function. That could be done with a vertical line test on the graph.

An inverse function would map $f(x)$ to x. In other words, x would be a function of y that can be checked with a horizontal line test.

A horizontal line test on the same graph reveals that x is not a function of y.

If the domain of f (therefore the range of f^{-1}) was restricted to just positive values, then the inverse function would be $f^{-1}(y) = \sqrt{y}$

Although it may seem confusing, it is important to note that $f^{-1}(y) = \sqrt{y}$ can be rewritten as $f^{-1}(x) = \sqrt{x}$, as x or y are only acting as go-betweens to describe what the function does. The advantage of swapping from y to x is that it allows us to draw a new graph showing $y = f^{-1}(x)$



Worked example 3.11

Given that the function $f(x) = 3 - 2x$ is one-to-one on the domain of real numbers, find a definition of the inverse function.

Solution

If $y = f(x)$, then y is a function of x. The inverse will be given by x as a function of y. So make x the subject of the equation.

$y = f(x) \Rightarrow x = f^{-1}(y)$

Let $y = f(x) = 3 - 2x$

Then $y + 2x = 3 \Rightarrow 2x = 3 - y \Rightarrow x = \dfrac{3 - y}{2}$

So $f^{-1}(y) = \dfrac{3 - y}{2}$

If needed, this can be restated as $f^{-1}(x) = \dfrac{3 - x}{2}$

Worked example 3.11 is about a function involving two steps: multiplication by 2 and then subtraction from 3. It is therefore sufficiently complicated to benefit from a formal approach.

Simpler functions can often be dealt with using prior knowledge.

For example, given $f(x) = x - 2$, we can state that the opposite of subtracting 2 is adding 2 and thus $f^{-1}(x) = x + 2$

We should always check the domain of f^{-1}. If the domain of f were positive integers, then its range would be integers greater than or equal to -1, and that would therefore become the domain of f^{-1}.

Practice questions 3.2.5

1 Answer these questions for each of the following graphs.

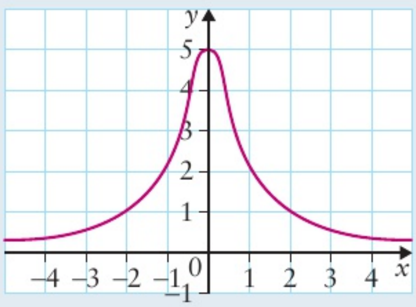
i Does the graph pass the vertical line test?

ii Does the graph represent y as a function of x?

iii Does the graph pass the horizontal line test?

iv If the graph represents y as a function of x, is it a one-to-one or a many-to-one function?

v Does the graph represent x as a function of y?

a


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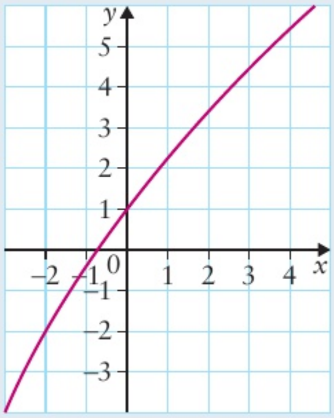
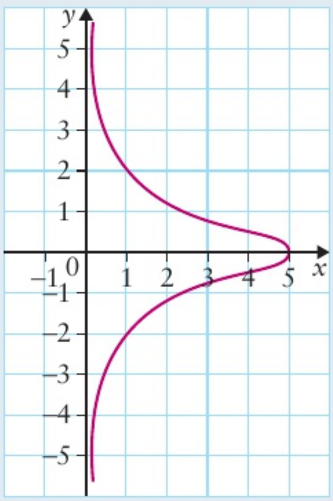
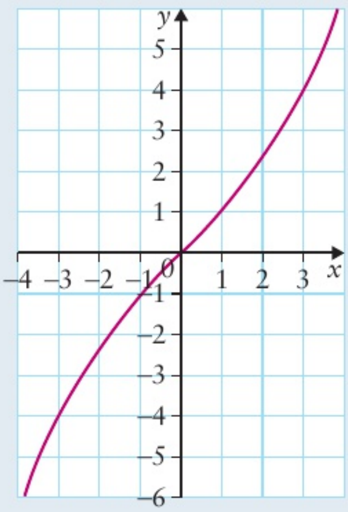


c


d


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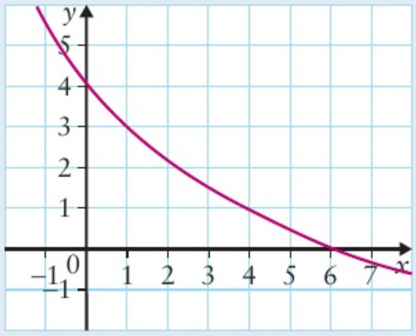

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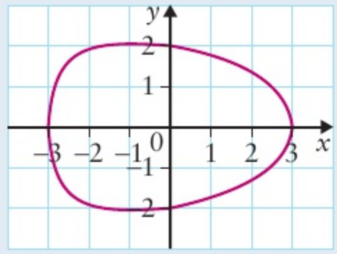


2 All the following functions are one-to-one and defined on the set of real numbers. Find the inverse function for each one, stating it in the form $f^{-1}(x)$.

a $f(x) = x + 1$ b $f(x) = x - 1$ c $f(x) = 3x$

d $f(x) = \frac{x}{5}$ e $f(x) = 2x - 1$ f $f(x) = 1 - x$

g $f(x) = 4 + 3x$

Challenge Q3

3 Given that $f(x) = x^4$ is defined on the set of integers, give examples to explain why the function does not have an inverse.

Challenge Q4

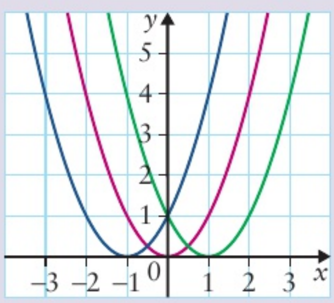
4 Given that $f(x) = x^2$ has an inverse function, give two examples of possible domains for f.

3.2.6 Transformations of graphs of functions

Explore 3.8

The graphs here are of $y = x^2$, $y = (x + 1)^2$ and $y = (x - 1)^2$

Use technology to check which is which.



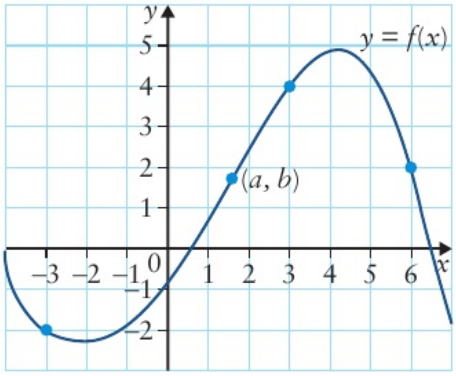
Continue to investigate graphs of the form $y = (x + a)^2$ using, for example, a constant controller, scrollbar (slider) or spin button to vary a.

What do you observe?

This graph shows $y = f(x)$.
For example, $b = f(a)$ since (a, b) is a point on the graph.

The definition of the function f is not important for what follows.

We can see that the graph passes through $(6, 2)$, $(3, 4)$ and $(-3, -2)$



We will now apply the function f, not to x, but to $3x$, and find the graph of $y = f(3x)$

This will transform the graph. The aim is to describe the **transformation**. We will do this by finding values in the domain, based on values in the range.

Using the point $(6, 2)$ as an example, 2 in the range comes from 6 in the domain (among other values, because the function is not one-to-one).

So we know $2 = f(6)$

What value of x would give $y = 2$ if we use $y = f(3x)$?

We would need $3x = 6$, so $x = 2$

It follows that $(2, 2)$ is a point on $y = f(3x)$ that corresponds to $(6, 2)$ on $y = f(x)$

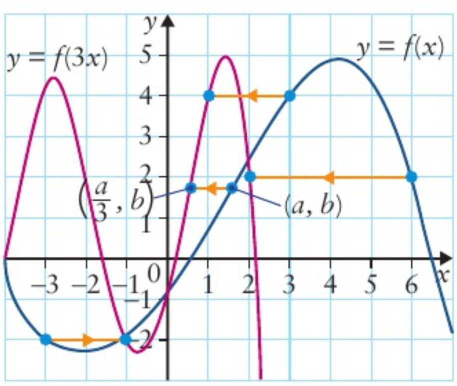
Follow the process again to see that:

- $(1, 4)$ on $y = f(3x)$ corresponds to $(3, 4)$ on $y = f(x)$
- $(-1, -2)$ on $y = f(3x)$ corresponds to $(-3, -2)$ on $y = f(x)$

Finally, the point (a, b) on $y = f(x)$ corresponds to $\left(\frac{a}{3}, b\right)$ on $y = f(3x)$

As (a, b) could be anywhere on $y = f(x)$ we have a general guide for where to plot points on $y = f(3x)$: for a given point on $y = f(x)$, the corresponding point with the same y-coordinate on $y = f(3x)$ will have an x coordinate $\frac{1}{3}$ of the original.

The result is that all points from the graph of $y = f(x)$ move closer to the y-axis to appear on the graph of $y = f(3x)$



Worked example 3.12

Sketch the graph of $2y = f(x)$ from the graph of $y = f(x)$ on the previous page.

Solution

This time the change is made in the range (from y to $2y$), not the domain, so we need to find where a value in the domain goes.

From the points on the graph, we see that $2 = f(6)$, $4 = f(3)$, $-2 = f(-3)$, $b = f(a)$

What value of y would come from $x = 6$ in the case of $2y = f(x)$?

Which point on the graph of $2y = f(x)$ corresponds to $(6, 2)$ on $y = f(x)$?

What does this and the transformations of the other identified points tell us about the new graph?

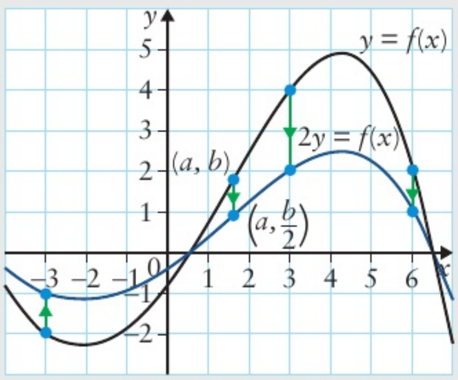
Given that $2 = f(6)$, and $2y = f(x)$, we need $2y = 2$, so $y = 1$

It follows that $(6, 1)$ is a point on $2y = f(x)$ that corresponds to $(6, 1)$ on $y = f(x)$.

Similarly:

- $(3, 2)$ on $2y = f(x)$ corresponds to $(3, 4)$ on $y = f(x)$
- $(-3, -1)$ on $2y = f(x)$ corresponds to $(-3, -2)$ on $y = f(x)$

(a, b) on $y = f(x)$ must become $\left(a, \frac{b}{2}\right)$ on $2y = f(x)$. From this, we see that all points from the graph of $y = f(x)$ move half of the way to the x-axis to appear on the graph of $2y = f(x)$



In general, we can think about these transformations in terms of compensation.

If the coefficient of x is doubled, then the value of x is halved to compensate and give the same total value for the term.

If 5 is added to x, then the value of x can be made smaller by 5 to compensate and give the same total value.

Connections

Inverse proportion: For ax to remain the same size when a is increased, x must decrease in inverse proportion. For example, $ax = 12$ with $a = 2$ gives $x = 6$, but with $a = 4$ it gives $x = 3$

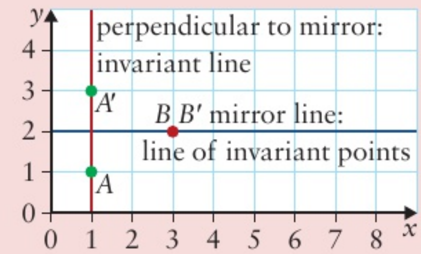
Similar statements can be made about the y-coordinates.

The transformations studied in this chapter are either **stretches**, **translations** or **reflections**.

Fact

Under a transformation:

- a **line of invariant points** is one where neither the line nor the points on the line change position
- an **invariant line** is one where the line does not change position but the points on the line might move to another position on the line.



For example, in the case of a reflection, the mirror line, $y = 2$, is a line of invariant points: B and its image B' remain in the same place.

Lines perpendicular to the mirror line, $x = 1$, are invariant lines: points on a perpendicular line, A and A', swap positions, but remain on and define the same line.

The transformation from $y = f(x)$ to $y = f(3x)$ is a stretch with scale factor $\frac{1}{3}$ parallel to the x-axis, with the y-axis as the line of invariant points.

The transformation from $y = f(x)$ to $2y = f(x)$ is a stretch with scale factor $\frac{1}{2}$ parallel to the y-axis, with the x-axis as the line of invariant points.

Fact

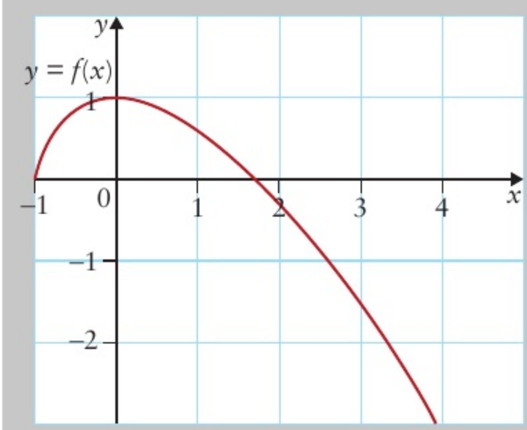
A one-way stretch has the effect of moving points perpendicularly to a fixed line. The new distance from the line is the original distance multiplied by the scale factor.

If the scale factor is greater than 1, the term *stretch* makes sense. We still use the term *stretch* when the scale factor is less than 1, even though the graph appears to shrink. This is similar to saying that division by 3 has the same effect as multiplication by $\frac{1}{3}$.

Worked example 3.13

From the given graph of $y = f(x)$, construct the graph of:

a $y = f(x - 1)$ **b** $-y = f(x)$



Solution

Understand the problem

a y remains the subject of the new graph, so we need to look at how the x-coordinates would change to give the same y-coordinates.

b x remains unchanged, so we need to look at how the y-coordinates would change to give the same x-coordinates.

Make a plan

The graph of $y = f(x)$ passes through $(0, 1)$, so that $1 = f(0)$

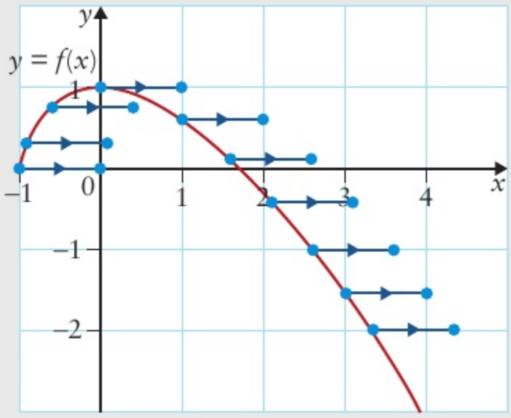
a Considering $y = f(x - 1)$, what value of x would produce the same result of $y = 1$? We will use this to understand the more general pattern of changes. Then draw guides to represent the change.

b Considering $-y = f(x)$, what value of y would be obtained from $x = 0$?

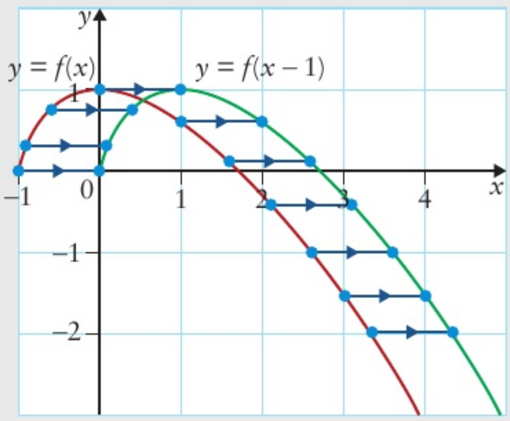
Carry out the plan

a $1 = f(0)$ and $y = f(x - 1)$, so choosing the value $x = 1$ would provide $x - 1 = 0$ as required.

We expect the point $(0, 1)$ on the original graph to correspond to $(1, 1)$ on the new graph. In general, x-coordinates would move 1 to the right as shown.

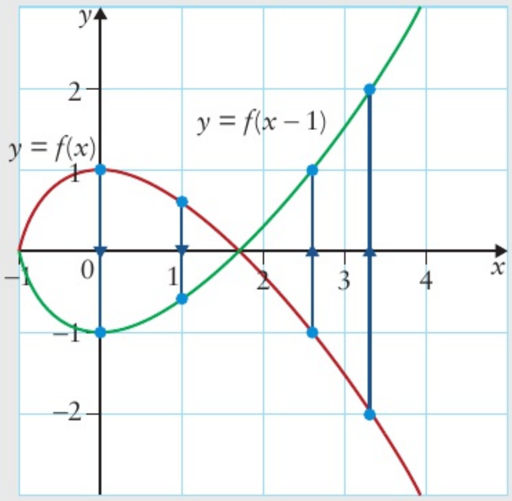


We can use this construction to trace the graph of $y = f(x - 1)$



b $1 = f(0)$ and $-y = f(x)$, so the value $y = -1$ would follow from $x = 0$ as required.

We expect the point (0, 1) on the original graph to correspond to (0, −1) on the new graph. In general, y-coordinates would change sign compared with the original graph, producing a reflection in the x-axis.



Look back

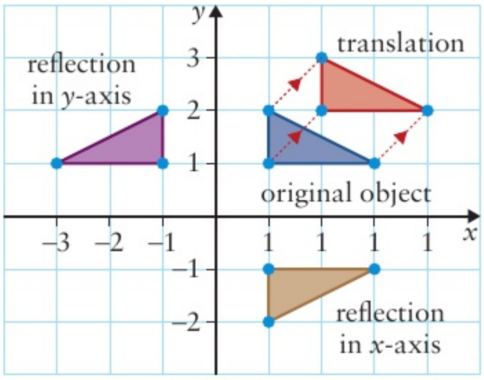
We can confirm our answers, again using the idea of compensation.

a Subtracting 1 from x (creating $x - 1$) while keeping the original value requires x itself to be increased by 1.

b Taking the negative of a negative reverts to the original value.

A transformation that moves all points a fixed distance in one direction is called a **translation**.

A transformation that swaps a y-coordinate for its negative is called a **reflection in the x-axis**. One that swaps an x-coordinate for its negative is called a **reflection in the y-axis**.

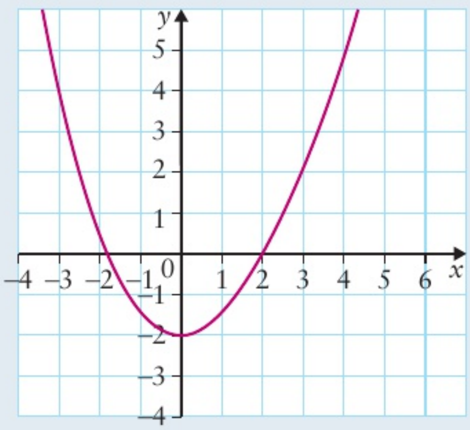


Practice questions 3.2.6

1 Make two accurate copies of this graph, which is of $y = f(x)$

To the first copy, add the graphs given in parts a and b.

To the second copy, add the graphs given in parts c, d and e.

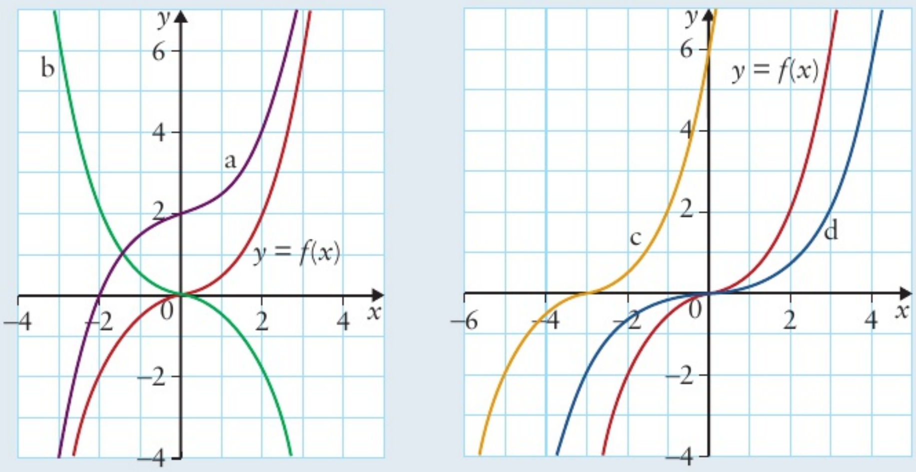


a $y = f(x + 1)$ **b** $-y = f(x)$

c $y - 1 = f(x)$ **d** $y = f\left(\frac{1}{2}x\right)$

e $2y = f(x)$

2 Find transformed functions to describe each of the graphs a, b, c and d in terms of the graph of $y = f(x)$. Where appropriate, give two possible solutions.



3 Without drawing graphs, find the functions that perform each of the following transformations on $f(x)$.

a A translation of 3 units upwards

b A translation of 5 units to the left

c A stretch of scale factor 4 in the direction of the x-axis, with the y-axis as the line of invariant points

d A stretch of scale factor $\frac{1}{4}$ in the direction of the y axis, with the x-axis as the line of invariant points

Thinking skills

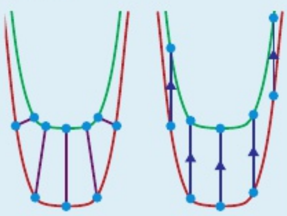
Fact Q2

When a graph has rotational or line symmetry it is possible for a particular transformation to be achieved in more than one way.

Hint Q2

Vertical translations can be difficult to identify because the eye is drawn to the curves apparently getting closer together. Draw vertical lines between corresponding points to clearly identify the behaviour.

For example, compare the shortest distances on the left, which are more likely to catch the eye than the vertical distances shown on the right. Those on the right are all the same length.



4 $y = f(x)$ describes a straight line given by $f(x) = x + 1$. Which of the following describe identical lines?

A $y = f(x - 2)$

B $y + 2 = f(x)$

C The original graph is translated 2 units to the right to produce the new graph.

D The original graph is translated 2 units down to produce the new graph.

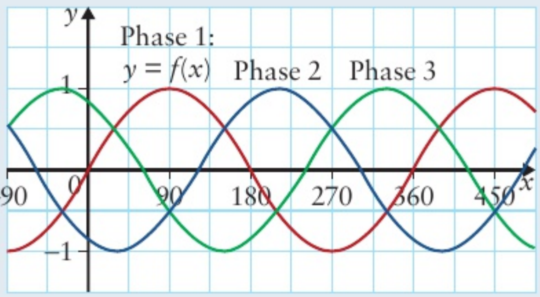
5 The graph shows the variation in (scaled) voltage for three-phase electricity based on the angle, x, of rotation of a generator. Phase 1 has the equation $y = f(x)$

Identify which of Phase 1, Phase 2 or Phase 3, if any, are described by the equations:

a $y = f(x + 120)$ b $y = f(x + 180)$ c $y = f(x + 240)$

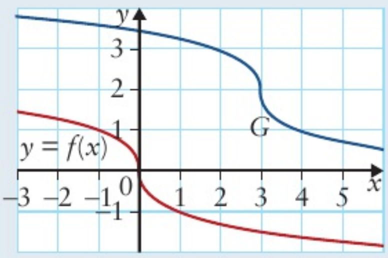
d $y = f(x - 120)$ e $y = f(x + 360)$ f $-y = f(x + 180)$

Challenge Q5f



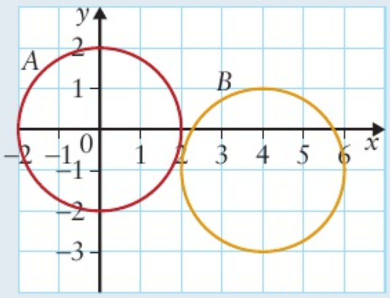
6 What transformed function would describe the graph G in terms of the original $y = f(x)$?

Challenge Q6



Challenge Q7

7 The methods used for transformations can be applied to relations that are not functions. The circle A has equation $x^2 + y^2 = 4$. Suggest an equation for circle B, which is the result of a translation of 4 units to the right and 1 unit down.



Self assessment

- I understand that a relation is a connection between an element in one set and an element in another set.
- I know that the initial set is called the domain and the target set is the range.
- I know that a relation is often described in terms of a rule, but may be defined by the connections themselves.
- I can illustrate a relation with a grid, a mapping diagram, a set of ordered pairs and a Cartesian graph.
- I understand that in the context of relations 'many' means 'more than one'.
- I can classify relations as one-to-one, many-to-one, one-to-many or many-to-many.
- I can recognise, use and apply one-to-one relations.
- I know that functions make connections that provide a unique output for any input within the domain.
- I know that functions may be defined as one-to-one or many-to-one relations.
- I know that inverse functions can be found for one-to-one functions.
- I can use $f(x)$ notation for functions and $f^{-1}(x)$ notation for inverse functions and recognise the efficiency of the notation in certain situations.
- I can identify functions from their graphs, but also recognise the limits of the method.
- I can find the equivalence between transformed functions and their graphs and describe corresponding stretches, translations and reflections in the axes.

Check your knowledge questions

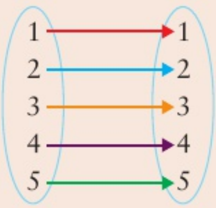
1 Use a mapping diagram to connect the largest cities in Europe to their countries:

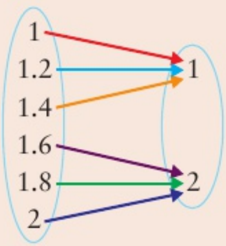
Moscow, London, Saint Petersburg, Berlin, Madrid, Kyiv, Rome, Paris, Bucharest, Minsk, Vienna, Hamburg, Warsaw, Budapest, Barcelona, Munich, Kharkiv, Milan

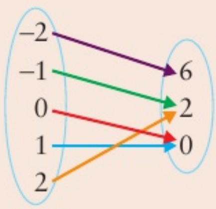
Austria, Belarus, France, Germany, Hungary, Italy, Poland, Romania, Russia, Spain, Ukraine, United Kingdom

Kyiv

2 Create a set of ordered pairs for each relation shown in these mapping diagrams.

a


b


c


3 In each case, decide whether the relation is one-to-one, many-to-one, one-to-many or many-to-many.

a Students in a class mapped to their birthdays

b Golfers in a championship mapped to their total scores for four rounds

c Winners of The French Open men's singles mapped to the year in which they won

d The perimeter of a square mapped to its area

e The perimeter of a rectangle mapped to its area

4 Represent each of the following sets of ordered pairs on a Cartesian graph and state whether the relation is one-to-one, many-to-one, one-to-many or many-to-many.

a $\{(-3, 4.5), (-2, 2), (-1, 0.5), (0, 0), (1, 0.5), (2, 2), (3, 4.5)\}$

b $\{(0, 10), (1, 8), (2, 6), (3, 4), (4, 2), (5, 0)\}$

c $\{(0, 0), (1, 0), (1, 1), (0, 1)\}$

d $\{(-3, -9), (-2, -3), (-1, 0), (0, 0), (1, 0), (2, 3), (3, 9)\}$

5 Which of the following relations are functions when applied to the domain of positive integers?

a is a multiple of

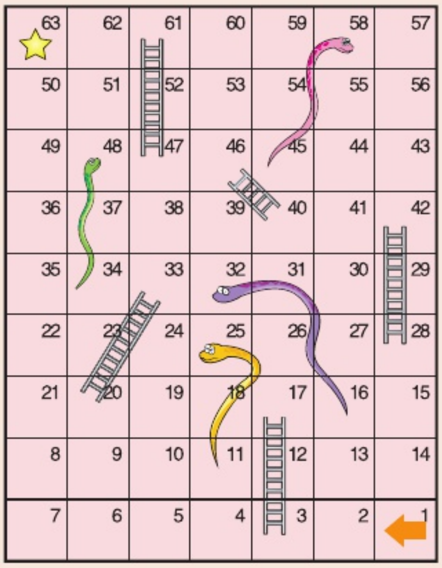
b is four more than

c is a factor of

d is the square of

e is the double of

6 In a game of Snakes and Ladders, you start with your counter on square 1, roll a die and advance the number of squares shown. You follow the sequence of numbers unless your counter lands at the foot of a ladder or on the head of a snake. If you land at the foot of a ladder then you climb immediately to the top of it. If you land on the head of a snake, you drop immediately to the end of the tail.

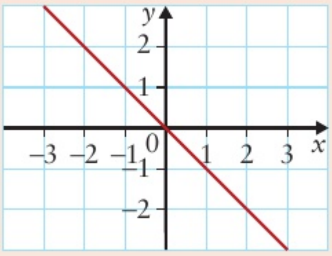


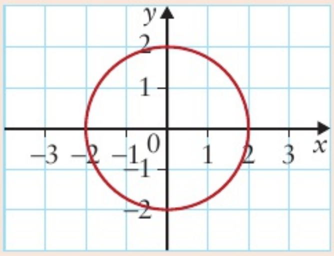
a What is the smallest possible total of scores on the die with which you could reach the star on the board shown on the previous page?

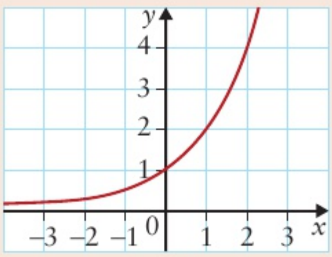
b Is the number of the square your counter is on a function of the total of scores of the die up to that point?

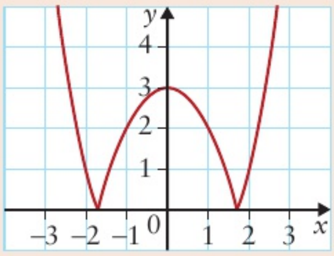
c Would your answer to part b be different if there were no snakes and no ladders?

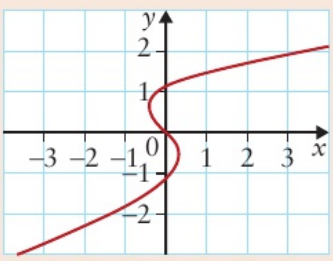
7 For each of the following graphs, decide whether or not y is a function of x. If it is a function, state whether it has an inverse function for the full graph as shown.

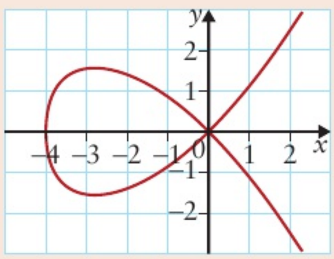
a 

b 

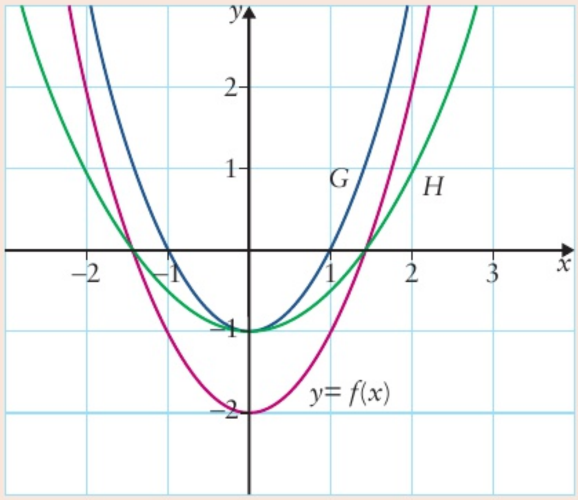
c 

d 

e 

f 

8 Describe graphs G and H in terms of $y = f(x)$



9 Write down the description of the transformations given by:

a $y = f(x)$ becoming $5y = f(x)$

b $y = f(x)$ becoming $y = f(x - 3)$

c $y = f(x)$ becoming $y = f\left(\frac{x}{2}\right)$

d $y = f(x)$ becoming $y + 10 = f(x)$

e $y = f(x)$ becoming $y = f(-x)$

4 Linear functions

4 Linear functions

KEY CONCEPT

Relationships

RELATED CONCEPTS

Change, Models, Representation

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Representing change as a relationship in innovative ways can assist in the decision-making processes in real-life situations, such as within business.

Factual

- What are the different forms for writing linear functions?

Conceptual

- How can we represent change?
- How can we use linear equations and their graphs to extract information about the real-life situations that they represent?

Debatable

- Will a system of inequalities always produce a feasible region?

Do you recall?

1 Are the lines $2x + 3y = 7$ and $3y + 2x = 7$ parallel? Give a reason for your answer.

2 Graph the lines $x = 3$, $y = -4$ and $x + y = 5$

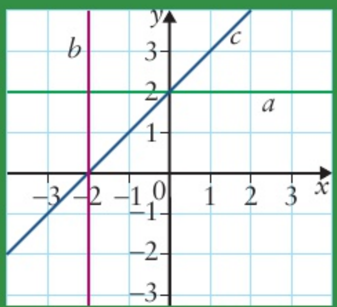
3 Find the x- and y-intercepts of the line $2x + 3y = 6$

4 Solve the equation $6(a + 2) - 4(a - 1) = 20$

5 Using the diagram below, write the equation of:

a line a **b** line b **c** line c

d the x-axis **e** the y-axis



6 Find an equation of a straight line with a gradient of 2 and a y-intercept of -1.

7 Find an equation of a line that passes through $(1, -2)$ and is perpendicular to the line with equation $y = 2x + 7$

8 Solve $2x - 3 < 7$ and show the solution on a number line.

4.1 Linear functions introduction

You are already familiar with the idea of a function and a linear equation. Explore 4.1 will help to refresh your memory.

Explore 4.1

It is easy to keep track of a child's growing vocabulary in their early years. Children typically recognise more words than they use themselves when speaking. For example, a toddler might only say a few different words (such as mama or papa) but they are able to understand many other words. For example, they might be able to point to a cat when asked 'Where is the cat?' Children learn many new words once they start going to school and reading. Vocabulary development doesn't stop as soon as a child has learned to talk.

On average, a typical 2-year old will have about a 300-word vocabulary and a 4-year old will have about a 1500-word vocabulary.

Plot the points representing this information on a grid with units of 1 year on the horizontal axis and 200 words on the vertical axis.

Can you tell how many words are added to a child's vocabulary every year?

Assuming the trend continues uniformly, can you find an equation representing vocabulary and age?

Can you think of a realistic domain for your function? Justify your answer.

By the age of 12, it is believed that a child will have a vocabulary of about 50 000 words. Do you think this is realistic?

The equation of a line with gradient m and y-intercept b is of the form $y = mx + b$. Written in function notation, this is $f{:}x \rightarrow mx + b$ or $f(x) = mx + b$

In general, both the domain and the range of a linear function is the set of real numbers. However, in practical situations, when you are describing functional relationships between natural numbers and other sets, the domain and range may be different. For example, when you describe the function of the number of visitors at a museum and the time they spent during a visit, the domain is the set of natural numbers and the range is the set of non-negative real numbers less than or equal to the maximum time the museum is open for visits.

Worked example 4.1

To stage a rock concert, a group of young musicians contracted a promoter who charges them €50,000 plus 15% of the tickets sold. They want to charge €20 per ticket.

a What is their break-even number of ticket sales?

b How many tickets do they need to sell to make a profit of €20,000?

Solution

a The break-even number is the number of tickets that will cover the costs.

We can find the break-even number by writing functions to calculate the revenue and costs.

Let x be the number of tickets sold. Then the revenue is the product of this number and the revenue per ticket. So the revenue function is $R(x) = 20x$

The cost function is made up of two parts: the fixed part (which is €50,000) and the variable part, which is 15% of the revenue: $C(x) = 50\,000 + 0.15 \times (20x) = 50\,000 + 3x$

To find the break-even number we equate the two functions and solve:

$$20x = 50\,000 + 3x \Rightarrow 17x = 50\,000 \Rightarrow x = \frac{50\,000}{17} = 2941.18$$

We are looking for the number of tickets sold, so an answer with a decimal part does not make sense. Normally we would round 2941.18 down to 2941, but this is also not feasible because the group would end up with a loss. Therefore we round up to 2942.

Hint

With 2941 tickets sold, the revenue would be $R(2941) = 20 \times 2941 = €58{,}820$ while the cost would be €58,823, which is more than the revenue.

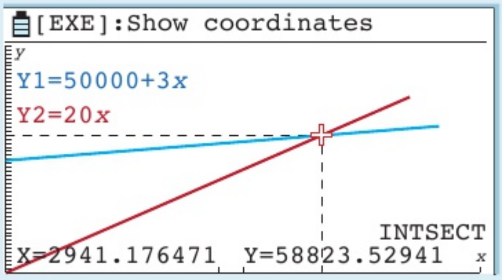
b A €20,000 profit means earning this amount on top of the cost.
We can write a new function, $F(x)$, to represent this.
$F(x) = 20\,000 + C(x) = 20\,000 + (50\,000 + 3x) = 70\,000 + 3x$

Thus $20x = 70\,000 + 3x \Rightarrow 17x = 70\,000 \Rightarrow x = \frac{70\,000}{17} = 4117.65$

So, they must sell at least 4118 tickets to make a profit of €20,000.

Hint

The solution can be found using your GDC. Here is the GDC solution for Worked example 4.1.



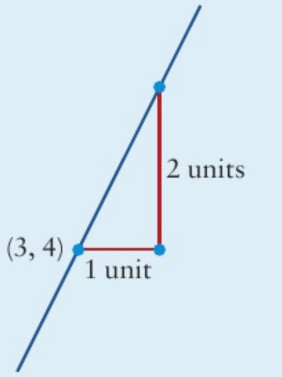
Note that you need to pay attention to the horizontal and vertical scales of your GDC.

Reminder

$Ax + By = C$ or $Ax + By + C = 0$ are usually called the standard form of the equation of a straight line. By solving for y in either of the equations you can find the gradient-intercept form.

Hint

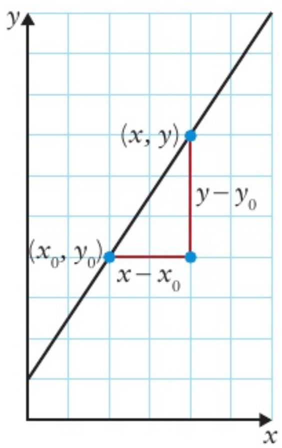
You can use point–gradient form to easily sketch the graph. In this case, start at the point (3, 4) and then go horizontally one unit and vertically the amount of the gradient, 2 in this case. Then join the two points.



The horizontal and vertical distances can be any numbers as long as $\frac{\text{vertical length}}{\text{horizontal length}} =$ gradient

Linear functions can be represented graphically by straight lines. The equation of a line can be found using different methods, depending on the information available.

- Given the gradient m and the y-intercept b, the function can be expressed as $f(x) = mx + b$
  This is called the **gradient-intercept** (or **slope-intercept**) form.
- Given one point on the line (x_0, y_0) and the gradient, the equation can be found using the definition of gradient:
  $m = \frac{y - y_0}{x - x_0} \Rightarrow y - y_0 = m(x - x_0)$
  This is known as the **point–gradient (point–slope)** form.
  For example, the equation of the line that has a slope of 2 and contains the point (3, 4) is $y - 4 = 2(x - 3) \Rightarrow y = 2x - 2$

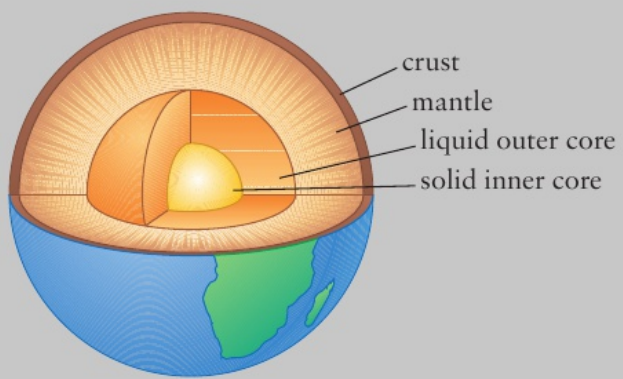


- Given two points on the line, the equation can be found using the definition of the gradient in two stages. First find the gradient. Then apply the point–slope form. For example, the equation of the line containing (3, 4) and (2, 2) can be found by first finding the gradient:

  $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{3 - 2} = 2$

  Then applying the point–slope form:
  $y - 2 = 2(x - 2) \Rightarrow y = 2x - 2$ or $y - 4 = 2(x - 3) \Rightarrow y = 2x - 2$
  Note here that you can use either of the two given points.

Worked example 4.2



The Earth's interior is hotter than its surface.

A simplified model of how temperature in the core of the Earth increases as the depth inside the Earth increases states that, on average, the temperature increases by about 10°C for every kilometre of depth.

a At a place where the temperature is 5°C on the surface of the Earth, find a formula for the temperature in the Earth's interior, T°C, as a function of depth, d km.

b Estimate the temperature in the Earth's interior at an approximate depth of 2900 km.

c At what depth is the temperature 100°C?

Thinking skills

Fact

The difference in temperatures in the interior of the Earth drives the flow of what is called geothermal energy. This allows the energy to be used for heating and electricity generation.

Solution

a We can deduce that the approximate relationship between temperature and depth is linear because the statement says that the temperature changes by a fixed amount for every km of depth. The slope is 10.

We are told that on the surface (where depth = 0 km) the temperature is 5°C. So one point on the function is (0, 5)

We will use gradient-intercept form for the linear function:

$$10 = \frac{T(d) - 5}{d - 0}$$

$$T(d) = 10d + 5$$

b We need to find the value of $T(2900)$:

$T(2900) = 10 \times 2900 + 5 = 29\,005°C$

c We need to solve for d in terms of T:

$T = 100$, thus $100 = 10d + 5 \Rightarrow 10d = 95 \Rightarrow d = 9.5\,\text{km}$

Fact

Note that these calculations are only rough approximations that may differ from the scientific estimates made by geophysics scientists. One source of error is the assumption that the change in temperature is an absolutely linear function of depth.

Reflect

By how much does the estimate in the example differ from scientists' estimates?

What is the source of discrepancy between the estimates?

Practice questions 4.1

1 Find each linear function using the given information.

a The gradient is $-\frac{1}{2}$ and $f(3) = 4$

b The graph of the function contains the points $(6, 1)$ and $(-3, 2)$

c The graph of the function is parallel to the x-axis and contains the point $(2, -3)$

d The x-intercept is $(-3, 0)$ and the y-intercept is $(0, 4)$

e The graph of the function contains the point $(-2, 3)$ and is parallel to the line $2x - 3y = 17$

f The graph of the function contains the point $(-2, 3)$ and is perpendicular to the line $2x - 3y = 17$

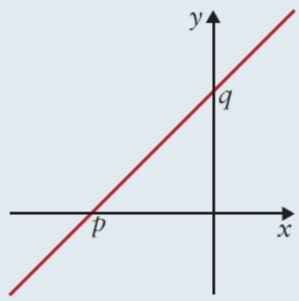
2 **a** Find the inverse of $f(x) = 2x - 9$

b Find $f^{-1}(x)$ if $f(x) = mx + b$

c Describe the relationship between the gradients and y-intercepts of a linear function and its inverse.

d The equation of a line is written in standard form $Ax + By = C$
Write down the equation of the inverse function in standard form.

3 The diagram shows a graph of the line $f(x) = mx + b$



a Write m in terms of p and/or q.

b Write b in terms of p and/or q.

c Hence, write $f(x)$ in terms of p and q.

4 Consider the equation of a line given in the form $Ax + By = C$

a How can you determine the gradient of the line?
What about the y-intercept?

b Find the gradient and the y-intercept of the line with equation $3x + 4y = 12$

c How can you determine the x- and y-intercepts directly from the given equation?

d Use the method in part c to find the x- and y-intercepts of the line with equation $3x + 4y = 12$

5 A large strawberry farm gives students summer jobs picking strawberries. They pay each student \$25 per day and \$4 for each baskets they fill.

a Write an equation for the income a student can make per day for picking x baskets.

b How much should a student expect to earn if she picks 50 baskets?

c How many baskets should she fill to receive at least \$200?

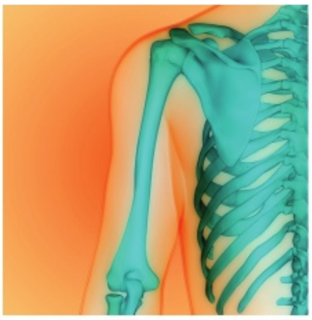
6 A ball thrown in space, where gravity is virtually zero, would travel in a straight line at a constant speed. Data collected for an experiment in space is given in the table.

Time (s)	Distance (m)
0.2	2.590
0.5	5.065
0.7	6.715
1.5	13.315

a Determine the gradient of the line that contains the points.

b Find an equation giving the distance travelled as a function of time.

c Find the time it takes the ball to travel 10 m.



The humerus bone runs from the shoulder to the elbow.

Fact

If you count the number of seconds between the flash of lightning and the sound of thunder, and then divide by 5, you'll get the distance in miles to the source of the lightning.

This rough calculation comes from the difference between the speed of sound (hearing the thunder) and the speed of light (seeing the lightning).

7 The height h of a human female can be approximated from the length l of her humerus bone using a linear function $h = 2.8l + 71.5$

a Graph this function.

b Estimate the length of the humerus bone of a female who is 160 cm tall.

c Can you interpret the h-intercept (that is, the y-intercept) in context? Hence, or otherwise, suggest a reasonable domain for the function.

8 The speed of sound depends on temperature. At 0°C, the speed is approximately 331 m/s. At 10°C, the speed is 337.06 m/s.

a Find an equation expressing the speed of sound v as a function of temperature T.

b Determine the speed of sound when the temperature is 27°C.

c At what temperature is the speed of sound 329 m/s?

9 Near the Earth's surface, air gets cooler the higher you climb. As you climb a mountain, you can expect the air temperature to decrease by 6.5°C for every kilometre of height you ascend.

a Assuming the temperature on the ground is 20°C, write an equation expressing the temperature, T°C, as a function of altitude, h km.

b Mont Blanc is the highest mountain in the Alps. The elevation from Geneva, Switzerland, to the summit of the mountain is approximately 4432 m. Estimate the temperature at the summit on a day when Geneva's temperature is 20°C.

c How would your equation from part a change on a day when the temperature in Geneva is 30°C?

d In 2019, the highest temperature recorded for Geneva was 34°C. What does this tell you?

10 It costs \$9750 to prepare the film and make the printing plates for a school yearbook. This cost is the same no matter how many yearbooks are printed. The cost of printing and binding is \$5.18 for each copy.

a Write an equation expressing the total printing cost, C dollars, as a function of the number of copies printed, n.

b Use your equation to determine:

i the total cost of 750 yearbooks

ii the number of yearbooks that can be printed for \$17,400.

Fact

Mont Blanc has an elevation of 4807 metres above sea level.

4.2 Graphing linear equations

Explore 4.2

Graph the four functions below, using the same coordinate plane. You may use a GDC or other available software.

$f(x) = \frac{1}{2}x + 2 \quad g(x) = -2x + 2 \quad h(x) = -2x - 3 + 5 \quad k(x) = \frac{1}{2}x + 5$

What do you notice? State as many observations as you can. Remember to justify each observation you make.

Reminder

For the two lines with equations $y = m_1x + b_1$ and $y = m_2x + b_2$

- if $m_1 = m_2$ then the lines are parallel
- if $m_1 \times m_2 = -1$, that is, $m_1 = -\frac{1}{m_2}$ then the lines are perpendicular.

Worked example 4.3

Consider these two functions:

$f(x) = 2x + 2 \qquad g(x) = -2x + 6$

a Graph them on the same coordinate plane.

b On the same set of axes, graph the functions $h(x) = f(x - 4)$ and $j(x) = g(x) + 5$

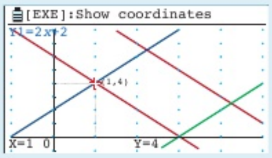
c What type of quadrilateral do the points of intersection make?

d Work out the area of the quadrilateral.

Challenge Qd

Hint

You can use your GDC or available software to graph the functions. Here is an example:

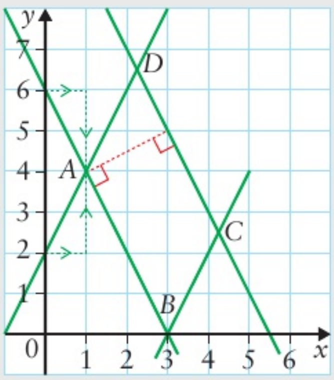


Solution

The first three parts are straight-forward applications of basic graphing methods. We could use a GDC or software, or we could sketch the graphs using the information given. Both $f(x)$ and $g(x)$ are given in the gradient-intercept form. We will use this form by starting at the y-intercept and moving horizontally and vertically so that the distance in the vertical direction divided by the distance in the horizontal direction will be equal to the gradient.

a $f(x)$ has a y-intercept of $(0, 2)$ and a gradient of 2. Starting at the y-intercept, we move right one unit and then up two units and mark a point, then we join the two points. The graph of $f(x)$ is the line AD. We do the same for $g(x)$: start at $(0, 6)$, go right one unit and down two units. The graph of $g(x)$ is AB.

b The graph of $h(x)$ is the graph of $f(x)$ translated horizontally by +4 units. The graph of $h(x)$ is the line BC. The graph of $j(x)$ is the graph of $g(x)$ translated vertically by +5 units. The graph of $j(x)$ is the line DC.



c The lines representing $f(x)$ and $h(x)$ are parallel, and the lines representing $g(x)$ and $j(x)$ are also parallel. Therefore, the opposite sides of the quadrilateral $ABCD$ are parallel, so $ABCD$ is a parallelogram.

Hint Qc

There are two explanations for why these pairs of lines are parallel:

- One line in each pair is a translation of the other.
- The two lines in each pair have the same gradient.

d Since $ABCD$ is a parallelogram, its area is the product of one of its sides and the vertical height, which is at right angles to that side.

For convenience, we choose AB or CD as the side and find its length. The height can be found by finding the shortest distance from point A to side CD.

$CD = AB = \sqrt{(1-3)^2 + (4-0)^2} = 2\sqrt{5}$

So from A we consider the line perpendicular to CD. This line passes through $A(1, 4)$ and is perpendicular to CD, and thus has a gradient of $-\frac{1}{-2} = \frac{1}{2}$

Using the point-slope form, the equation of this line is:

$y - 4 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}x + \frac{7}{2}$

Now we find the intersection of this line with CD:

$\frac{1}{2}x + \frac{7}{2} = -2x + 6 + 5 \Rightarrow \frac{5}{2}x = \frac{15}{2} \Rightarrow x = 3$ and thus $y = \frac{1}{2} \times 3 + \frac{7}{2} = 5$

Therefore, height $= \sqrt{(1-3)^2 + (4-5)^2} = \sqrt{5}$

Hence, area = base × height $= 2\sqrt{5} \times \sqrt{5} = 10$ square units

Hint Qd

The coordinates of A can be found either by inspection or by finding the intersection between the graphs $f(x)$ and $g(x)$.

$2x + 2 = -2x + 6 \Rightarrow 4x = 4 \Rightarrow x = 1$, then substitute this in any of the equations to get $y = 4$

Reflect

How else could you have worked out the solution to part c in Worked example 4.3?

Investigation 4.1

Consider the equation $y = 2x + 5$

1 Graph the equation.

2 Consider a few cases of the equation $y = mx + b$ where $b - m = 3$ For example, $b = 7$ and $m = 4$, and so on. What do you notice? Can you justify your observations?

3 Consider the situation where $m + b = 8$, or any other constant. What do you notice?

4 Finally, consider the situation where $b = m^2$. What do you notice?

Hint

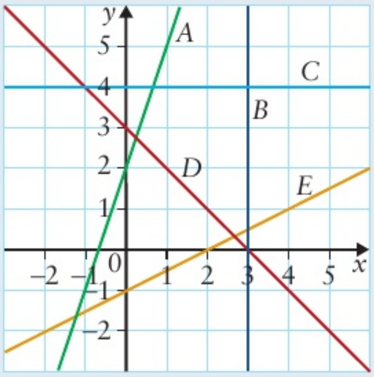
It is advisable to use a GDC or software for Investigation 4.1. Use positive as well as negative values for m, including decimals.

Reflect

What if $b = \frac{1}{m}$?

Practice questions 4.2

1 Find the equation of each line in the diagram.



2 Consider the point $A(2, -3)$ and the linear function $f(x) = -\frac{2}{5}x + 7$
Find the equation of a line through A:

a parallel to the graph of $f(x)$

b perpendicular to the graph of $f(x)$.

3 Determine whether the given pairs of lines are parallel, perpendicular, or neither. Give reasons for your answers.

a $2x + 3y = 3$ and $y = -\frac{3}{2}x - \frac{7}{10}$

b $5y - 15x + 4 = 0$ and $3y + x = 5$

c $4y + 3x = 7$ and $4x + y = 7$

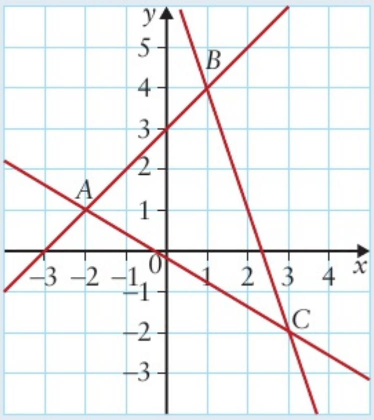
4 The electrical resistance R of a certain resistor has been measured for different temperatures t in Celsius. The measurements are given in the table.

a Graph this data with temperature on the horizontal axis.

b Determine an equation giving the resistance as a function of the temperature.

Temperature (°C)	Resistance (ohms)
10	5.70
13	5.79
15	5.85
16	5.88
19	5.97

5 A is the point $(-2, 1)$, B is the point $(1, 4)$ and C is the point $(3, -2)$.



a Find the equation of the line BC.

b Find the equation of a line through A perpendicular to BC.

c Find the point of intersection of the new line with BC.

d Find the area of $\triangle ABC$.

6 a Show that $y - 4 = 2(x - 5)$ is a linear function.

Thinking skills

b What does the number 2 tell you about the graph of the function?

c The coordinates of a point on the graph are concealed in the equation. What are the coordinates of the point? Justify your answer.

7 Determine whether or not there is a linear function that contains all the points in each of these groups. If there is, work out the equation of the function. If not, explain why not.

a $(6, 2)$, $(5, 3)$, $(1, 7)$

b $(-3, 16)$, $(1, 10)$, $(9, -3)$

c $(1, 4)$ $(3, 7)$, $(5, 10)$, $(7, 13)$

Hint Q9

$\frac{x}{a} + \frac{y}{b} = 1$ is called the intercept form of the equation of a line.

8 Another form of the linear function equation is $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are non-zero constants.

a What do a and b represent?

b Convert the equation $\frac{x}{3} + \frac{y}{5} = 1$ to the following forms:

i $Ax + By = C$, where A, B, and C are integers

ii $y = mx + b$

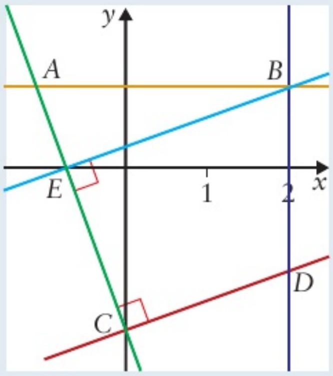
c Convert the equation $y = 4x - 12$ to intercept form, then give the x- and y-intercepts.

Challenge Q9

9 In the diagram, the line $5x + 2y + 5 = 0$ intersects the x-axis and y-axis at E and C respectively.

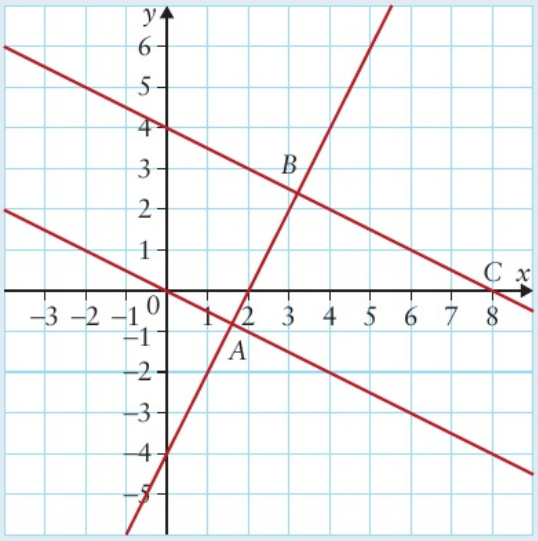
BD is the line with equation $x = 2$, AB is parallel to the x-axis and BE and CD are perpendicular to AC.

Find the coordinates of the points A, B, C, D and E.



Challenge Q10

10 A, B and C are three vertices of a rectangle. The graph shows the lines containing three of the sides of the rectangle.



a What do you think the equations of the lines are? Justify your answer.

b Find the equation for the fourth side.

11 The equations of two lines are:

$A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$

a If the lines are perpendicular, how are the coefficients A_1, B_1, C_1, A_2, B_2 and C_2 related? Justify your answer.

b Apply what you discovered in part a to the line with equation $3x + 2y + 5 = 0$ by finding another line perpendicular to it.

Challenge Q11

4.3 Graphing linear inequalities

When we consider linear functions represented by linear equations of the form $Ax + By + C = 0$, or alternatively $Ax + By = C$, we look for all the points (x, y) in the coordinate plane whose coordinates satisfy that equation. All such points lie on a straight line, hence the name 'linear'. This section deals with situations where instead of an equation we have an inequality such as $Ax + By > C$ or equivalent forms.

Explore 4.3

Consider the inequality $x < 2$ and show the solution on a number line. What about $x \leqslant 2$?

Now consider less familiar examples. Given a coordinate plane, can you show all points (x, y) such that $x < 2$? Explain. What about $x \leqslant 2$?

Can you show all points (x, y) such that $y < 2$? Explain. What about $y \leqslant 2$?

Hint

Such inequalities are also known as **half planes**.

Explore 4.4

Consider all the points with coordinates (x, y).

Can you describe what it means for those points to satisfy $y = 2x + 1$?

Can you describe what it means for those points to satisfy $y < 2x + 1$?

Can you describe what it means for those points to satisfy $y > 2x + 1$?

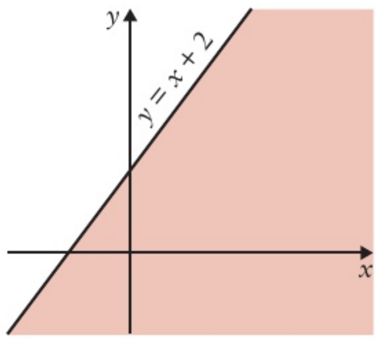
Can you describe what it means for those points to satisfy $y \geqslant 2x + 1$?

Use graphs in your explanations.

You have worked with inequalities before. For example, to solve the inequality $3x - 7 \geqslant 0$ we isolate the variable in the same manner that we do with equations, so $x \geqslant \frac{7}{3}$. The graph of the solution is shown on the number line.

In general, the solution is a set of numbers and not a single value.

Consider the case where we are asked to solve $y \leqslant x + 2$. That is, find the points where the coordinates (x, y) satisfy this inequality. For example, the point $(3, 4)$ is in the solution because $4 < 3 + 2$, and the point $(3, 5)$ is also in the solution because $5 = 3 + 2$. However, the point $(3, 6)$ is not because $6 \nleqslant 3 + 2$

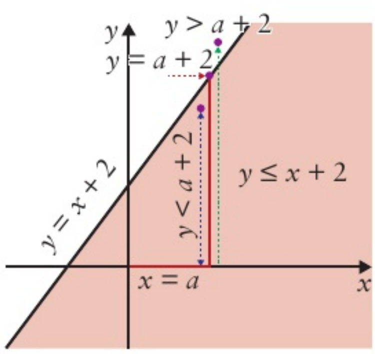


If you use your GDC other software, the solution it gives you is similar to that shown in the diagram on the right.

Let us examine why this is so.

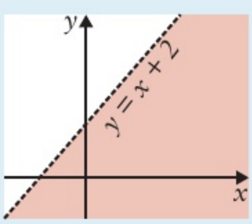
Take any point with x-coordinate $x = a$, where a is any value. The point on the line $y = x + 2$ has a distance from the horizontal axis of $a + 2$

Take any point on the vertical line at a in the shaded region. Its y-coordinate is less than $a + 2$ because the segment representing its y-coordinate is shorter than the segment representing $a + 2$. If you take any point in the non-shaded region, that is, on the opposite side of the line $y = x + 2$, its distance from the x-axis is more than $a + 2$. So, all points in the shaded region have the property that their y-coordinate is shorter than $x + 2$, and thus, this half-plane, including the line itself, represents all points such that $y \leqslant x + 2$



Hint

If the inequality was $y < x + 2$ then the solution would not include the line itself.



This observation allows us to make the following statements.

- On the coordinate plane, all points satisfying the equation $y = mx + b$ lie on one straight line.
- All points satisfying the inequality $y < mx + b$ lie on one side of the line.
- All points satisfying the inequality $y > mx + b$ lie on the other side of the line.

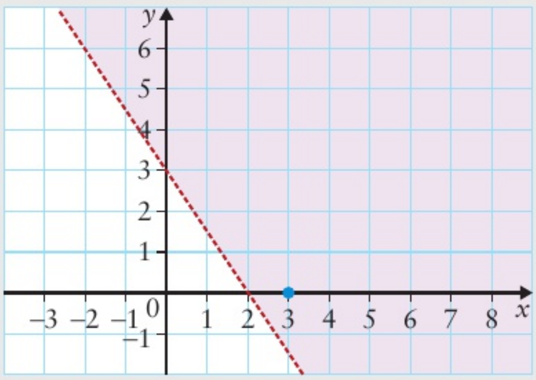
Worked example 4.4

Graph the region $3x + 2y > 6$ in the coordinate plane.

Solution

We know that the points satisfying $3x + 2y = 6$ lie on the straight line represented by that equation.

We can choose a point on one side of the line. Point $(3, 0)$ is such a point because $3 \times 3 + 2 \times 0 = 9$, and $9 > 6$. So this point satisfies the inequality. Therefore all points on the same side of the line as $(3, 0)$ will represent the solution to this inequality.



Hint

One way of finding solution sets for linear inequalities such as $Ax + By > C$ (similarly for $\geqslant$, $\leqslant$, or $<$) is to write the inequality in gradient-intercept form, $y > mx + b$, and then consider which half plane your solution would be in according to the following criteria:

- If $y > mx + b$ the solution is the half plane above the line.
- If $y \geqslant mx + b$ the solution is the half plane above the line and also all the points on the line.
- If $y < mx + b$ the solution is the half plane below the line.
- If $y \leqslant mx + b$ the solution is the half plane below the line and also all the points on the line.

Reflect

Could you have done this without using the graph? That is, without evaluating the inequality at a specific point?

Worked example 4.5

An appliance salesperson earns a fixed weekly salary of €300 plus a commission of €50 for each washing machine they sell and €120 for each refrigerator they sell. How many washing machines and refrigerators must the salesperson sell to make at least €1500 in a week?

Solution

Let x = number of washing machines sold and
y = number of refrigerators sold.

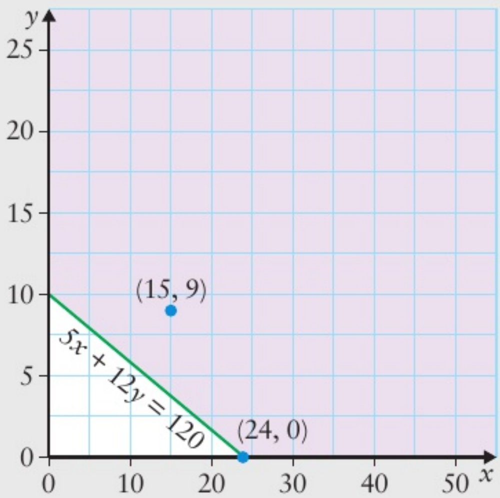
We start with the fixed salary: The salesperson will earn €300 even if they do not sell anything. Plus, for every washing machine sold they earn €50, and for every refrigerator sold they earn €120.

The salesperson's total earnings are therefore $300 + 50x + 120y$

If they were to earn more than €1500, then: $300 + 50x + 120y \geqslant 1500$

The inequality can be simplified to: $5x + 12y \geqslant 120$

We can show the solution set on a graph:



Note that:

- By arbitrarily choosing (15, 9) we end up above the function. This means $300 + 50x + 120y = 300 + 50 \times 15 + 120 \times 9 = 2130$, which is more than 1500. Thus the 'upper' half plane is the solution set.
- There are two other 'hidden' inequalities, $x \geqslant 0$ and $y \geqslant 0$, because the salesperson can only sell a non-negative number of appliances.
- If the salesperson sells 10 refrigerators and no washing machines, they will earn $300 + 120 \times 10 =$ €1500. If they sell 24 washing machines and no refrigerators, they will earn €1500.

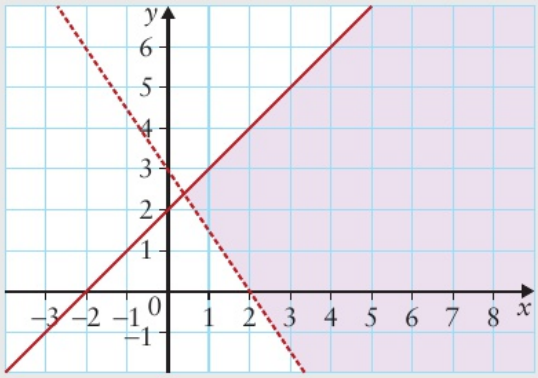
Worked example 4.5 raises another question. What if we want to solve two inequalities simultaneously? Worked example 4.6 takes a look at this.

Worked example 4.6

Find the solution common to $3x + 2y > 6$ and $y \leqslant x + 2$

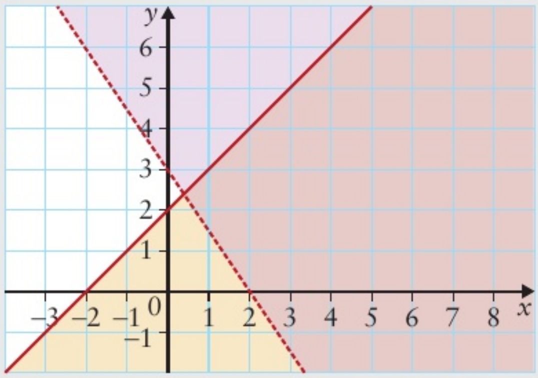
Solution

Since we want two conditions to be satisfied simultaneously, the solution is the intersection of the individual solutions of the inequalities. Recall that the solution to $y \leqslant x + 2$ is the half plane under the line while the solution to $3x + 2y > 6$ is above the line. If sketched together we have the following diagram.



Note that the solution set is the intersection between two half planes: the half plane above $3x + 2y = 6$ and the half plane below $y = x + 2$

Looking back, if the question were to find the solution to $3x + 2y > 6$ or $y \leqslant x + 2$, then we would need the union of the two solutions, which is shown in this diagram.



Hint

The intersection of solutions of two or more inequalities is known as the **feasible region**.

Reflect

For which cases would the boundary line be represented by a dashed line (or solid line) when graphing an inequality?

What is a method to help you determine which half plane to colour when graphing an inequality?

Can you graph a solution set for $|x| \leqslant 3$ or for $|y| \leqslant 3$?

Practice questions 4.3

1 Graph each inequality in a coordinate plane.

a $y \geqslant -5$

b $y < 4x + 3$

c $3x - 4y \geqslant 12$

d $5y - 6x \geqslant -15$

e $y + 5 \leqslant 10 - 4x$

f $y > -x$

2 Find the solution region of the following systems of inequalities.

a $x - y < -6$
$2y \geqslant 3x + 17$

b $4y - 5x < 8$
$-5x \geqslant 16 - 8y$

c $5x - y \geqslant 5$
$2y - x \geqslant -10$

Challenge Q3

Hint Q3

An alloy is a combination of metals.

3 A kilogram of gold costs $55,553 and a kilogram of silver costs $842. Find all possible weights of silver and gold that make an alloy that costs less than $24,200 per kilogram.

Challenge Q4

4 A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and night-time minutes could you use while still paying less than $20 over a 24-hour period?

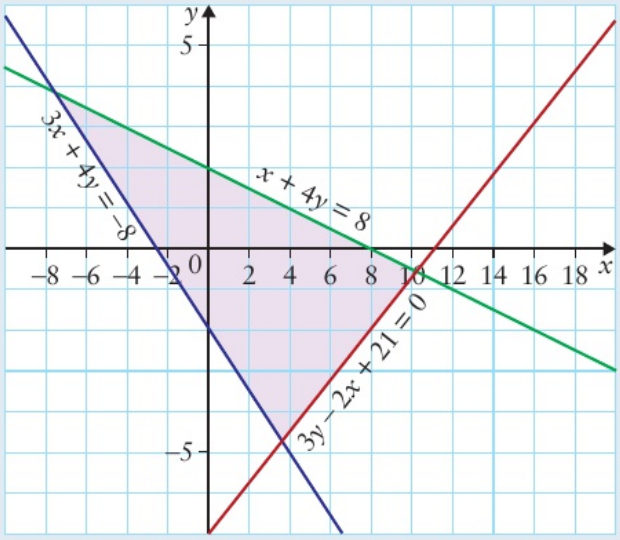
Challenge Q5

5 A coffee roastery makes different blends of coffee by using two types of coffee beans: a large bean that costs €9.00 per kg and a small bean that costs €7.00 per kg. They need to know all possible mixtures of weights of the two different coffee beans for which the blend will not cost them more than €8.50 per kg. Set up a model to help offer the roastery some options.

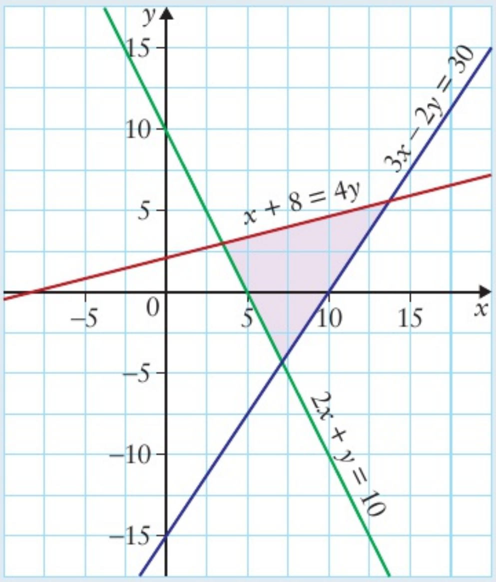
6 Write the system of inequalities for each feasible region.

Challenge Q6

a



b



Self assessment

- I can find the gradient of a linear function given two points on its graph.
- I can explain what the gradient-intercept form is for a linear function.
- I can use different forms of a straight line to draw its graph.
- I can work with the gradient of parallel and perpendicular lines.
- I can find the intersection of two lines.
- I can change an equation from gradient-intercept form to standard form, and vice versa.
- I can work with inequalities in the coordinate plane.
- I can find the feasible region for simultaneous inequalities.
- I can apply linear functions and inequalities to real-life situations.

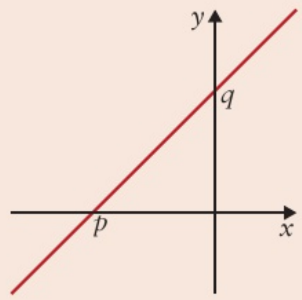
Check your knowledge questions

1 The gradient of the line $f(x) = mx + b$ is m.
Find the gradient of the line:

a $y = -f(x)$ **b** $y = 3f(x)$

c $y = f(x) + 5$ **d** $y = f(2x)$

2 The diagram shows a graph of the line $f(x) = mx + b$



Make a copy of the graph then complete the following tasks.

a Add the line $y = 2f(x)$ to the graph, and write its equation in terms of p and/or q.

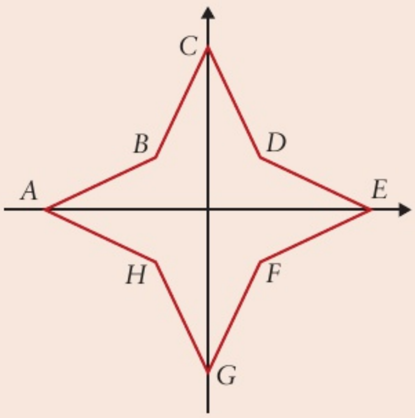
b Add the line $y = -f(x)$ to the graph, and write its equation in terms of p and/or q.

c Add the line $y = f(x) - 3q$ to the graph, and write its equation in terms of p and/or q.

d Add the line $y = f^{-1}(x)$ to the graph, and write its equation in terms of p and/or q.

Challenge Q3

3 The graph of a four-pointed star has four lines of symmetry.



Let the equation of the line segment BC be $y = g(x)$

a Which line segment has equation $y = -g(x)$?

b Which line segment has equation $y = g(-x)$?

c Which line segment has equation $y = -g(-x)$?

d Which line segment has equation $y = g^{-1}(x)$?

The equation of one of the line segments forming the star is $x - 2y = 6$

e Which line segment has this equation?

f Hence, find the equation of each of the other line segments.

4 Find the equation of each line described.

a The line contains the point $(-4, 1)$ and is parallel to the graph of $2x - 9y = 57$

b The line contains the point $(-4, 1)$ and is perpendicular to the graph of $2x - 9y = 57$

c The line has an x-intercept of $(5, 0)$ and a y-intercept of $(0, -6)$

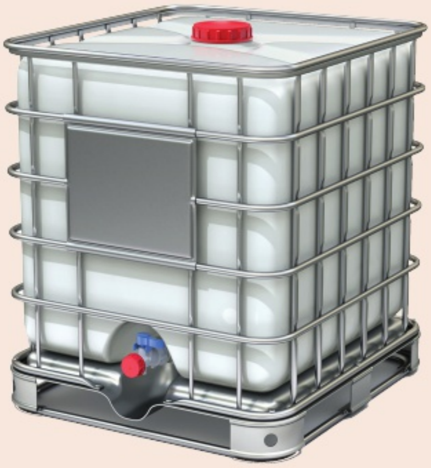
d The line is vertical and contains the point $(-13, 8)$

5 Sketch the feasible region for each system of inequalities.

a $3x + 4y \leqslant 24$
$x - y > 5$

b $3x + 2y \leqslant 12$
$5y - 2x > -10$
$x - y \geqslant -1$

6 In several rural areas, farmers gather rainwater for use during the summer. In one area, the containers used are strong plastic containers that hold up to 1000 litres when full. When winter comes, these containers need to be emptied completely so that the water will not freeze and damage the containers.



When a tap is opened, water starts to drain from the tank. One farmer estimated that after 5 minutes there are 900 litres left. After 10 minutes 800 litres are left. Assume that the rate of drainage does not change with time.

a Write down an equation that expresses the volume of water left as a function of the number of minutes after opening the tap.

b What does the gradient represent?

c Sketch the graph of this function, using a suitable domain.

d How many litres will be left after 30 minutes?

e What does the volume-intercept represent?

f When will the tank be empty?

Hint

A millisecond (ms) is $\frac{1}{1000}$ of a second(s).

7 When you are pricked with a pin, there is a short delay before you say 'Ouch!' This reaction time varies linearly with the distance between your brain and the place you are pricked. Neurologists use such a test to check whether a patient suffers from any nerve problems.

A doctor pricks a patient's finger and toe, and measures reaction times of 15.2 and 22.9 milliseconds (ms), respectively. This patient's finger is 100 cm away from the brain and their toe is 170 cm from the brain.

a Write an equation expressing time delay as a function of distance from the brain.

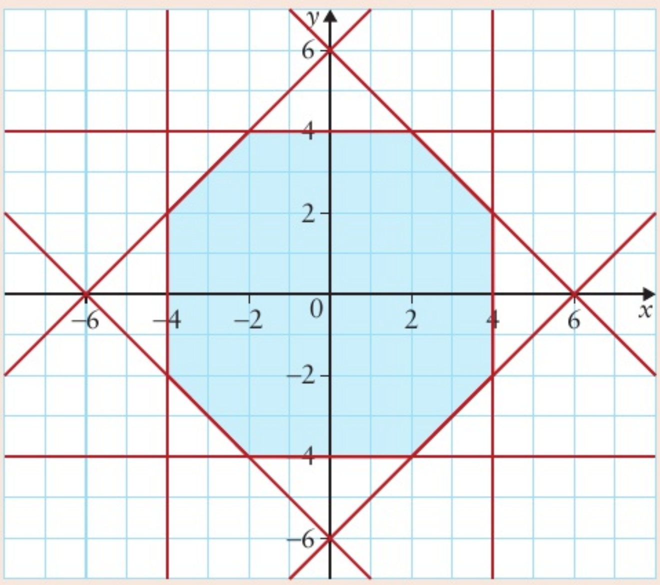
b How long will it take this patient to say 'Ouch!' if pricked in the neck, 10 cm from the brain?

c What does the time-intercept represent?

d Sketch the graph of this function.

e Since the gradient is in milliseconds per centimetre (ms/cm), its reciprocal is in centimetres per millisecond (cm/ms). Interpret the reciprocal of the gradient in real life.

8 The graph below shows eight lines containing the sides of an octagon.



a What are the equations of the eight lines?

b Work out the perimeter of the octagon.

c Work out the area of the octagon.

5

Systems of equations

5 Systems of equations

KEY CONCEPT

Form

RELATED CONCEPTS

Equivalence, Models, Representation

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Complex real-life problems can be solved using technically innovative methods by representing the information in equivalent algebraic and graphical forms.

Factual

- Are solutions to systems of equations unique?
- What determines whether a system of equations has a solution?

Conceptual

- What do we mean by 'a solution to a system of equations'?
- How are solutions interpreted and justified?

Debatable

- Is it always realistic to represent real-life situations with a system of equations?

Do you recall?

1 Write an equation in gradient-intercept form for the line through $(1, 3)$ and $(-2, 4)$.

2 How can you graph the function defined by $y = -2x + 3$ using the gradient and intercept only?

3 Is the point $(2, 3)$ a solution of $y = -2x + 3$?

4 Find the gradient of the line with equation $2x + 3y = 6$

5 Are the lines $2x + 3y = 7$ and $3x - 2y = 9$ parallel?

5.1 Solving systems of equations by graphing

Many real-world situations can be modelled with the use of algebraic functions. Some can be modelled by just one equation, but others, where we wish to explore how multiple factors interact, are more complex. In this chapter we explore only linear functions.

Fact

'Simultaneous' means 'at the same time'.

Functions may be used as mathematical models of the real world. Two or more functions with the same variables form what is called a **system** of equations. A solution of such a system is an ordered pair, for example (x, y), that satisfies all the equations at the same time. For this reason, the equations in a system are often called **simultaneous equations**. Finding a solution is called 'solving the system'. There are several ways to find solutions, the first of which is graphical.

Explore 5.1

A fitness centre offers two options for its users:

1. Membership: A one-time membership fee of €100 and then a cost of €2 per visit.
2. Daily rate: No membership fee but a cost of €12 per visit.

If you visit the centre twice, the cost will be €104 using the membership option, but €24 using the daily rate option. It is obviously advisable not to become a member in this case. However, if you visit 20 times, the costs will be €140 and €240 respectively, in which case it is better to become a member.

Without listing all the possibilities, can you work out the number of visits you need to make before membership becomes the better option?

Worked example 5.1

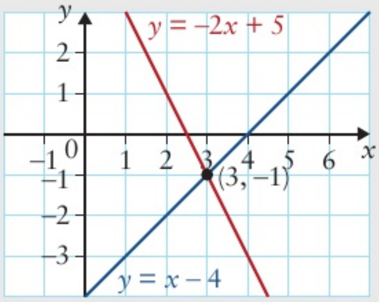
Solve the following system of equations by graphing.

$$y = -2x + 5$$

$$y = x - 4$$

Solution

To solve a system of equations graphically, we need to find the point(s) where both equations are satisfied. By graphing both lines we can find where the lines intersect. The solution will be the coordinates of the point of intersection.



The solution to this system of equations is $x = 3$ and $y = -1$

To check whether this solution is correct, we substitute these values into the equations and check whether both are satisfied.

Start with the original system.

$y = -2x + 5$ $\qquad\qquad$ $y = x - 4$

Substitute in the coordinates $(3, -1)$ and simplify:

$(-1) = -2(3) + 5$ $\qquad\qquad$ $(-1) = (3) - 4$

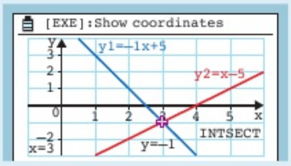
$-1 = -1$ $\qquad\qquad$ $-1 = -1$

Since both equations are satisfied, we know the solution $x = 3$ and $y = -1$ is correct.

Hint

For graphical solutions you can use your GDC.

Here is a sample solution:



Worked example 5.2

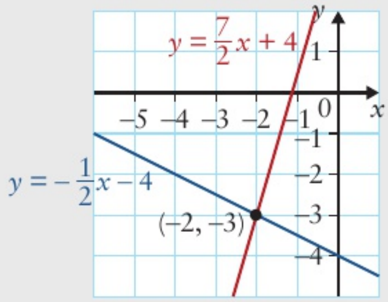
Solve the following system of equations by graphing.

$$y = \frac{7}{2}x + 4$$

$$x + 2y = -8$$

Solution

We first write the second equation in gradient-intercept form.



Since the lines intersect at $(-2, -3)$ the solution to this system of equations is $x = -2$ and $y = -3$

To check whether this solution is correct, we substitute the values into the original equations to make sure both are satisfied.

Start with the original system:

$y = \frac{7}{2}x + 4 \qquad x + 2y = -8$

Substitute in the coordinates $(-2, -3)$:

$(-3) = \frac{7}{2}(-2) + 4 \qquad (-2) + 2(-3) = -8$

Simplify:

$-3 = -3 \qquad -8 = -8$

Since both equations are satisfied, we know the solution $x = -2$ and $y = -3$ is correct.

Worked example 5.3

Solve the following two systems of equations by graphing.

a $6y = 3x + 18$
$2x - 4y = -12$

b $3y = 2x - 15$
$4x - 6y = 12$

Solution

a First we write the equations in gradient-intercept form.

$$y = \frac{1}{2}x + 3$$

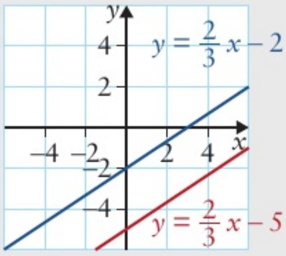
$$y = \frac{2}{4}x + \frac{12}{4} = \frac{1}{2}x + 3$$

Note that both equations are equivalent. That is, every solution to one is also a solution to the other. On a graph of the system, we would only 'see' one line even though both are graphed. This means that every point on the lines is a solution. Hence, there are *infinitely many solutions* to this system.

b In the second system of equations, we follow the same procedure as above. The gradient-intercept forms of the equations are:

$y = \frac{2}{3}x - 5$ and $y = \frac{2}{3}x - 2$

Since the two gradients are equal, the lines are parallel.



The two lines do not intersect. Hence, there is *no solution* to this system.

Fact

A system of equations is called an **inconsistent** system of equations if there is no solution to the system.

Worked example 5.4

Veronica and David work for different companies repairing photocopiers. Veronica charges a €150 service fee plus €50 per hour for labour. David charges €80 per hour for labour, but does not charge a service fee. On one day, they each did one job. The charge for both jobs was the same. How many hours did each work on that day? What was the charge?

Solution

To solve this problem, we need to identify the variables to use. Then we can write an equation for each person's company and graph the equations to find the point of intersection. This point of intersection will be the solution to the system and the answer to the question.

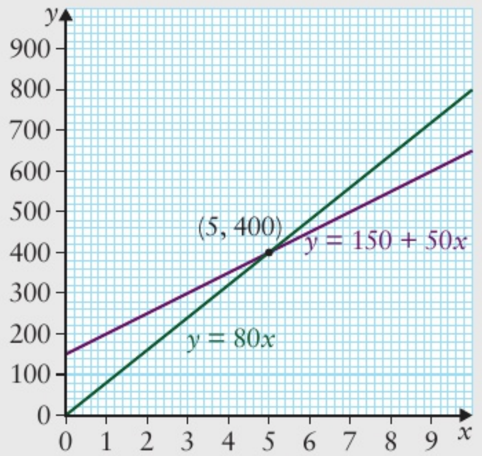
Let x = the number of hours worked

Let y = the total cost

Veronica's company: $y = 150 + 50x$

David's company: $y = 80x$

Now we can graph the system and determine the point of intersection.



The solution to the system is $x = 5$ and $y = 400$. Therefore, Veronica and David each worked 5 hours and the cost for each job was €400.

To check whether this solution is correct, we substitute the coordinate point into both equations and simplify:

Start with the original system:

$y = 150 + 50x \qquad y = 80x$

Substitute in the coordinates (5, 400), then simplify:

$(400) = 150 + 50(5) = 400 \qquad (400) = 80(5) = 400$

Since both equations are satisfied, we know the solution of $x = 5$ and $y = 400$ is correct.

Fact

- When two lines intersect, the point of intersection is the solution for the system. In other words, when you substitute the coordinates of the solution point into both equations, it is true for both equations. Hence, they are true at the same time (or **simultaneously**).
- If the lines are parallel, they do not intersect, and there is no solution.
- If the lines intersect at every point, they are identical lines and there are infinitely many solutions.

Reflect

When you look at a system of equations, how can you determine the number of solutions it might have?

Practice questions 5.1

Hint

Check your work algebraically or by using technology.

Work out the solution to each of the systems of equations in questions 1–13 by graphing.

1 $y = 2x - 1$
$y = \frac{1}{2}x + 2$

2 $y = -\frac{3}{2}x + 5$
$y = x$

3 $y = 6x - 4$
$y = -3x + 5$

4 $y = 4x + 1$
$y = 4x - 3$

5 $y = \frac{2}{5}x - 1$
$y = -\frac{3}{5}x - 6$

6 $y = \frac{2}{3}x + 5$
$y = -3x - 6$

7 $y = -\frac{4}{7}x + 3$

$y = -\frac{8}{14}x + 1$

8 $y = \frac{4}{5}x + 4$

$y = -\frac{1}{5}x - 1$

9 $y = -\frac{1}{6}x - 2$

$y = -2x + 9$

10 $y = -\frac{7}{2}x + 3$

$4y = -14x + 12$

11 $y = \frac{5}{2}x + 6$

$y = \frac{5}{2}x - 10$

12 $y = \frac{3}{5}x - 8$

$y = -x$

13 $y = -\frac{2}{3}x + 4$

$3y = -2x - 6$

14 Matthew is collecting information on the heights of two different plants for his biology class. When he began collecting the data, the first plant had a height of 12 cm and grew at a rate of 0.5 cm per week. The second plant had a height of 7 cm and grew at a rate of 1 cm per week. After how many weeks will the plants have the same height?

15 Two professional snowboarders, Kate and Tony, are racing down a hill. Kate travels at a rate of 11 m/s and Tony travels at a rate of 15 m/s. The track is 1600 m long. Kate starts 5 seconds before Tony.

a Will Tony pass Kate?

b How long does it take each snowboarder to finish the track?

16 Morgana is given the first equation in a system as $y = \frac{2}{3}x + 4$
How should she write a second equation so that the system has:

a one solution

b no solutions

c infinitely many solutions?

17 Carlee and Ashley are walking through different parts of a nature trail. Carlee gets on the trail 8 km from the beginning of the trail and walks at a rate of 5 km/h. At the same time, Ashley starts walking on the trail 5 km from the beginning and walks at a rate of 4 km/h. Determine when and where Carlee and Ashley will meet. Is your solution meaningful? Explain.

18 The student council bought 28 one-litre bottles of juice for an event. The apple juice cost €1.75 per bottle and the orange juice cost €2.25 per bottle. They spent a total of €55. How many bottles of each type of juice did they purchase?

Hint Q18

Try writing the equations in standard form.

19 The table shows the population of Starksville and Rogerstown over 25 years. Assuming the growth rates are linear, determine how many people were in each city when they had the same population.

	Starksville	Rogerstown
1990	110 000	103 000
2015	90 000	115 000

Connections

It is important to know how to work out profit and loss in the day-to-day running of a business.

20 The cost in dollars to produce x packages of organic dog treats each day is given by $C = 6x + 150$. The profit for selling x packages is represented by $P = 12x$. Graph both equations and interpret the point of intersection in context.

21 Gary's repair shop charges £90 for parts and £50 per hour for labour. For the same job, Frank's repair shop charges £115 for parts and £40 per hour for labour.

a Which repair shop is cheaper if the job takes:

i 2 hours ii 5 hours?

b How many hours does the job take if the cost is the same at both repair shops?

22 Students in Mrs Harte's art class created greeting cards and sold them for a local charity. The number of stamped cards was three times the number of mixed media cards. The stamped cards sold for €2 each and the mixed media cards sold for €4 each. The students raised €260. How many of each type of card was sold?

5.2 Solving systems of equations by substitution

As well as solving systems graphically, we can also use algebraic methods for solving systems. We will look at the substitution method first.

Explore 5.2

Your school hosted a fundraiser over three days. Each day, the fundraiser earned a total of $1200. Adult tickets sold for $8 and youth tickets sold for $6. On the first night, a total of 250 people attended. On the second night, there were three times as many adult tickets sold as youth tickets. On the final night of the event, there were twice as many youth tickets sold as adult tickets.

How can you determine which night had the most tickets sold? How many tickets were sold for all three days?

When using the substitution method, the goal is to choose one equation and solve it for one variable in terms of the other, and then substitute that expression into the other equation. This gives us one equation with one unknown variable, which we can then solve. We then substitute the solution back into one of the original equations to determine the value of the other variable.

Worked example 5.5

Solve the following system of equations using the substitution method.

$$x = 4y + 12$$
$$\frac{1}{2}x + y = 3$$

Solution

In this system, we already have an expression for x in terms of y in the first equation. We can substitute that expression for x into the second equation and then solve for y.

$$\frac{1}{2}(4y + 12) + y = 3$$

Expand the brackets, then simplify:

$$2y + 6 + y = 3$$

So, $3y + 6 = 3$

Therefore, $y = -1$

To determine the value of x, we substitute the y-value into the first equation and solve for x.

$$x = 4(-1) + 12 = 8$$

The solution is $x = 8$ and $y = -1$

To check our solution we can substitute it back into both of the original equations and simplify.

$x = 4y + 12$	$\frac{1}{2}x + y = 3$
$(8) = 4(-1) + 12 = 8$	$\frac{1}{2}(8) + (-1) = 3$

Because the left-hand side and the right-hand side are equal for both equations, we know that $x = 8$ and $y = -1$ is the correct solution.

Worked example 5.6

Solve the following system of equations using the substitution method.

$$y = \frac{3}{4}x - 5$$

$$4y + 8 = 3x$$

Solution

In this system, we already have an expression for y in terms of x. We can substitute that expression for y into the second equation and then solve for x.

$$4\left(\frac{3}{4}x - 5\right) + 8 = 3x$$

$$3x - 20 + 8 = 3x$$

$$3x - 12 = 3x$$

$$-12 = 0$$

This gives us a false statement, which means there is no solution to this system. The result is confirmed when we write the second equation in gradient-intercept form.

$$4y + 8 = 3x$$

$$y = \frac{3}{4}x - 2$$

When comparing the two equations, we can see they have the same gradient, but different y-intercepts. Hence, the lines are parallel.

Worked example 5.7

Solve the following system of equations using the substitution method.

$$y = -\frac{3}{5}x + 2$$

$$y = \frac{1}{2}x + 13$$

Solution

In this system, both equations already have expressions for y in terms of x. We can set the two expressions equal to each other and solve for x. One way to start is to multiply both sides by 10 in order to clear the fractions.

$$-\frac{3}{5}x + 2 = \frac{1}{2}x + 13$$

$$10\left(-\frac{3}{5}x + 2\right) = 10\left(\frac{1}{2}x + 13\right)$$

$$-6x + 20 = 5x + 130$$

$$20 = 11x + 130$$

$$-10 = x$$

To determine the value of y, we substitute the x-value into the first equation and solve for y.

$$y = -\frac{3}{5}(-10) + 2$$

$$y = 8$$

The solution is $x = -10$ and $y = 8$

To check our solution, we substitute it back into both of the original equations and simplify.

$y = -\frac{3}{5}x + 2$	$y = \frac{1}{2}x + 13$
$(8) = -\frac{3}{5}(-10) + 2$	$(8) = \frac{1}{2}(-10) + 13$
$8 = 8$	$8 = 8$

Because the left-hand side and the right-hand side are equal for both equations, we know that $x = -10$ and $y = 8$ is the correct solution.

Worked example 5.8

In a chemistry lab, there is a bottle of 6% hydrochloric acid solution and a bottle of 16% hydrochloric acid solution. For an experiment, 250 ml of a 10% hydrochloric acid solution is needed. How much of each type of solution should be used to make up the solution required for the experiment?

Solution

This question is asking how much of each of the given acid solution mixtures should be combined to create the amount and concentration of acid solution required for the experiment.

First we identify what variables to use. Then we can write a system of equations. One equation can represent the amount of solution needed and the other equation can represent the amount of hydrochloric acid in the solution. We can then solve the system using the substitution method and check our answer.

Let x = number of ml needed of the 6% solution.

Let y = number of ml needed of the 16% solution.

The first equation represents the number of ml needed:

$x + y = 250$

The second equation represents the amount of hydrochloric acid in each quantity of the solution. Note that we are multiplying the percentage by the amount of the solution to get the amount of hydrochloric acid:

$0.06x + 0.16y = 0.10(250)$

Now we can solve the system by writing the first equation as an expression for y in terms of x and then substituting into the second equation:

$y = 250 - x$

$0.06x + 0.16(250 - x) = 25$

$0.06x + 40 - 0.16x = 25$

$-0.10x + 40 = 25$

$x = 150$

Then we substitute this value for x into the first equation to find y:

$150 + y = 250$

$y = 100$

The solution is $x = 150$ and $y = 100$, which means we will need 150 ml of the 6% hydrochloric acid solution and 100 ml of the 16% hydrochloric acid solution to have 250 ml of a 10% hydrochloric acid solution.

We can check our answer by substituting the values into both equations and simplifying:

$x + y = 250$	$0.06x + 0.16y = (0.10)(250)$
$(150) + (100) = 250$	$0.06(150) + 0.16(100) = 25$

Because both equations simplify so that the right-hand sides are equal to the left-hand sides, we know we have solved the system correctly.

Reflect

Look back at Explore 5.2. How did you write your equations to represent the number of tickets sold each day? What method did you use to answer the questions?

The method of substitution should be a familiar concept. When checking an answer to an equation, we substitute the value into the equation and simplify. With a system of equations, we substitute an expression into an equation to simplify two equations with two variables into one equation with one variable.

Practice questions 5.2

Use substitution to work out the solutions to each of the following systems of equations.

1 $y = 4x - 1$
$2x + y = 5$

2 $y = -\frac{1}{4}x + 3$
$-3x = 4y + 8$

3 $x = 2y + 5$
$x = -6y - 20$

4 $x = -2y - 7$
$-2x - y = -4$

5 $y = \frac{4}{3}x - 1$
$y = -\frac{1}{3}x + 5$

6 $-2x = \frac{1}{3}y - 1$
$-y = 6x - 2$

7 $x = \frac{1}{2}y - 4$
$x = -\frac{3}{2}y - 6$

8 $-4x = -10y - 4$
$2x - 5y = 2$

9 $2x + y = -4$
$-\frac{1}{4}x - 2y = -1$

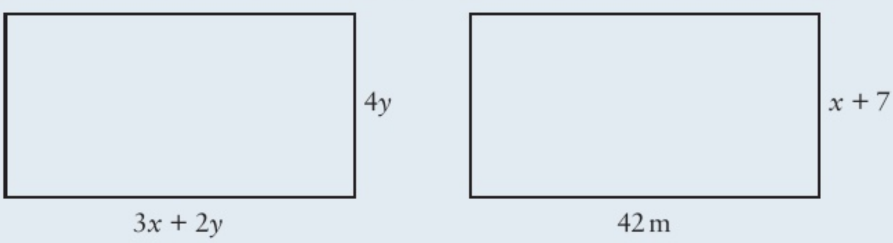
10 $y = 2x + 3$
$y = -2x - 9$

11 $x = -\frac{5}{2}y + 8$
$6x = -15y + 48$

12 $y = \frac{4}{3}x + 4$
$\frac{1}{4}y = -5 - x$

Work out the solution to each of the following questions.

13 What values of x and y will makes the rectangles shown below congruent? Hence, what is the perimeter and area of the rectangles?



Reminder

Congruent means the shapes are identical in shape and size.

14 Write a system of equations to find the values of the first and second numbers described in the following number puzzle. Five less than two times the first number is the second number. Six less than ten times the second number is four times the first number.

15 The sum of the ages of Stephanie and her father is 123. Her father is three years less than twice her age. How old are Stephanie and her father?

16 Picky Pets pet food store sells one mixture of seafood flavoured cat food for \$6.00 per pound and another mixture of poultry flavours for \$4.50 per pound. How much of each type must Jennifer mix in order to have 75 pounds of a seafood and poultry mixture that will cost her \$5.00 per pound?

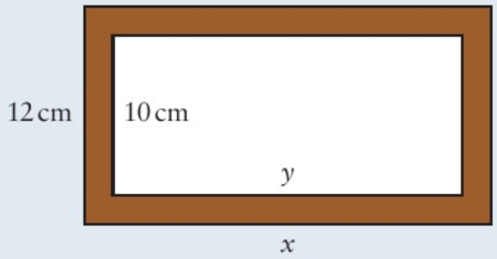
17 The student council is trying to book a location for the Spring Dance. The Red Hall will charge a fee of €2000 plus €50 per person for the dinner. The White Lounge will charge a fee of €2800 plus €42 per person for the dinner. How many people will need to attend for the Red Hall to be the better deal?

18 Simon earns £3.50 more per hour than Florian. If they both work 40 hours in one week they earn £1200 in total. How much money per hour do Simon and Florian earn?

19 If the difference of two numbers is 51 and the sum of the numbers is 23, what are the numbers?

Challenge Q20

20 A wooden picture frame is to be made by cutting out a rectangular hole. The hole will be 10 cm wide and y cm long. It will be cut from a board that measures 12 cm wide and x cm long. The length of the hole is 3 cm less than the length of the board. After the hole is cut, the remaining area for the picture frame is 66 cm^2. Find the length of the hole and the length of the frame.



5.3 Solving systems of equations by elimination

Systems of equations can also be solved using the arithmetic operations of addition, subtraction and multiplication. The goal of the elimination method is to eliminate one of the variables by either adding or subtracting the equations. However, you may need to multiply one or both equations before you can eliminate a variable.

Explore 5.3

A farmer sells fresh produce by weight. All tomatoes weigh the same, and all cucumbers weigh the same. Given the information in the image, can you determine the weight of one tomato and one cucumber?

Worked example 5.9

Solve the following system of equations using the elimination method.

$2x + 5y = 3$

$3x - 5y = -8$

Solution

In this system, we can see that the coefficients for y are the same, but have opposite signs. If we add the equations, we can eliminate y and solve for x:

$2x + 3x + 5y - 5y = 3 - 8$

$x = -1$

To determine the value of y, we substitute the x-value into the first equation and solve for y:

$2x + 5y = 3$

$2(-1) + 5y = 3$

$y = 1$

The solution is $(-1, 1)$

To check our solution, we substitute it back into both of the original equations and simplify:

$2x + 5y = 3$	$3x - 5y = -8$
$2(-1) + 5(1) = 3$	$3(-1) - 5(1) = -8$
$3 = 3$	$-8 = -8$

Because the left-hand side and the right-hand side are equal for both equations, we know that $x = -1$ and $y = 1$ is the correct solution.

Worked example 5.10

Solve the following system of equations using the elimination method.

$-4x + 11y = 3$

$-4x + 8y = -6$

Solution

In this system, we see that the coefficients for x are the same and have the same sign. If we subtract the equations, we can eliminate x and solve for y:

$-4x - (-4x) + 11y - 8y = 3 - (-6)$

$3y = 9$

$y = 3$

To determine the value of x, we substitute the y-value into the first equation and solve for x:

$-4x + 11y = 3$

$-4x + 11(3) = 3$

$x = 7.5$

The solution is $(7.5, 3)$

To check our solution, we substitute it back into both of the original equations and simplify:

$-4x + 11y = 3$	$-4x + 8y = -6$
$-4(7.5) + 11(3) = 3$	$-4(7.5) + 8(3) = -6$
$3 = 3$	$-6 = -6$

Because the left-hand side and the right-hand side are equal for both equations, we know that $x = 7.5$ and $y = 3$ is the correct solution.

Worked example 5.11

Solve the following system of equations using the elimination method.

$-2x + 3y = 12$

$3x + 5y = 1$

Solution

In this system, none of the coefficients are the same. We need to multiply both equations by some numbers before adding or subtracting in order to eliminate one of the variables.

We can eliminate x by multiplying the top equation by 3 and the bottom equation by 2, then adding the equations:

$-6x + 9y = 36$

$6x + 10y = 2$

Now we add the equations to eliminate x:

$19y = 38$

$y = 2$

To determine the value of x, we substitute the y-value into the first equation and solve for x:

$-2x + 3y = 12$

$-2x + 3(2) = 12$

$x = -3$

The solution is $x = -3$, $y = 2$

To check your solution, we substitute it back into both original equations and simplify:

$-2x + 3y = 12$	$3x + 5y = 1$
$-2(-3) + 3(2) = 12$	$3(-3) + 5(2) = 1$
$12 = 12$	$1 = 1$

Because the left-hand side and the right-hand side are equal for both equations, we know that $x = -3$, $y = 2$ is the correct solution.

Worked example 5.12

Use the elimination method to work out the solution to the following system of equations.

$$6x - 9y = -12$$

$$2x - 3y = -4$$

Solution

In this system, none of the coefficients are the same. However, if we multiply the second equation by 3, we will be able to eliminate x by subtracting the equations and then solve for y.

Multiplying the second equation by 3 changes the system to:

$$6x - 9y = -12$$

$$6x - 9y = -12$$

Now the coefficients for both x and y are the same. If we subtract the equations, we will eliminate both x and y:

$$0 = 0$$

This is a true statement, which means there are infinitely many solutions to this system. The two equations have the same gradient and y-intercept. Hence, the lines are identical, and there are infinitely many solutions.

Worked example 5.13

Melissa has booked a holiday with a local tour company. She purchased 3 nights in a hotel and 4 theatre tickets for a total of €369. John used the same tour company to book 5 nights in the same hotel and 14 theatre tickets for a total of €824. David also wants to book his holiday plans with this tour company, but he plans to purchase 6 nights in the hotel and 6 theatre tickets. How much will David pay?

Solution

This question concerns two different holiday plans. We know the total cost for each holiday booking and the quantity of hotel nights and theatre tickets. We need to determine the cost of one night in the hotel and one theatre ticket. Once we know that, we can calculate the total cost for 6 nights in the hotel with 6 theatre tickets.

First we need to identify the variables to use. Then we can write an equation describing Melissa's booking, and a second equation that describes John's booking. Next we solve the system of equations using the elimination method.

Let x = cost of one night in the hotel

Let y = cost of one theatre ticket

Melissa: $3x + 4y = 369$

John: $5x + 14y = 824$

Multiply Melissa's equation by 5 and John's equation by -3:

$15x + 20y = 1845$

$-15x - 42y = -2472$

Now we can add the equations to eliminate x:

$-22y = -627$

$y = 28.50$

We substitute $y = 28.50$ into the first equation to solve for the value of x:

$3x + 4(28.50) = 369$

$x = 85$

The solution is $x = 85$ and $y = 28.50$

To check our solution, we substitute $x = 85$ and $y = 28.50$ into the other equation and simplify:

$5(85) + 14(28.50) = 824$

$824 = 824$

Now that we know that our solution is correct, we can answer the question. For David to book a holiday with 6 nights in a hotel and 6 theatre tickets, it will cost:

$6(85) + 6(28.50) = €681$

Reminder

Always check that you have answered the question that was asked.

Reflect

You have now seen different ways to apply the elimination method, along with several different techniques to solve systems of equations. When looking at a system of equations, how do you decide which method would be the most efficient to use? Which method do you prefer? Why?

Practice questions 5.3

For questions 1–15, use elimination to work out the solutions to the systems of equations.

1 $2x - 3y = 1$
$x + 3y = -4$

2 $12y - 5x = 22$
$9y - 5x = 10$

3 $-6x - y = 7$
$6x - 2y = 14$

4 $5x - 6y = 17$
$-x - 6y = -7$

5 $\frac{2}{3}x + 2y = -6$
$-2x - 6y = 18$

6 $-4x - 9y = 23$
$2x + 9y = -7$

7 $x - 5y = -6$
$6x - 2y = 20$

8 $-3x + 2y = -16$
$5x + 4y = 1$

9 $5x + 8y = 12$
$5x + 8y = 1$

10 $-5x + 6y = -7$
$3x + 2y = 14$

11 $\frac{4}{3}x + \frac{1}{2}y = -5$
$8x + 3y = 2$

12 $3x - 7y = 18$
$3x + y = -6$

13 $-4y + 3x = -20$
$3y = -2x - 19$

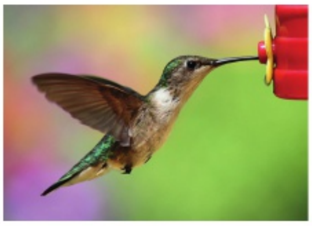
14 $2x + 3y = 4$
$4x + 6y = -2$

15 $-4x + y = 4$
$\frac{5}{4}x + \frac{3}{8}y = -4$

16 Hakim invested a total of \$12,000 in two different accounts. Account A earns 4.5% simple interest and account B earns 6% simple interest. After one year, the combined interest earned is \$699. How much of the \$12,000 did Hakim invest in each account?

17 Judy, the owner of Sweet-n-Salty Treats, is making bags containing candies and nuts to sell. She needs a total of 10 kg of the mixture, and she wants to pay €3.50 per kg. The candies cost €4.40 per kg and the nuts cost €2.60 per kg. How many kilos of candies and nuts does she need?

18 Matt and Sarah drove a total of 1800 km in 21 hours. Sarah drove the first part at an average rate of 95 km/h. Matt drove the rest of the way at an average speed of 80 km/h. How many hours did Sarah drive for? How many hours did Matt drive for?



19 To refill her hummingbird feeders, Marian needs to make 4 litres of a 40% sugar-water solution by mixing together a 70% sugar-water solution and a 20% sugar-water solution. How much of each solution should she use?

20 A tour boat travelled 240 km each way downstream and back. The trip downstream took 6 hours, and the return trip took 10 hours. What was the speed of the current? How fast was the boat travelling in each direction?

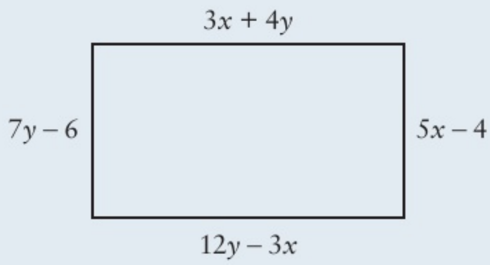
Hint Q20

You can use the formula distance = speed × time

21 Anneke rents a bouncy castle for a party. She paid €191, which included the set-up fee as well as the cost to rent the bouncy castle for 2 hours. The next month, her neighbour Elliott rented the same bouncy castle from the same company for 7 hours and paid €331. How much would it cost to rent the bouncy castle for 5 hours?

22 The Lee family and the Park family go skiing for the day. The Lees purchase three adult passes and four youth passes for €360. The Parks purchase four adult passes and seven youth passes for €555. What is the cost of two adult passes and two youth passes?

23 Determine the length and width of the rectangle shown.



24 A speciality paintbrush company is able to supply personalised paint brushes. The cost of manufacturing can be modelled by the equation $P = 28 + 0.8x$, where x is the number of paintbrushes produced per hour and P is the price in dollars. The demand for these paintbrushes is modelled by $P = 37 - 0.1x$. Determine the solution for the supply and demand equations and interpret your solution in context.

Thinking skills

Communication skills

Self assessment

- I can solve systems of equations using different techniques, such as graphing, substitution, elimination and technology.
- I can select the most efficient method for solving a system of equations.
- I can apply problem-solving strategies to solve systems of equations.
- I can check my solution to verify my answer.
- I can interpret my solution in context.

Check your knowledge questions

Solve the following systems of equations by graphing.

1 $-2x + 2 = y$
$y = \left(-\frac{3}{2}\right)x - \frac{1}{2}$

2 $y = -x + 5$
$-4x + 8y = 4$

3 $-4x + 8y = 24$
$2x - 4y = -16$

4 $x + 4y = 11$
$6y = -4x + 4$

Solve the following systems of equations by substitution.

5 $-12x + 9y = 27$
$4x - 3y = -9$

6 $-4x - 5y = 72$
$x = -6y - 75$

7 $3y - 13 = x$
$6x - 6y = 6$

8 $2x + 7y = -36$
$5x - 21 = y$

Solve the following systems of equations by elimination.

9 $-5x - 8y = -68$
$x - 8y = -92$

10 $-3x + 3y = -54$
$3x + 7y = -36$

11 $-7x - 2y = 63$
$3x + y = -26$

12 $-6x + 7y = 3$
$-12x + 14y = 18$

Solve the following systems of equations using any method.

13 $-x - 4y = 43$
$7x + 2y = -41$

14 $-3x + 3y = 105$
$x - 7y = -155$

15 $8x + y = 30$
$-2x + 7y = 36$

16 $-4x - y = -54$
$7x + 3y = 107$

17 $-x - 6y = 4$
$-\frac{1}{2}x - 3y = 4$

18 $-2x + 6y = 40$
$6x - 3y = 75$

19 $-x + 3y = -71$
$2x + 2y = 6$

20 $2x + 2y = -10$
$x + y = -5$

21 $7x + 2y = 4$
$-6x + 7y = 136$

22 $8x + 6y = -36$
$5x + 5y = -30$

23 $3x + 4y = -8$
$12x + 16y = -16$

24 $-3x - 2y = -7$
$-4x + y = 53$

25 $7x - 4y = -45$
$-4x + 3y = 35$

26 $-4x - 2y = 36$
$-x + 3y = 65$

27 A prize of €5000 was invested in two stock accounts. One account pays 3% annual interest and the other pays 5% annual interest. At the end of the year, the combined interest earned was €190. How much was invested in each account?

28 Helena and Sindhu are entered in a timed cross-country race. Helena leaves first and runs with an average speed of 8 km/h. Sindhu leaves 10 minutes later and runs with an average speed of 12 km/h. How long will it take for Sindhu to pass Helena?

29 Tara buys some school supplies. She purchases 10 notebooks and 4 pens for \$32.50. Later, she returns to the same store and buys 8 notebooks and 5 pens for \$35. What is the cost of 3 notebooks and 2 pens?

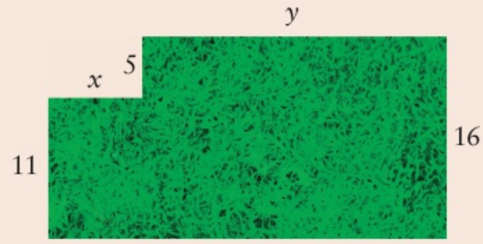
30 The sides of a parallelogram include 2 equal acute angles and 2 equal obtuse angles. The measure of each obtuse angle is 31° greater than the measure of each acute angle. How large are the four angles? Draw a diagram to help understand the situation.

Hint

Drawing a diagram will help you to visualise the shape in the question.

31 In preparing his field for planting, farmer Bob spends \$300. He plants green beans and estimates it costs him \$1.45 per pound to harvest the beans. He sells the beans for \$3.50 per pound. How many pounds must he sell to break even?

32 A pet exercise area is being constructed in a local park in the shape shown below. The area will be 366 m^2 and require 80 m of fencing to surround it. Find the values for x and y and hence the overall dimensions of the park.



33 Adam is mixing two brands of fertiliser for his garden. He needs a mixture that is 30% of Brand A and 70% of Brand B. He currently has 3.5 litres of a mixture that is 50%–50% of the two brands. He adds 100% of Brand B to the 50%–50% mix. Does Adam have enough of the 50%–50% mixture to make 7 litres of the 30%–70% mixture?

6

Matrices

6 Matrices

KEY CONCEPT

Form

RELATED CONCEPTS

Generalisation, Representation, Simplification, Systems

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Representing systems of equations in matrix form helps us to simplify calculations and enables us to perform them at the same time by using an innovative technical application method.

Factual

- What are the advantages of having a unified description of many different phenomena?
- How is finding the inverse of a matrix similar to division of real numbers?

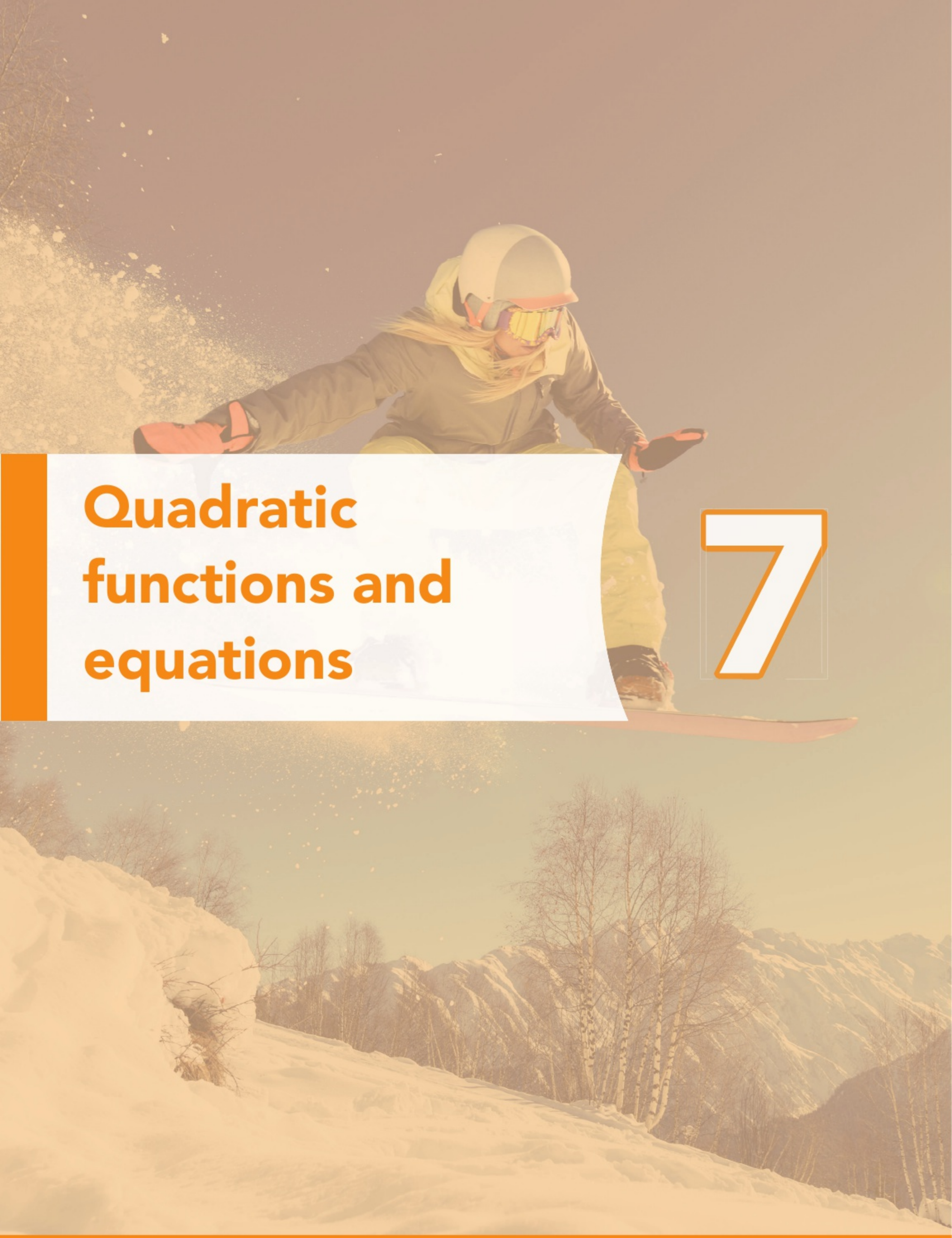
Conceptual

- How are data stored in matrices?
- How do we use matrices to solve systems of equations?

Debatable

- What advances are made possible by faster computation techniques?

Chapter 6 is an enrichment chapter at Standard level. Please access the chapter via the link on this page of your eBook.



Quadratic functions and equations

7

7 Quadratic functions and equations

KEY CONCEPT

Form

RELATED CONCEPTS

Equivalence, Models, Representation, Simplification

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Representing a quadratic relationship in different forms allows engineers to model complex structures more simply and facilitates the use of technology in innovative ways.

Factual

- How do the parameters of a quadratic function determine the characteristics of the curve that represents it?

Conceptual

- How many forms can you use to express a quadratic function?

Debatable

- If a quadratic function is created by a combination of transformations from the basic quadratic model, which transformation has precedence?

Do you recall?

1 Graph these straight lines, and find their x- and y-intercepts.

$y = 2x + 5$

$3x - 2y + 3 = 0$

2 Find the point where $y = 3x + 5$ and $y = -x + 1$ intersect.

3 Which of the points (1, 2) and (2, 4) are on the graph of $f(x) = 3x - 1$? Why?

7.1 The graphs of $y = x^2$ and $y = ax^2$

Explore 7.1

Use a GDC or other suitable software to graph the functions
$y = f(x) = x^2$ and $y = h(x) = -x^2$

Describe the shape of each graph.

Copy and complete the following table.

x	−3	−2	−1	0	1	2	3
$y = f(x)$							
$y = h(x)$							

What can you say about the connection between the graphs of these two functions as x increases from −3 to 3?

Explore 7.2

Consider the graphs of $y = f(x) = x^2$ and $y = g(x) = ax^2$ when $a = 1.5, -1.5, 3$ and -3

Graph the functions using technology. Then copy and complete the table.

x	−3	−2	−1	0	1	2	3
$y = f(x)$							
$y = g(x)$ $a = 1.5$							
$y = g(x)$ $a = -1.5$							
$y = g(x)$ $a = 3$							
$y = g(x))$ $a = -3$							

What can you conclude about the graphs of $f(x)$ and $g(x)$?

Can you predict the shape of the graph of $h(x) = 4x^2$?

Reminder

Be careful with the order of operations. $2x^2 \neq (2x)^2$ and, in general, $ax^2 \neq (ax)^2$

Worked example 7.1

Consider the graphs of the following functions.

$y = \pm x^2$ $\quad$ $y = \pm 2x^2$ $\quad$ $y = \pm 0.5x^2$

Compare the graphs. Make as many observations as you can.

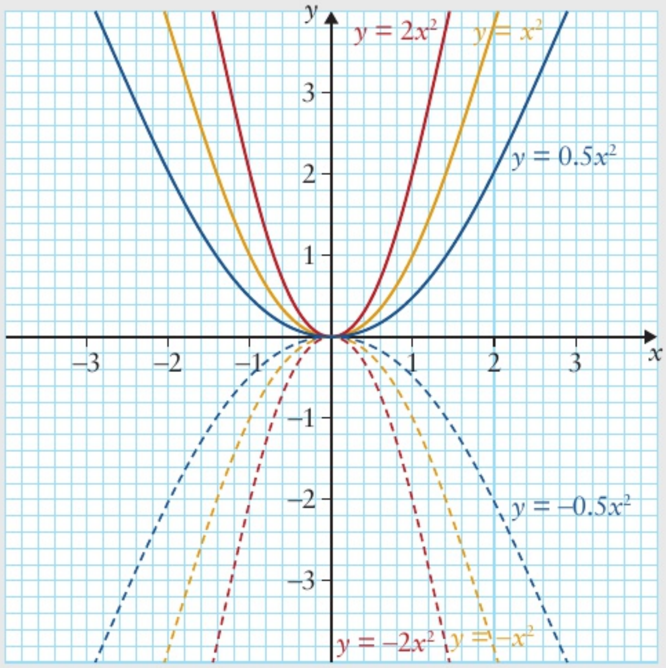
Can you generalise your observations?

Fact

The symbol ± reads as 'plus or minus'. The notation $y = \pm x^2$ is shorthand for the functions $y = x^2$ or $y = -x^2$

Solution

We need to compare the graphs of six different functions. We can start by using technology to graph the functions.



We start with the basic quadratic model $y = x^2$ and compare how the different coefficients affect the graph.

When the coefficient is negative the graphs are the reflections of their positive counterpart over the x-axis. For example, $y = -2x^2$ is the graph $y = 2x^2$ reflected in the x-axis.

The graphs become narrower as the absolute value of the coefficient increases. For example, as the coefficient increases from 0.5 to 1 to 2, the graphs get closer to the y-axis.

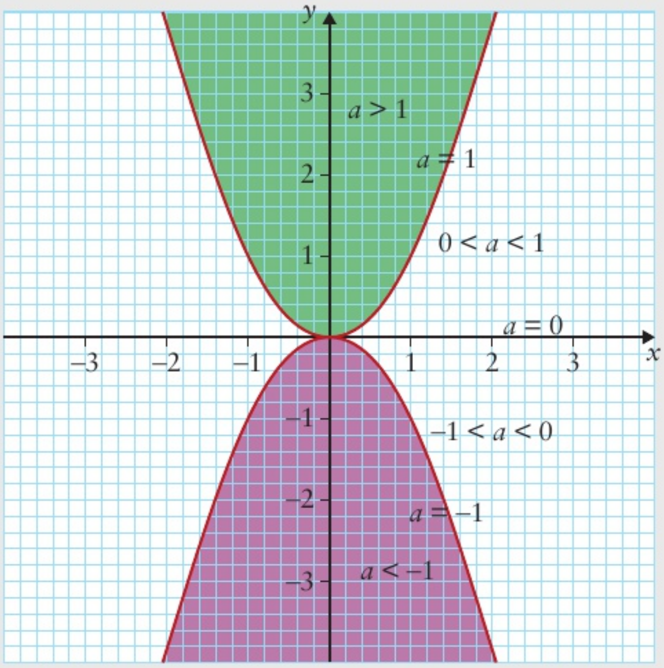
Fact

When parabolas **open upwards** (when $a > 0$) they are called **concave up**.

When parabolas **open downwards** (when $a < 0$) they are called **concave down**.

In both cases, the **vertex** of the parabola is the point $(0, 0)$.

We can also generalise our observations for $y = ax^2$ as in the graph shown.



Hint Q1

When plotting the parabolas, choose a y-scale from -25 to $+25$

Connections

How does the choice of scale affect the readability of a graph with many curves?

Practice questions 7.1

1 For each parabola, copy and complete a table similar to the one below. Plot all the parabolas on the same set of axes.

x	−3	−2	−1	0	1	2	3
y							

a $y = -3.5x^2$ b $y = -1.2x^2$ c $y = 3x^2$

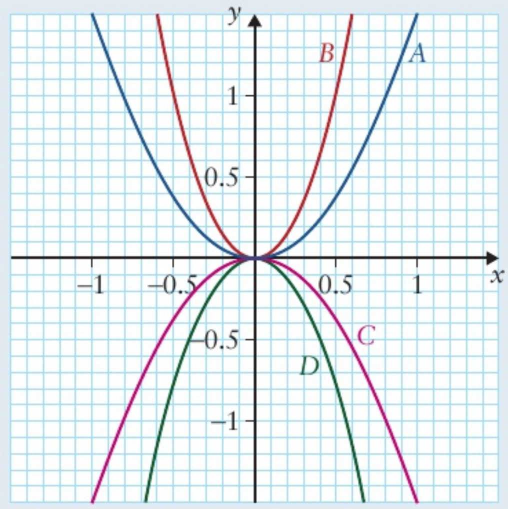
d $y = 2.5x^2$ e $y = -5x^2$

2 State whether each parabola is concave up or down, then plot each graph to confirm your statement.

a $y = 5x^2$ b $y = 1.4x^2$ c $y = -0.4x^2$

d $y = -\frac{1}{3}x^2$ e $y = \frac{x^2}{5}$ f $y = -\frac{x^2}{4}$

3 Identify the equation that matches each graph.



a $y = 4x^2$ b $y = -1.5x^2$ c $y = -3x^2$

d $y = -2x^2$ e $y = 1.5x^2$

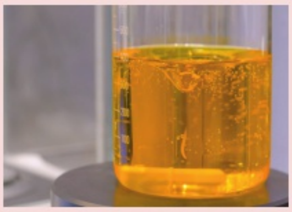
Challenge Q3

7.2 Intersections of parabolas and straight lines

In this section we look at how to find the coordinates of the point (or points) where two graphs intersect. A point belongs to the graph of a function when the coordinates of the point satisfy the equation of the function.

An immediate consequence of this statement is that a point belongs to the graphs of two functions when its coordinates satisfy the equations of both curves.

Fact



When a container with a liquid inside is made to rotate, the surface of the liquid in it can be described with parabolas.

Explore 7.3

It is recommended that you use a GDC or other suitable software for this exploration.

a Consider the graphs of the functions $f(x) = x^2$ and $g(x) = 1$
Draw the graphs of $f(x)$ and $g(x)$ on the same set of axes, then find the coordinates of the intersection points.
Can you obtain the same results through a calculation? How?

b Now consider the graphs of the functions $f(x) = x^2$ and $h(x) = x + 1$
Draw the graphs of $f(x)$ and $h(x)$ on the same set of axes, then find the coordinates of the intersection points.
Can you obtain the same results through a calculation?

Connections

Explore 7.3 considers the important link between a **geometry** fact (one point belongs to two graphs, that is, the two graphs intersect) and an **algebra** fact (the coordinates of a point satisfy two equations simultaneously, that is, they are a solution to a system of equations).

Reflect

How many intersections can a parabola and a straight line have? Consider all relative positions of the two graphs.

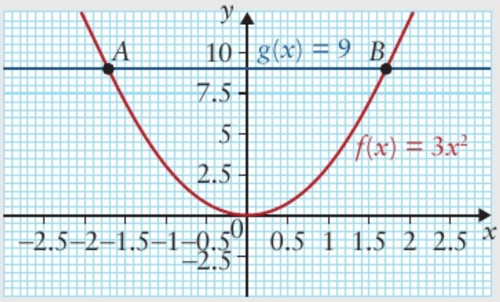
Worked example 7.2

Find the intersection points of the graphs of $f(x) = 3x^2$ and $g(x) = 9$ both graphically and with a calculation.

Solution

To address the first question we need to find both the x- and the y-coordinates of all points where the graphs of the two given functions intersect.

We can use a GDC or other suitable software to plot the graphs $f(x) = 3x^2$ and $g(x) = 9$, and then locate the intersection points.



We can see that there are two distinct intersection points, A and B. From the graph, we can estimate their coordinates as $A(-1.7, 9)$ and $B(1.7, 9)$.

The values obtained from the graph are estimates. Any graphical method, even with the help of technology, will only produce approximate results.

To obtain the same results with a calculation we need to find an algebraic way to represent the geometric idea of intersection.

We set up an equation that says 'for the same unknown value of x, the y-coordinate of the point on the graph of f is equal to the y-coordinate of the point on the graph of g'. Of course this can happen for more than one value of x, or for no values of x at all.

In other words:

$f(x) = g(x)$

We can also picture this as scanning the x-y plane with a vertical line until we find one or more points where the vertical distance $f(x) - g(x)$ between the two graphs is zero: the equations of the vertical lines where $f(x) - g(x) = 0$ are the solutions to $f(x) = g(x)$

Either way, setting $f(x) = g(x)$ gives the equation $3x^2 = 9$. This equation simplifies to $x^2 = 3$, with solutions $x = \pm\sqrt{3} \approx \pm 1.73$

The answers obtained algebraically confirm our estimates from the graphical method.

Fact

The notation $x = \pm\sqrt{3}$ is shorthand for 'either $x = \sqrt{3}$ or $x = -\sqrt{3}$'. We read it as 'x equals plus or minus the square root of three'.

Worked example 7.3

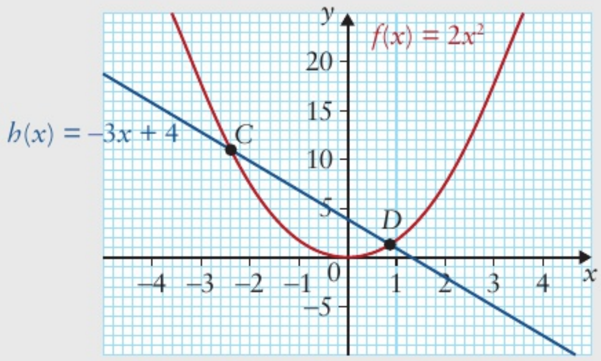
Find the intersection points of the graphs of $f(x) = 2x^2$ and $h(x) = -3x + 4$ to one decimal place.

Confirm your estimate with a calculation.

Solution

Graphing both functions gives two distinct points, C and D, where the graphs intersect.

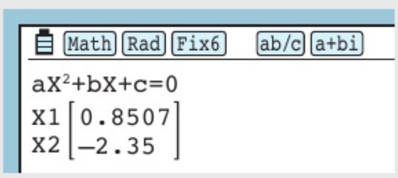
We can estimate their coordinates as $C(-2.4, 11.1)$ and $D(0.9, 1.4)$



To confirm this result with a calculation, let $f(x) = h(x)$ and set up the equation $2x^2 = -3x + 4$.

This quadratic equation cannot be solved by taking the square root of both sides, so we cannot yet confirm our estimate algebraically.

We can, however, use a GDC or other software to solve the equation. Here is an example.



Note that we wrote the equation in the form $2x^2 + 3x - 4 = 0$

Fact

A quadratic equation contains non-negative integer powers of the unknown variable up to the second order.

We will discuss how to solve a general quadratic equation in the next sections.

Practice questions 7.2

Fact

Curve is a collective term, which includes straight lines.

1 Find all intersections between each pair of curves.

a $y = x^2$ and $y = 4$

b $y = 3x^2$ and $y = 5$

c $y = -2x^2$ and $y = 3$

d $y = -2x^2$ and $x = -1$

2 Draw graphs of each pair of curves on the same set of axes. Estimate the coordinates of all intersections between them.

a $y = x^2$ and $y = 2x + 1$

b $y = -x^2$ and $y = x - 1$

c $y = -x^2$ and $y = x + \frac{1}{4}$

d $y = 2x^2$ and $y = -x - 1$

e $y = -x^2$ and $y = 2x - 2$

f $y = x^2$ and $y = 4x - 4$

3 Draw graphs of each pair of curves using software or a GDC. Classify the number of intersections as either two distinct points, two coinciding points, or no points.

a $y = 2x^2$ and $y = x + 1$

b $y = x^2$ and $y = 2x + 1$

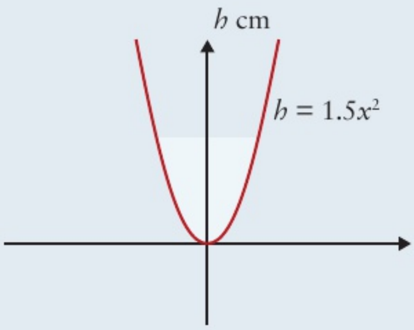
c $y = -x^2$ and $y = 2x + 1$

d $y = -x^2$ and $y = x + 1$

e $y = 3x^2$ and $y = -2x - \frac{1}{3}$

f $y = -3x^2$ and $y = 0$

4 Drinking glasses can be shaped as parabolas, as shown below.



If the equation of the parabola is $h = 1.5x^2$ and the water in the glass has a depth of 4 cm, find the radius of the circular surface of the water.

7.3 The graph of a quadratic function

You have already seen how the graph of the basic parabola $f(x) = x^2$ can be vertically stretched by transforming the function to $y = ax^2$

We can now extend this using further function transformations.

We can **shift** the graph of $y = ax^2$ both vertically and horizontally by transforming the function ax^2.

Explore 7.4

Can you predict what the graph of the three parabolas $y = x^2 + 1$, $y = x^2 - 2$ and $y = 2x^2 + 4$ will look like?

How could you find the coordinates of the vertices of the three parabolas?

Consider the function $y = ax^2 + k$, where k can be any real number, positive or negative. Can you predict what the graph of $y = ax^2 + k$ looks like? Make sure that you consider both $k > 0$ and $k < 0$

Can you predict the coordinates of the vertex of the parabola $y = ax^2 + k$?

Worked example 7.4

Find the coordinates of the vertex for the parabola with equation:

a $y = x^2 + 3$ **b** $y = x^2 - 1$ **c** $y = 3x^2 - 1$

Solution

a Starting from $f(x) = x^2$, we can see that $y = x^2 + 3 = f(x) - 3$

This is an example of a function transformation that produces a **vertical shift upwards** by three units. The parabola will have its vertex shifted from (0, 0) to (0, 3).

b Starting from $f(x) = x^2$, we can see that $y = x^2 - 1 = f(x) - 1$

This is an example of a **vertical shift downwards** by one unit, so the parabola will have its vertex shifted from (0, 0) to $(0, -1)$.

c Starting from $f(x) = x^2$, we can see that $y = 3x^2 - 1 = 3f(x) - 1$

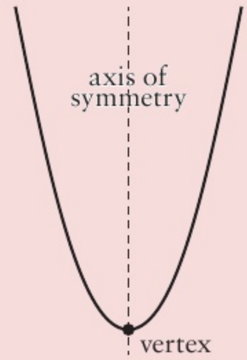
This is an example of a **vertical stretch** by a factor of three, followed by a **vertical shift downwards** by one unit. The vertical stretch does not affect the coordinates of the vertex, so the parabola will have its vertex shifted from (0, 0) to $(0, -1)$.

Connections

Parabolic trough collectors are used in solar power plants to focus light rays from the sun.

Fact

The vertex of a parabola is the point where the axis of symmetry crosses the parabola.



Reminder

The order in which transformations are performed is important. Would you get the same final result if the graph of $f(x) = x^2$ was first shifted, then stretched?

Given a composite function transformation for example, $f(x) \to 3f(x) + 1$, the order in which the individual transformations are performed is given by the standard order of operations: first multiplication then addition, therefore first vertical stretch then vertical shift.

Investigation 7.1

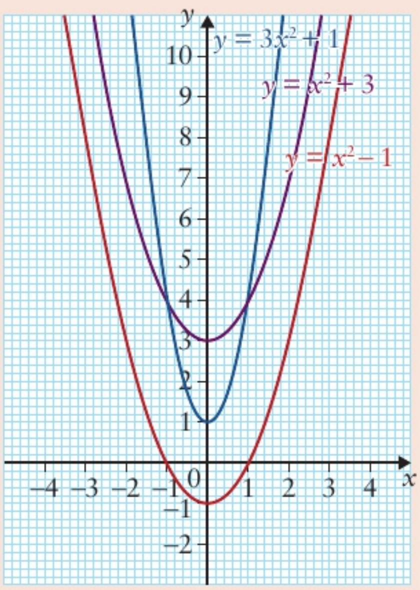
Research skills

Online graphing software that allows for the use of sliders to vary the values of parameters (variables in the equation other than x and y) is helpful for seeing the effects of transformations on graphs.

Check with your teacher or librarian which resources you can use for this task.

Find out how to set up a graph and sliders that will allow you to change the values of a and k in the function $ax^2 + k$ and immediately see the changes in the graph of $ax^2 + k$.

Use your sliders to graph the functions from Worked example 7.4.
The graphs you should expect to see are shown below.



Use your graph to confirm the results from Worked example 7.4.

Hint

Two widely used free resources are Desmos and GeoGebra.

The graph of $y = ax^2 + k$ is obtained from the graph of $f(x) = x^2$ by a vertical stretch of factor a followed by a vertical shift of k units.
The vertex has coordinates $(0, k)$.

Reminder

Be careful with negative values of k. What does a vertical shift of −3 units mean?

Reflect

Communication skills

This section gives an example of the link between the language of **geometry** (graphs, vertex, stretch, shift, vertical) and the language of **algebra** (equations, multiplication, addition).

Can you establish a dictionary that links these two languages? How many words do you know in each of them? Can you translate words and sentences from one language to the other?

Worked example 7.5

Find the coordinates of the vertex of the parabola with equation:

a $y = (x - 3)^2$ **b** $y = (x + 1)^2$

Solution

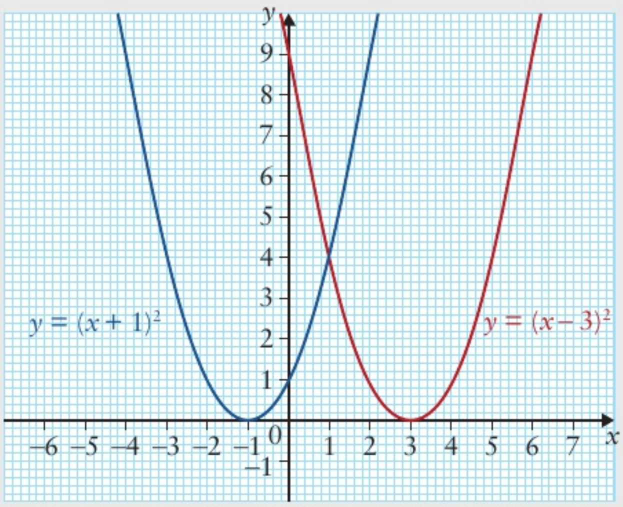
a Starting from $f(x) = x^2$, we can see that $y = (x - 3)^2 = f(x - 3)$

This is an example of a function transformation that produces a **horizontal shift to the right** by three units, so the parabola will have its vertex shifted from (0, 0) to (3, 0).

b Starting from $f(x) = x^2$, we can see that $y = (x + 1)^2 = f(x + 1)$

This is an example of a function transformation that produces a **horizontal shift to the left** by one unit, so the parabola will have its vertex shifted from (0, 0) to (−1, 0).

The graphs of both $y = (x - 3)^2$ and $y = (x + 1)^2$ are shown below.



The graph of $y = (x - h)^2$ is obtained from the graph of $f(x) = x^2$ by a horizontal shift of h units. The vertex has coordinates $(h, 0)$.

Hint

When $h > 0$, the shift is to the right, that is, $y = (x - 5)^2$ is shifted by 5 units to the right. When $h < 0$, the shift is to the left, that is, $y = (x + 5)^2 = (x - (-5))^2$ is shifted 5 units to the left.

Explore 7.5

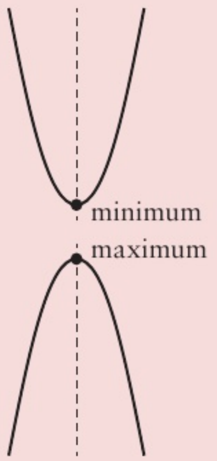
What information is needed to accurately describe the graph of $y = 2(x - 1)^2 + 3$?

Mention as many observations as possible.

Confirm your observations by graphing the function $y = 2(x - 1)^2 + 3$ with the use of technology.

Fact

A **concave up** parabola has its **minimum** value at the vertex. A **concave down** parabola has its **maximum** value at the vertex.



A parabola with equation $y = a(x - h)^2 + k$ in **vertex form** has vertex $V(h, k)$ and axis of symmetry $x = h$

Practice questions 7.3

1 Write the equation of the parabola obtained from $y = x^2$ by:

a shifting it 2 units to the left

b shifting it 1 unit up

c shifting it 3 units to the right

d shifting it 4 units down

e shifting it −1 unit to the right

f shifting it −2 units up

g stretching it vertically by a factor of 2, then shifting it up by 3 units

h stretching it vertically by a factor of 2, then shifting it 2 units to the right

i compressing it vertically by a factor of 2, then shifting it up by 3 units

j stretching it vertically by a factor of 2, then shifting it to the left by 1 unit

k turning it upside down, then stretching it vertically by a factor of 2, then shifting it to the left by 1 unit

l stretching it vertically by a factor of 3, then shifting it to the right by 2 units, then shifting it down by 1 unit.

2 Describe which transformations of the graph of $y = x^2$ correspond to each equation.

a $y = -x^2$ b $y = 2x^2$ c $y = \frac{1}{3}x^2$

d $y = -0.5x^2$ e $y = (x - 1)^2$ f $y = -x^2 + 3$

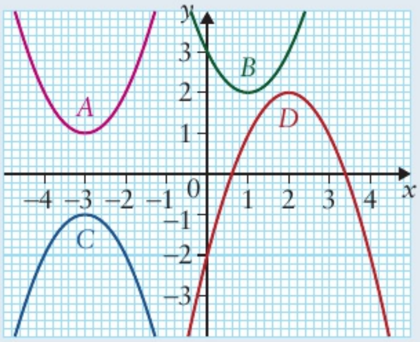
g $y = x^2 - 3$ h $y = (x + 2)^2$ i $y = (x - 2)^2 + 3$

j $y = (x + 1)^2 - 2$ k $y = 2(x - 2)^2 - 2$ l $y = -(x - 2)^2 + 4$

m $y = -\frac{1}{3}(x + 2)^2 + 3$

3 Work out the coordinates of the vertex for each parabola in question 2.

4 Match each graph with its equation.



a $y = -(x + 3)^2 - 1$

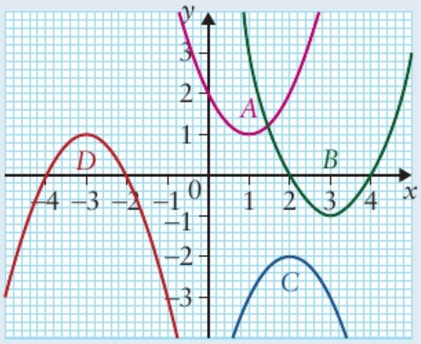
b $y = (x - 1)^2 + 2$

c $y = -(x - 3)^2 - 2$

d $y = (x + 3)^2 + 1$

e $y = -(x - 2)^2 + 2$

5 Write the equation of each parabola in vertex form.



7.4 Quadratic equations

A **quadratic equation** in **standard form** is an equation written as $ax^2 + bx + c = 0$, where x is the unknown and a, b, c are **coefficients** that can take any real value. The only exception is that a cannot be equal to zero.

If you separate the quadratic part ax^2 from the linear part $bx + c$, you can understand (as you saw in Section 7.2) why we need to solve a quadratic equation when we find intersections between a parabola and a straight line. Quadratic equations often arise in many other contexts, including geometry, economics and physics.

A quadratic equation can have **two distinct** solutions, one solution (or, more accurately, **two coinciding** solutions) or **no** solutions. In the next sections we will see three methods to find solutions, when they exist.

Connections

Satellite dishes used in telecommunications have a parabolic shape.

Fact

a is called the **quadratic** coefficient, because it multiplies x^2.
b is called the **linear** coefficient, because it multiplies x.
c is called the **constant** term.

7.4.1 Solution by factorising

Reminder

Expanding means removing the brackets. That is, changing a product into a sum, such as $3(x + 2) = 3x + 6$
Factorising (or factoring) is the opposite of expanding. It means introducing brackets. That is, changing a sum into a product, such as $a^2 + ab = a(a + b)$

Explore 7.6

Expand and simplify the product $(x + 1)(x + 3)$. What do you notice about the linear coefficient in the resulting expression? What do you notice about the constant term?

Can you solve the equation $x^2 + 4x + 3 = 0$?

The link between solving a quadratic equation and factorising a quadratic expression is given by the **null factor law**, which says that if **the product of two factors equals zero then at least one of the factors must equal zero**.

In symbols, if $A \times B = 0$ then either $A = 0$ or $B = 0$ (or both).

Reflect

How might the null factor law help in solving a quadratic equation?
Can you solve $x^2 - 5x + 6 = 0$?

Worked example 7.6

Solve the following quadratic equations.

a $x^2 - x - 6 = 0$ **b** $2x^2 + 3x = 0$ **c** $3x^2 + 7x + 2 = 0$

Solution

a **Understand the problem**

We need to find the numbers that, when replaced for x, make the expression $x^2 - x - 6$ equal to zero.

Make a plan

If we can factorise the expression $x^2 - x - 6$, we will have a product of two factors that multiply to give zero. We can then use the null factor law and set each of the two expressions in the factors equal to zero. This way, we will find two solutions to the original quadratic equation.

Carry out the plan

Factorising $x^2 - x - 6 = 0$ gives $(x + 2)(x - 3) = 0$

Therefore:

- either $x + 2 = 0 \Rightarrow x = -2$
- or $x - 3 = 0 \Rightarrow x = 3$

Look back

We can check the we have found the correct solutions by evaluating $x^2 - x - 6$ for $x = -2$ and $x = 3$

$(-2)^2 - (-2) - 6 = 4 + 2 - 6 = 0$ and $(3)^2 - (3) - 6 = 9 - 3 - 6 = 0$

Both expressions evaluate to 0, so both solutions are correct.

b This quadratic expression is missing the constant term, so we can factorise it by pulling out the common factor, x.

$2x^2 + 3x = 0$

$x(2x + 3) = 0$

Therefore either $x = 0$ or $2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$

c This quadratic expression is harder to factorise because $3x^2 + 7x + 2$ has a quadratic coefficient $\neq 1$

$3x^2 + 7x + 2 = 0$

$(3x + 1)(x + 2) = 0$

So either $3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$

or $x + 2 = 0 \Rightarrow x = -2$

When the two linear factors are the same, the equation has two **coinciding** solutions.

Explore 7.7

Can you solve $x^2 - 2x + 1 = 0$ by factorising?

Can all quadratic equations be solved by factorising?

Practice questions 7.4.1

1 Solve these quadratic equations.

a $(x - 1)(x - 2) = 0$

b $x(x + 3) = 0$

c $(1 - x)(-2 - x) = 0$

d $\left(x + \frac{1}{2}\right)\left(x + \frac{1}{2}\right) = 0$

e $(x - 4)^2 = 0$

f $2x^2 = 0$

g $2(x - 1)(x + 1) = 0$

h $(2x - 1)(x + 1) = 0$

i $-3\left(3x - \frac{1}{2}\right)\left(2 - \frac{3}{5}x\right) = 0$

2 Solve these quadratic equations by factorising.

a $3x^2 - x = 0$ b $x^2 - 4 = 0$ c $x^2 + 5x + 6 = 0$

d $x^2 - 5x + 6 = 0$ e $x^2 + 7x + 10 = 0$ f $x^2 - 10x + 21 = 0$

g $x^2 - 12x + 35 = 0$ h $x^2 - x - 56 = 0$ i $3x^2 + 17x + 10 = 0$

3 Write, in the form $ax^2 + bx + c = 0$, a quadratic equation with solutions:

a 2 and 3 b 0 and 1 c −2 and 1

d −2 and −3 e −2 and −2 f 1 and 1

4 Solve these quadratic equations by factorising.

a $2x^2 - x - 1 = 0$ b $2x^2 + x - 1 = 0$

c $6x^2 - x - 1 = 0$ d $5x^2 - 17x + 6 = 0$

e $2x^2 - 11x + 5 = 0$ f $4x^2 - 3x - 7 = 0$

g $3x^2 + 17x + 10 = 0$ h $3x^2 - 13x + 14 = 0$

5 Find the intersections between each pair of curves.

a $y = x^2$ and $y = 3x$ b $y = x^2$ and $y = 5x - 4$

c $y = -x^2$ and $y = -2x - 15$ d $y = -x^2$ and $y = 2 - 3x$

e $y = 4x^2$ and $y = -21x - 5$ f $y = 4x^2$ and $y = 19x + 5$

g $y = 5x^2$ and $y = -16x - 3$ h $y = 2x^2$ and $y = x + 10$

Connections

Parabolas make a striking design feature and are popular in architecture.

7.4.2 Solving quadratics by completing the square and the quadratic formula

Explore 7.8

Can you show how knowing the solution to $x^2 = 9$ can help you to work out the solution to $(x + 1)^2 = 9$?

Worked example 7.7

Use the solution to $x^2 = 10$ to solve $(x + 1)^2 = 10$ without expanding the left-hand side.

Solution

By taking the square root of both sides of $x^2 = 10$, we have $x = \pm\sqrt{10}$

The question tells us that we should not expand the brackets $(x + 1)^2$, so we must look for an alternative method.

The equation $(x + 1)^2 = 10$ is similar to the equation $x^2 = 10$. The only difference is that here we have $(x + 1)^2$ instead of x^2. Therefore, we can follow the same steps and take the square root of both sides.

$(x + 1)^2 = 10$

$x + 1 = \pm\sqrt{10}$

This gives us two solutions. One comes from choosing the plus sign in $\pm\sqrt{10}$ and the other comes from choosing the minus sign.

$x + 1 = \sqrt{10} \Rightarrow x = -1 + \sqrt{10}$

$x + 1 = -\sqrt{10} \Rightarrow x = -1 - \sqrt{10}$

Fact

More formally, taking the square root of both sides gives $|x| = \sqrt{10}$, which in turn gives $x = \pm\sqrt{10}$. The notation $|x|$ reads **the absolute value** of x, and it means the distance of x from zero. This intermediate step is often skipped, but be aware that $\sqrt{x^2} \neq x$

You can check this by graphing $y = \sqrt{x^2}$ and $y = x$ on your GDC: For what values of x do the two graphs overlap? Can you figure out why?

If we can write a quadratic equation in the form $(x + a)^2 = b$, we can solve it using the method in Worked example 7.7. There are some limitations to this. The right-hand side must be positive, otherwise we cannot take its square root and the equation has no solutions.

Reflect

Try solving $(x - 3)^2 = 4$ in two different ways.
Which method do you prefer?

If we expand and simplify the equation $(x + 1)^2 = 10$, we obtain $x^2 + 2x - 9 = 0$

If we were asked to solve the equation $x^2 + 2x - 9 = 0$, we could try to factorise the left-hand side, but we would need to find two numbers whose product is -9 and whose sum is 2. If we could write it in the form $(x + 1)^2 = 10$ then we could use the method in Worked example 7.7. We will now look at how to write a quadratic expression in this form.

Explore 7.9

What must be added to $x^2 + 2x$ to make an expression that is the **square of a binomial**? That is, an expression in the form $(x + a)^2$?

Can you use the result to solve $x^2 + 2x = 3$?

Hint Qa

This process is called **completing the square.**

Fact

To complete the square for a difference, we use the same approach but with $(x-a)^2 = x^2 - 2ax + a^2$

For example, completing the square for $x^2 - 8x$, gives us $x^2 - 8x + 16 = (x-4)^2$

Worked example 7.8

a What number must be added to the expression $x^2 + 6x$ to make the new expression the square of a binomial?

b Use the method of completing the square to solve the quadratic equation $x^2 + 6x = 5$

Solution

a We need to add a number to $x^2 + 6x$ so that the new expression we obtain can be written in the form $(x+a)^2$

We know that $(x+a)^2 = x^2 + 2ax + a^2$, so we can compare

$x^2 + 6x + \dots$ and $x^2 + 2ax + a^2$

These two expressions are very similar. We can work out the value of a that makes them identical by comparing the linear terms $6x$ and $2ax$.

Since $2a$ must be equal to 6, this gives us $a = \frac{6}{2} = 3$. So the number we are looking for is $a^2 = 9$ and the expression is $x^2 + 6x + 9 = (x+3)^2$

b To answer this question we can use our result from part a. The left-hand side of $x^2 + 6x = 5$ is a perfect square if we add 9 to it.

We can add the same number to both sides and obtain an equivalent equation that has the same solutions as the first.

$x^2 + 6x = 5$

$x^2 + 6x + 9 = 5 + 9$

$(x+3)^2 = 14$

We can now solve this by taking the square root of both sides:

$x + 3 = \pm\sqrt{14}$

$x = -3 \pm \sqrt{14}$

We can generalise our findings as follows:

To complete the square for the expression $x^2 + bx$ we need to add the square of half the linear coefficient, $\left(\frac{b}{2}\right)^2$. That is, $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

Worked example 7.9

Solve $2x^2 + 6x = 5$ by the method of completing the square.

Solution

In this case the quadratic coefficient is not equal to 1. We can make it equal to 1 by dividing both sides by two.

$$2x^2 + 6x = 5$$

$$x^2 + 3x = \frac{5}{2}$$

Then we can complete the square.

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \frac{5}{2} + \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{5}{2} + \frac{9}{4}$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{19}{4}$$

$$x = -\frac{3}{2} \pm \sqrt{\frac{19}{4}} = \frac{-3 \pm \sqrt{19}}{2}$$

Investigation 7.2

Consider the equation $ax^2 + bx + c = 0$, where $a \neq 0$

1 Show how it can be transformed to:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

2 Show how, by completing the square, you can obtain

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2 \times a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2 \times a}\right)^2$$

3 Show that:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

4 Show that:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5 Apply the result in part 4 to solve the following equations.

a $x^2 + 3x - 10 = 0$

b $2x^2 + x - 1 = 0$

Fact

The solution to the equation $ax^2 + bx + c = 0$, $a \neq 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is called the **quadratic formula**. The expression under the square root, $b^2 - 4ac$, is called the **discriminant** of the equation $ax^2 + bx + c = 0$ (see Section 7.4.4).

Reminder

It is very important that you keep track of signs in expressions like this. You should use brackets whenever two operation signs are close to each other, such as $-(-1)$ or $2 \times (-5)$

Connections

How many intersections do you expect between $y = 2x^2$ and $y = x + 5$? Why?
Check with a GDC.

Fact

When we need to distinguish between the two solutions obtained with the quadratic formula, we can label them explicitly as

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Connections

How many intersections do you expect between $y = x^2$ and $y = 2\sqrt{2}x - 2$? Why?
Check with a GDC.

Connections

How many intersections do you expect between $y = x^2$ and $y = -x - 1$? Why?
Check with a GDC.

Worked example 7.10

Apply the quadratic formula to solve these equations:

a $2x^2 - x - 5 = 0$

b $x^2 - 2\sqrt{2}x + 2 = 0$

c $x^2 + x + 1 = 0$

Solution

a The quadratic formula for the equation $2x^2 - x - 5 = 0$ requires correct identification of a, b, c. We have $a = 2$, $b = -1$ and $c = -5$

Putting these values into the quadratic formula gives:

$$x_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-5)}}{2a}$$

The **two** distinct solutions are:

$$x_1 = \frac{1 + \sqrt{1 + 40}}{4} = \frac{1 + \sqrt{41}}{4} \text{ and } x_2 = \frac{1 - \sqrt{41}}{4}$$

b Repeating the same process for the equation $x^2 - 2\sqrt{2}x + 2 = 0$, we identify $a = 1$, $b = -2\sqrt{2}$ and $c = 2$

Replacing these values gives:

$$x_{1,2} = \frac{-(-2\sqrt{2}) \pm \sqrt{(-2\sqrt{2})^2 - 4(1)(2)}}{2}$$

Simplifying gives $x_{1,2} = \dfrac{2\sqrt{2} \pm \sqrt{(8-8)}}{2} = \dfrac{2\sqrt{2} \pm 0}{2}$

Here, $x_1 = x_2 = \sqrt{2}$

There are two coinciding solutions, so only one distinct solution.

c For the equation $x^2 + x + 1 = 0$, $a + b = c = 1$

Replacing these values in the quadratic formula gives:

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

Here we encounter the problem that we cannot take the square root of a negative number. We cannot evaluate $\sqrt{-3}$ using only real numbers.

So, the equation has **no real** solutions.

Practice questions 7.4.2

1 Complete the squares in these expressions.

a $x^2 + 6x + \square = (x + \square)^2$ b $x^2 + 8x + \square = (x + \square)^2$

c $x^2 - 2x + \square = (x - \square)^2$ d $x^2 - x + \frac{\square}{\square} = \left(x - \frac{\square}{\square}\right)^2$

e $x^2 + 3x + \frac{\square}{\square} = \left(x + \frac{\square}{\square}\right)^2$ f $2x^2 - 8x + \frac{\square}{\square} = 2(x - \square)^2$

2 Solve the following quadratic equations, giving your answer in exact form.

a $(x - 3)^2 = 5$ b $(x + 3)^2 = 8$ c $\left(x - \frac{1}{2}\right)^2 = \frac{9}{4}$

d $(x - 1)^2 = \frac{5}{9}$ e $\left(x + \frac{3}{2}\right)^2 = 3$

Hint Q2

'In exact form' means that you should write your answer as a surd, or as an expression involving surds, instead of as a decimal number.

3 State the number of distinct solutions to each equation.

a $(x + 1)^2 = 3$ b $(x - 1)^2 = 4$

c $(x + 3)^2 = 0$ d $\left(x - \frac{1}{2}\right)^2 = 0$

e $(x + 1)^2 = -2$ f $\left(x - \frac{2}{3}\right)^2 = -\frac{2}{3}$

4 Solve the following equations by completing the square. Give your answers to three significant figures.

a $x^2 - 2x = 4$ b $x^2 + 12x = 1$ c $x^2 - 12x = 0$

d $x^2 - 2x - 1 = 0$ e $x^2 + 4x - 8 = 0$ f $x^2 + 4x + 8 = 0$

g $x^2 - 5x = 3$ h $x^2 - 7x - 1 = 0$ i $2x^2 - 4x - 2 = 0$

j $-2x^2 + 3x - 1 = 0$

5 Solve these equations using the quadratic formula.

a $x^2 - 2x - 15 = 0$ b $x^2 - 6x + 8 = 0$

c $3x^2 - 7x + 2 = 0$ d $2x^2 - 11x + 5 = 0$

6 Solve these equations using the quadratic formula. Give your answers in exact form.

a $3x^2 + 8x + 4 = 0$ b $x^2 - 2x - 2 = 0$

c $x^2 - x + 2 = 0$ d $2x^2 - 6x + 1 = 0$

Hint Q7

This question is an extension to finding intersections between a parabola and a straight line. All GDCs can find intersections between curves, so you are encouraged to use software or a GDC as an alternative to algebraic methods.

7 Find the intersections between each pair of parabolas.

a $y = x^2 + 2x - 1$ and $y = -3x^2 - 5x + 2$

b $y = x^2 - 5x + 3$ and $y = -2x^2 + 7x - 4$

c $y = x^2 + 12x + 23$ and $y = x^2 + 2x - 2$

d $y = 2x^2 - x + 2$ and $y = x^2 - 5x - 2$

e $y = x^2 + 2$ and $y = -x^2 + x - 2$

7.4.3 The graph of a quadratic in standard form

Explore 7.10

What does the graph of the parabola $y = ax^2 + bx + c$ look like?

Given the quadratic equation $x^2 - 4x + 3 = 0$ and the quadratic function $f(x) = (x - 2)^2 - 1$, list as many connections between the graph of the function and the equation as you can.

Worked example 7.11

Consider the parabola $y = x^2 + 2x - 8$

a Find the y-intercept of the parabola.

b Find the x-intercepts of the parabola.

c Use your answer to part b to find the equation of the axis of symmetry and the coordinates of the vertex.

d Sketch by hand the graph of $y = x^2 + 2x - 8$

Solution

a To find the y-intercept, we can set up a system of equations:

$y = x^2 + 2x - 8$
$x = 0$

where $x = 0$ is the equation of the y-axis.

Substituting $x = 0$ in the equation of the parabola, the y-intercept is $(0, -8)$.

b To find the x-intercepts, we can set up a system of equations:

$y = x^2 + 2x - 8$
$y = 0$

where $y = 0$ is the equation of the x-axis.

This system becomes the quadratic equation $x^2 + 2x - 8 = 0$

Using the quadratic formula gives us:

$$x_{1,2} = \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm 6}{2} = -1 \pm 3, \text{ so } x_1 = 2 \text{ and } x_2 = -4$$

c To find a mathematical relationship between the x-intercepts and the equation of the axis of symmetry, we can use a geometry fact to establish an algebraic relationship.

The axis of symmetry passes halfway between the x-intercepts, so it has equation $x = \dfrac{x_1 + x_2}{2}$

In this case, the equation of the axis of symmetry is $x = -1$

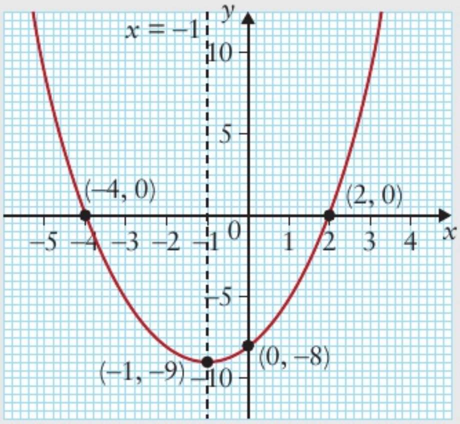
To find the coordinates of the vertex, remember that the vertex lies on the axis of symmetry, so the x-coordinate of the vertex is $x_V = -1$

To find y_V, the y-coordinate of the vertex, we use the fact that the vertex lies on the parabola, so its coordinates must satisfy the equation of the parabola.

$$\begin{aligned} y_V &= x_V{}^2 + 2(x_V) - 8 \\ &= (-1)^2 + 2(-1) - 8 \\ &= -9 \end{aligned}$$

The vertex is $V(-1, -9)$

d We can plot the x- and y-intercepts, the vertex and the axis of symmetry and use them to sketch the parabola.



Hint Qd

After drawing a sketch, you can confirm your results using software or a GDC.

Reflect

Look back at parts b and c in Worked example 7.11. How does $x_{1,2} = -1 \pm 3$ from part b tell us that the axis of symmetry has the equation $x = -1$?

We now have the following general results for the graph of $y = ax^2 + bx + c$ that we can use for similar problems.

- The parabola is concave up if $a > 0$, and concave down if $a < 0$
- The y-intercept is $(0, c)$.
- The x-intercepts are given by the solution of the quadratic equation: $ax^2 + bx + c = 0$
  We can find them using any of the methods we have learned for solving quadratic equations.
- One method for solving quadratic equations is the quadratic formula: $x_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$
- The x-coordinate of the vertex is $x_V = -\frac{b}{2a}$
- The y-coordinate of the vertex is obtained by substituting x_V in the equation of the parabola: $y_V = f(x_V)$

Practice questions 7.4.3

1 Where possible, estimate the x-intercepts, the y-intercept and the coordinates of the vertex for each parabola.

a $y = x^2 - 6x + 5$ b $y = 2x^2 + 8x - 10$

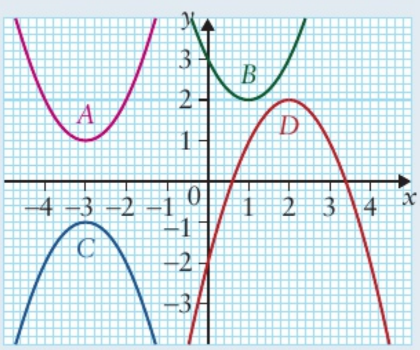
c $y = -x^2 - 6x - 9$ d $y = x^2 + 6x$

e $y = 2x^2 + 3x - 2$ f $y = 2x^2 - 4x + 2$

g $y = 2x^2 + x + 1$ h $y = 2x - x^2$

Challenge Q2

2 Calculate the x-intercepts, the y-intercept, the coordinates of the vertex, and the equation of the axis of symmetry for each graph.



3 Expand the equation of each parabola, then use the quadratic formula to find the x-intercepts. What do you observe?

a $y = (x + 1)(x - 4)$ **b** $y = (1 - x)(x + 3)$

c $y = (2x - 1)(x + 1)$ **d** $y = (1 - 3x)(2x - 1)$

e $y = 2(x + 1)(x - 4)$ **f** $y = 2(3x + 1)(2 - 3x)$

g $y = (x - 1)(x - 1)$ **h** $y = -2(x + 1)(x - 1)$

4 Calculate the x-intercepts, the y-intercept and the coordinates of the vertex for each parabola, then sketch its graph.

a $y = x^2 - 3$ **b** $y = x^2 + 8x + 12$

c $y = x^2 - 2x + 2$ **d** $y = x^2 - 8x + 16$

5 Calculate the maximum value of y in the graph of:

a $y = -x^2 - 6x - 2$ **b** $y = -4x^2 - 4x + 6$

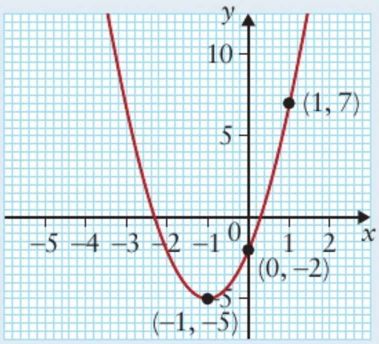
c $y = -9x^2 - 6x + 2$

6 Calculate the minimum value of y in the graph of:

a $y = x^2 + 6x + 2$ **b** $y = 1 - 2x + x^2$

c $y = 4x^2 + 16x + 27$

7 Consider the parabola $y = ax^2 + bx + c$ shown in the graph.



a Use the y-intercept to show that $c = -2$

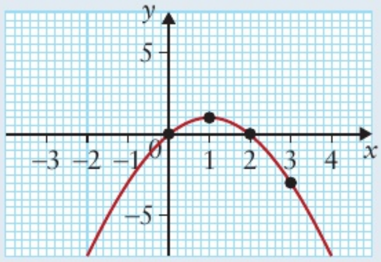
b Use the equation of the axis of symmetry to show that $b = 2a$

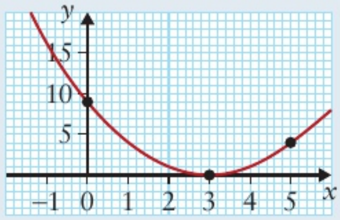
c Show that the equation of the parabola is $y = ax^2 + 2ax - 2$

d Use the coordinates of the vertex or the coordinates of the point $(1, 7)$ to find the value of a.

e Find the equation of the parabola.

8 Repeat the steps in question 7 to find the equation of each of the following parabolas.

a 

b 

Investigation 7.3

This investigation is an application of systems of equations and matrices to find the equation of a parabola.

1 Consider the parabola with equation $y = ax^2 + bx + c$

a The parabola passes through the point $A(1, 1)$. Show that $a + b + c = 1$

b The parabola passes through the point $B(-1, 1)$. Show that $a - b + c = 1$

c The parabola passes through the point $C(0, 3)$. Show that $c = 3$

d Solve the system of equations for the variables a, b, and c, and write the equation of the parabola. It is a good idea to use a GDC or other software to do this.

2 Find the equation of the parabola through the points $(1, 3)$, $(-1, 2)$ and $(2, 4)$.

3 Find the equation of the parabola through the points:

a $(2, 0)$, $(3, 0)$ and $(1, 1)$

b $(1, 2)$, $(-1, 2)$ and $(0, 4)$

c $(1, 2)$, $(1, 3)$ and $(0, 0)$

d $(1, 2)$, $(0, 0)$ and $(2, 4)$.

Connections

Matrices are a powerful tool to solve systems of linear equations in a variety of contexts.

7.4.4 The discriminant

Explore 7.11

Can you work out how many distinct solutions each of these equations has without solving them?

$x^2 + 2x = 0$

$x^2 + 2x + 1 = 0$

$x^2 + 2x + 2 = 0$

Consider the numbers x_1 and x_2 given by the quadratic formula.

When are they different? That is, when is $x_1 \neq x_2$?

When are they equal? That is, when is $x_1 = x_2$?

When is it impossible to evaluate them?

Hint

Consider how the coefficient c affects the graphs of a quadratic function.

Worked example 7.12

Using the quadratic formula in the form $x_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, solve these equations.

a $x^2 - 4x + 3 = 0$ **b** $x^2 - 4x + 4 = 0$ **c** $x^2 - 4x + 5 = 0$

Solution

a Applying the quadratic formula, we have $x_{1,2} = \frac{4}{2} \pm \frac{\sqrt{16 - 12}}{2} = 2 \pm \frac{\sqrt{4}}{2}$

The solutions x_1 and x_2 are different, because they are obtained from 2 by adding or subtracting the number $\frac{\sqrt{4}}{2}$, which is different from zero.

b Applying the quadratic formula, we have $x_{1,2} = \frac{4}{2} \pm \frac{\sqrt{16 - 16}}{2} = 2 \pm \frac{\sqrt{0}}{2}$

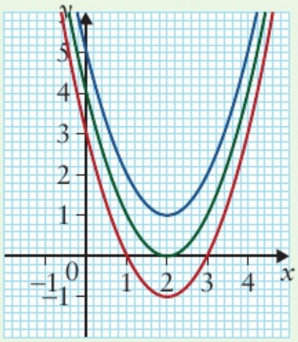
The solutions x_1 and x_2 are going to be equal, because they are obtained from 2 by adding or subtracting the number $\frac{\sqrt{0}}{2}$, which is equal to zero.

c Applying the quadratic formula, we have $x_{1,2} = \frac{4}{2} \pm \frac{\sqrt{16 - 20}}{2} = 2 \pm \frac{\sqrt{-4}}{2}$

The solutions x_1 and x_2 are going to be impossible to find, because there is no real number equal to the square root of a negative number.

Reflect

You can use a GDC or other suitable software to graph the three parabolas in Worked example 7.12.



Can you use the y-intercepts to identify the equation of each curve?

Which curve is tangent to the x-axis?

How does the number of solutions relate to the position of the curve relative to the x-axis?

The quantity that determines whether the quadratic equation $ax^2 + bx + c = 0$ has two, one or no solutions is the expression under the square root in the quadratic formula. It is called the **discriminant** of the equation, and it is denoted with the symbol Δ:

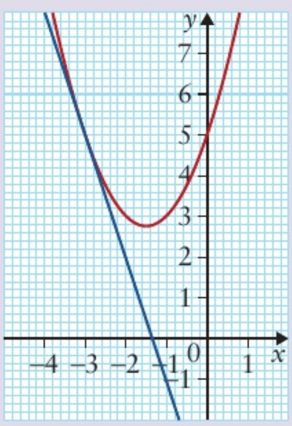
$$\Delta = b^2 - 4ac$$

- when $\Delta > 0$, the equation has **two real and distinct** solutions ($x_1 \neq x_2$)
- when $\Delta = 0$, the equation has **two real and coinciding** solutions ($x_1 = x_2$)
- when $\Delta < 0$, the equation has **no real solutions** (x_1 and x_2 are not real numbers)

Explore 7.12

The graph of $y = -3x - 4$ is tangent to the graph of $y = x^2 + 3x + 5$ at a single point.

Can you describe how this fact is related to the calculation of a discriminant?



Practice questions 7.4.4

1 Determine the number of solutions to each equation by evaluating the discriminant.

a $x^2 + 2x - 1 = 0$

b $2x^2 - x + 5 = 0$

c $x^2 + 2x + 1 = 0$

d $x^2 - 2x + 1 = 0$

e $3x^2 + 2x + 1 = 0$

f $3x^2 + 2x - 1 = 0$

2 Write in standard form:

a a quadratic equation with no real solutions

b a quadratic equation with two coinciding solutions

c a quadratic equation with two distinct solutions.

3 Determine the value of m for which the equation $x^2 + 2x + m = 0$ has two coinciding solutions.

4 Determine the values of m for which the equation $x^2 + mx + 1 = 0$ has two coinciding solutions.

5 Determine the value of m for which the straight line $y = x + m$ is tangent to the parabola $y = x^2$

Challenge Q5

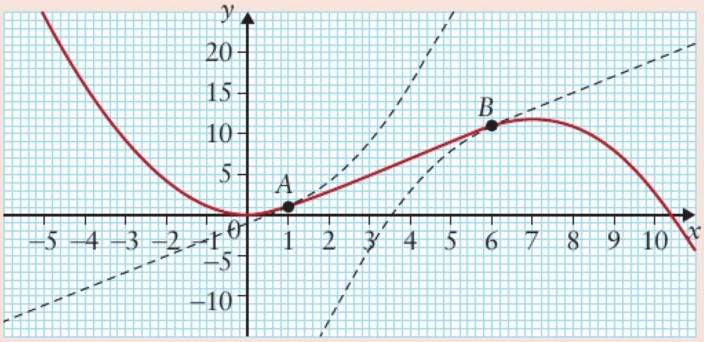
6 Determine the values of m for which the straight line $y = mx - 1$ is tangent to the parabola $y = x^2$

Challenge Q6

Investigation 7.4

When engineers design railways, they often have to join straight and curved portions of rail. In this case, they design them so that they are tangent to each other and join smoothly.

Part of a railway is shown below (top view).



1 The shape of the railway before point A is given by the equation $y = x^2$, and the x-coordinate of A is 1.

- **a** Find the y-coordinate of A.
- **b** Show that the equation of the straight line passing through A with gradient m is $y = mx + 1 - m$
- **c** Find the value of m such that the straight line $y = mx + 1 - m$ is tangent to $y = x^2$ at A.
- **d** Write down the equation of the railway portion AB.

2 The shape of the railway after point B is a parabola.

- **a** Using the graph, find the coordinates of the vertex of this parabola.
- **b** The parabola passes through the point (9, 8). Find the equation of the parabola in vertex form.
- **c** Show that the equation of the parabola in standard form is $y = -x^2 + 14x - 37$
- **d** Check that the straight and curved portions of the railway join smoothly at B.
- **e** Find the coordinates of B.
- **f** Find the length of the straight portion of railway.

3 a Extend these results to design another smooth curved-straight-curved portion of railway.

b Using a GDC or other suitable software, produce a graph that illustrates the three portions of railway.

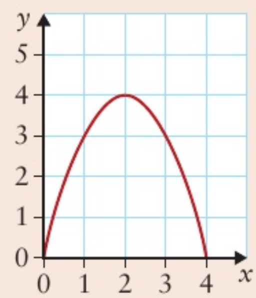
Self assessment

- I can recognise the important features of the graph of a parabola (vertex, x- and y-intercepts, axis of symmetry, concavity).
- I can find the intersections of a parabola and a straight line.
- I can determine when a parabola and straight line do not intersect.
- I can determine the number of intersections of a parabola and a straight line.
- I can use the correct terminology for the terms and coefficients in a quadratic expression.
- I can factorise a quadratic in standard form when $a = 1$ and when $a \neq 1$
- I know the null factor law.
- I can solve quadratic equations by factorising.
- I can solve quadratic equations by completing the square.
- I can solve quadratic equations with the quadratic formula.
- I can relate the graph of $y = x^2$ to the graph of $y = a(x - h)^2 + k$ through geometric transformations.
- I can relate the important features of the graph of a parabola to the coefficients of a quadratic function:
  - in vertex form
  - in standard form
  - in factorised form.
- I can sketch the graph of a parabola from a knowledge of its important features.
- I can calculate the discriminant of a quadratic expression and use it to determine the number of distinct real solutions.

Check your knowledge questions

1 A basketball is thrown from the ground at an angle to the horizontal as shown in the diagram.

The ball is thrown from the point (0, 0). The height, h m, of the ball when it is at a horizontal distance x m from the origin is given by $h(x) = ax^2 + bx + c$



a Using the graph, find:

i the maximum height reached by the ball

ii the horizontal distance from the origin at maximum height

iii the horizontal distance from the origin at landing.

Hint

These questions require you to apply the skills you have learned about quadratic functions and equations to practical contexts. Follow the four steps of problem solving:

- understand the problem
- make a plan
- carry out the plan
- look back.

b Find the equation of the parabola $h(x)$

c Find:

i the height of the ball when it is at a horizontal distance of 3.5 m from the origin

ii the values of the horizontal distance from the origin when the height is 3 m.

2 The length of a rectangle is 4 cm more than its height. The area of the rectangle is 28.49 cm^2. Find the length and height of the rectangle.

3 The cost, c dollars, of producing x litres of lemonade per day at a lemonade stand is given by the quadratic function $c(x) = 0.1x^2 - x + 20$

a Evaluate the cost of producing 8 litres of lemonade a day.

b If on one day you have 40 dollars to invest in lemonade, how much lemonade can you produce?

c How much does it cost to keep the stand open without producing any lemonade?

d How much lemonade should you produce each day to minimise production costs?

4 When a stone is dropped from a cliff, the remaining distance, d m, to the sea after t seconds is given by the quadratic function $d(t) = -5t^2 + 80$

a Evaluate the stone's distance from the sea after:

i 1 second

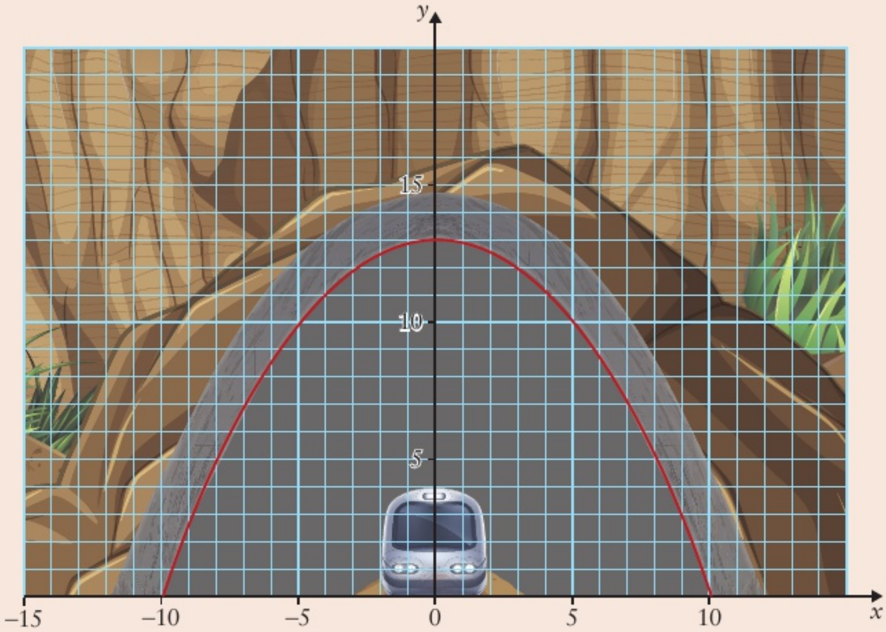
ii 2 seconds

iii 3 seconds

b What is the height of the cliff?

c How long does it take the stone to reach the sea?

5 Consider a parabolic arch used in the design of a tunnel, as shown in this illustration.



The width of the arch at ground level is 20 m, and the maximum height of the arch is 13 m.

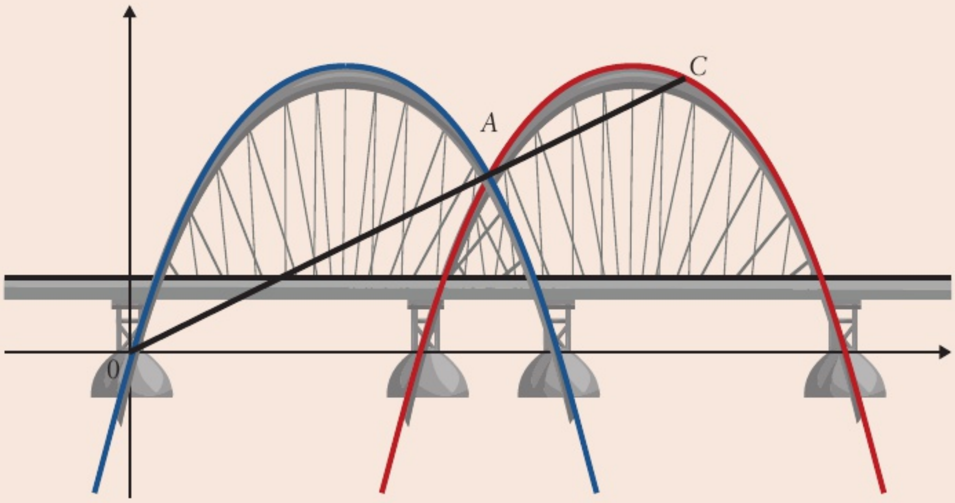
a Show that the equation of the parabola that describes the arch is $y = -0.13(x - 10)(x + 10)$

b A horizontal beam has to be added at a height of 8 m to support the arch.

i Determine the coordinates of the two points on the arch connected by the beam.

ii Calculate the length of the beam.

6 A bridge is suspended on two intersecting parabolic supports.



On a set of axes where one unit corresponds to one metre, the equations of the two parabolas are $y = -0.11x(x - 21.2)$ and $y = -0.11(x - 14)(x - 35.2)$

a Calculate:

i the maximum height of the two supports

ii the span of each support (the distance between its two bases).

b The road is 3 m above the base of the bridge. Calculate the length of the portion of the road suspended on the parabolic supports.

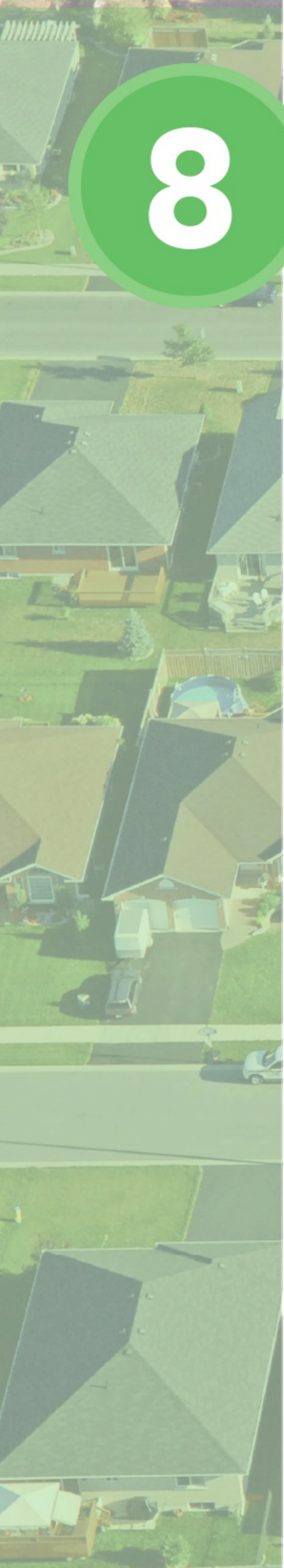
c A straight cable from $O(0, 0)$ to C is used to reinforce the parabolic structures. The cable passes through A, the point where the two parabolas intersect. Find:

i the coordinates of A

ii the coordinates of C

iii the total length of the cable.

8 Similarity



8 Similarity

KEY CONCEPT

Relationships

RELATED CONCEPTS

Equivalence, Generalisation, Models

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Understanding relationships between similar shapes allows us to model measures that are not easily accessible.

Factual

- What does it mean for two shapes to be similar?

Conceptual

- Does the perimeter of a shape always change as the area changes?
- How can similar shapes be used to solve problems?

Debatable

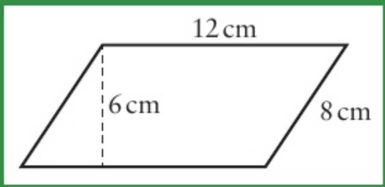
- Can visualisation help determine relationships between objects?
- Can giants really exist?

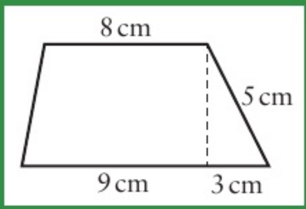
Do you recall?

1 Solve each of these equations for x.

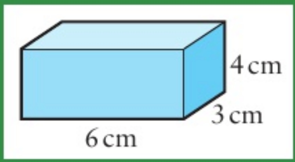
a $\frac{36}{48} = \frac{x-1}{5}$ b $3x^2 - 10 = 65$

2 Find the area of each of the following shapes.

a


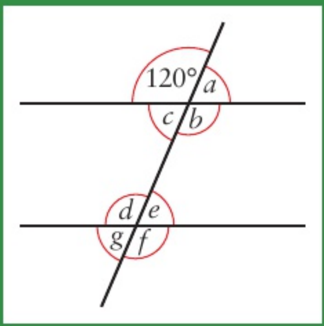
b


3 Find the surface area and volume of this cuboid.



4 The diagram shows a pair of parallel lines cut by a transversal.

One angle has a measure of 120°. Find the measures of the rest of the angles, giving reasons for your answers.



5 Solve this system of equations.

$2x - 3y = 8$

$3x + 2y = -1$

6 Find the equation of a line that goes through the point (1, 3) and is perpendicular to the line with equation $2x - y = 7$

8.1 Similar shapes

To describe the images above we can say that they are **similar**. If two objects are similar then they have the same shape. In mathematics, we need a 'well-defined' description of similar shapes. We need to have criteria that we can use to judge whether objects are similar.

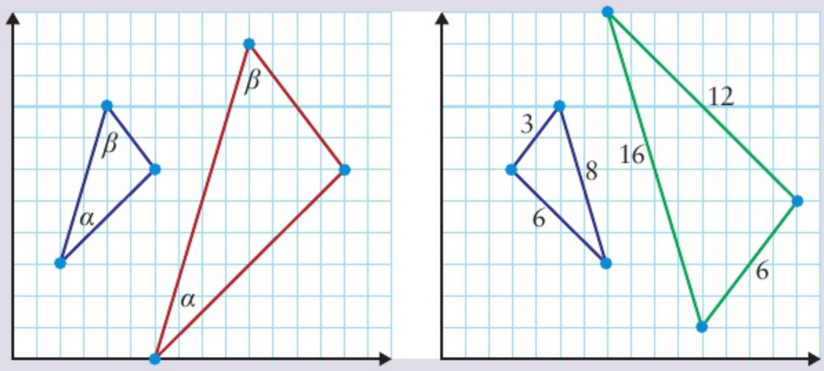
Explore 8.1

The following diagram contains two sets of similar triangles.

What do you think *similar* might mean?

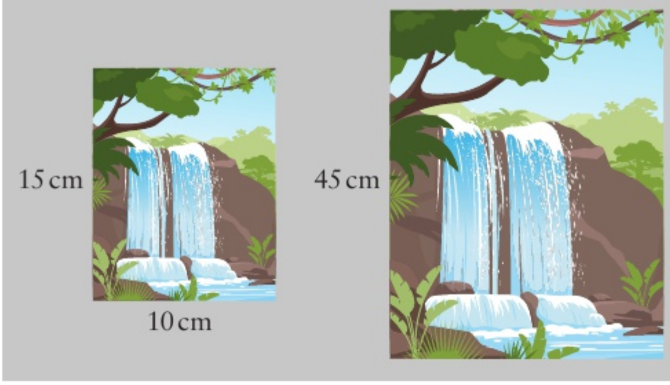
Can you formulate a definition for when two triangles are similar?

Can you create your own two similar shapes?



Worked example 8.1

A picture measuring 10 cm wide by 15 cm long is enlarged such that its new length is 45 cm. What is the scale factor of the enlargement and what is the new width?



Solution

To determine the scale factor, we need to write a ratio of the lengths of matching sides. Then to find the new width, we can write and solve a proportional relationship.

$\dfrac{\text{enlarged length}}{\text{orignal length}} = \dfrac{45}{15} = \dfrac{3}{1} = 3$ — Set up the ratio and simplify it to find the scale factor.

$\dfrac{3}{1} = \dfrac{w}{10} \Rightarrow w = 30$ — Write and solve a proportional relationship to determine the new width.

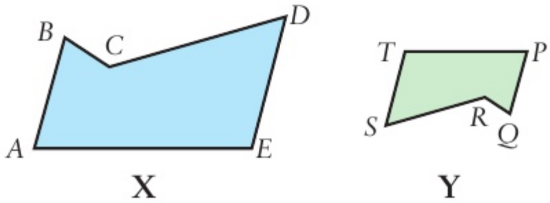
Hence, the width of the enlargement is 30 cm.

To check our solution, we can substitute the value of w back into the ratio and confirm that it reduces to the same scale factor.

$$\frac{30}{10} = \frac{3}{1}$$

We get the same scale factor, so we know that our answer is correct.

Two shapes are **similar** if they have the same shape. Two shapes are similar when one shape can be enlarged and superimposed onto the other such that they coincide exactly. For example, shape Y can be rotated and enlarged to fit exactly over shape X.



> **Hint**
>
> The symbol ≅ means that the two quantities are congruent. The symbol $\sphericalangle A$ represents the angle with vertex A.

That is, two shapes are similar if their **corresponding or matching** angles are congruent, and the ratios of the lengths of their corresponding sides are equal. This common ratio is called the **scale factor** (or sometimes the ratio factor). That is, $\sphericalangle A \cong \sphericalangle P$, $\sphericalangle B \cong \sphericalangle Q$, $\sphericalangle C \cong \sphericalangle R$, $\sphericalangle D \cong \sphericalangle S$, $\sphericalangle E \cong \sphericalangle T$ and $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$

Corresponding sides are said to be **proportional**.

Worked example 8.2

A girl who is 170 cm tall stands in the sun and casts a 290 cm shadow. When she stepped into the shadow of a tree, 12 m away from the tree, the tip of her shadow coincided with that of the tree. She is now completely in the shadow of the tree.

How tall is the tree? Give your answer to the nearest metre.

Solution

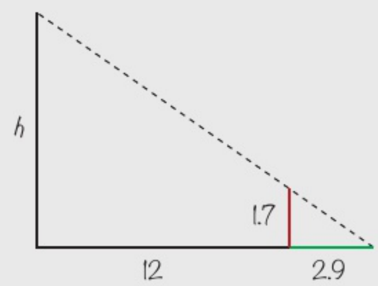
Understand the problem

We know the girl's height and the length of her shadow. We also know how far the girl is from the tree. We need to find the height of the tree.

> **Reminder**
>
> Make sure that all measurements use the same units.

Make a plan

We can draw a sketch to represent the problem. Assuming that the girl and the tree are at right angles with the ground, we use the given information to find the length of the tree's shadow. Finally, we can use similar triangles to find the height of the tree.



Carry out the plan

Make sure all measurements are in metres.

girl's height = 1.70 m, girl's shadow length = 2.90 m

The total tree shadow length is the sum of the girl's shadow length and the distance the girl is from the tree:

$$\frac{\text{girl's height}}{\text{tree's height}} = \frac{\text{girl's shadow}}{\text{tree's shadow}} \Rightarrow \frac{1.70}{h} = \frac{2.90}{14.90}$$

By cross multiplying and then simplifying:

$$h = \frac{(1.70)(14.90)}{2.90} \approx 8.73$$

The height of the tree is 9 metres, to the nearest metre.

Look back

To check our solution, we can substitute this value back into the original proportional relationship and verify that the two ratios are equal to each other.

$$\frac{1.7}{8.73} = \frac{2.9}{14.9} \Rightarrow 0.195 = 0.195$$

As both ratios simplify to the same scale factor, we know our answer is correct.

Reminder

When checking your solution, remember to use the full value, not the rounded answer.

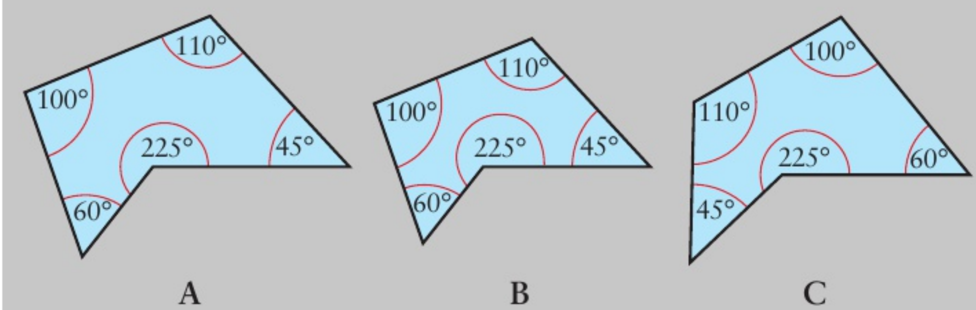
Reflect

Architects use similarity to make models of buildings and other projects. Can you think of other applications?

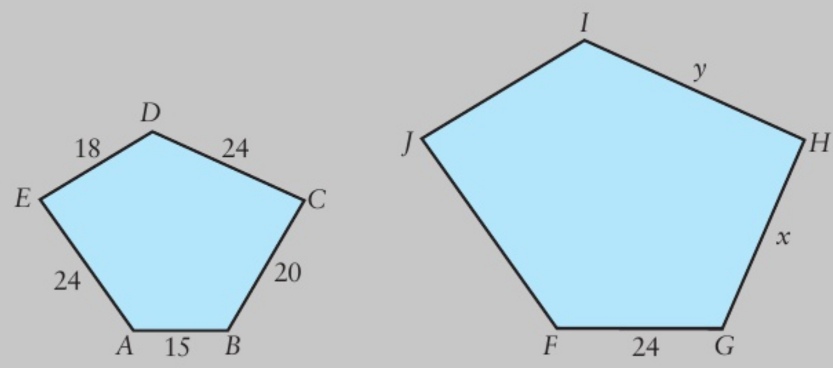
Research skills

Worked example 8.3

a Which of the shapes below cannot be similar to the other two?



b The two shapes shown are similar. Calculate the scale factor and use it to find the value of x and y.



Solution

a C is not similar to the others. Its angles are the same size as those of A and B, but they are not in matching (corresponding) positions. For shapes to be similar, corresponding angles must be congruent.

b enlargement factor = ratio of matching sides

As we know, 'similar' means that the shapes have the same form. Thus, the correspondence between the two polygons is $ABCDE$ to $FGHIJ$. So, FG is the image of AB.

$$\text{scale factor} = \frac{FG}{AB} = \frac{24}{15} = 1.6$$

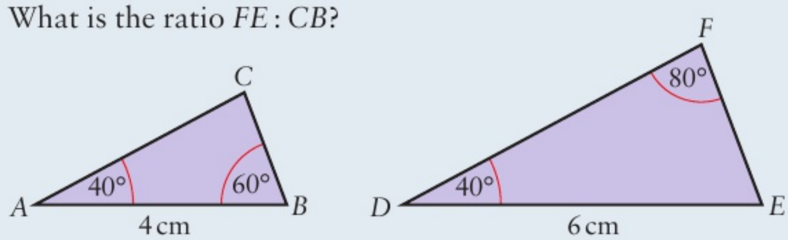
Now, x corresponds to BC and y corresponds to CD.

$x = 1.6 \times 20 = 32$ and $y = 1.6 \times 24 = 38.4$

Practice questions 8.1

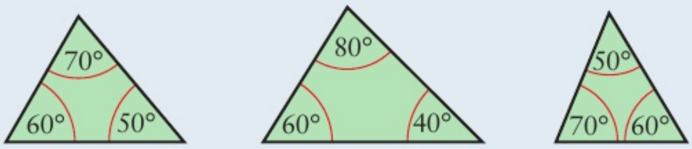
1 For the two triangles shown, answer the following questions.

- **a** What is the size of angle ACB?
- **b** What is the size of angle DEF?
- **c** Are the triangles similar?
- **d** What is the ratio $DE : AB$?
- **e** What is the ratio $FE : CB$?



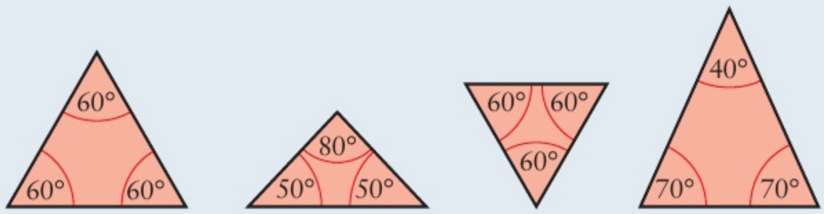
2 Identify the triangles that are similar for each of the following.

a



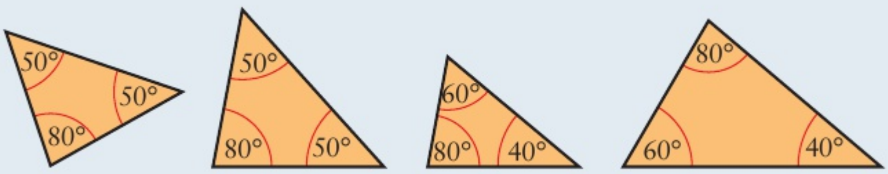
A B C

b



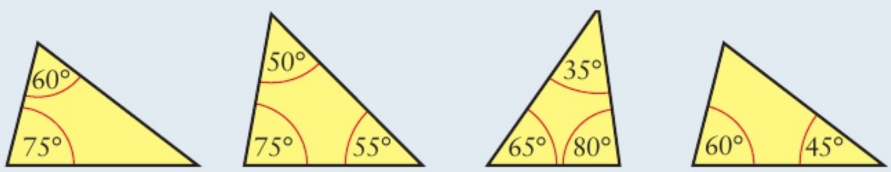
A B C D

c



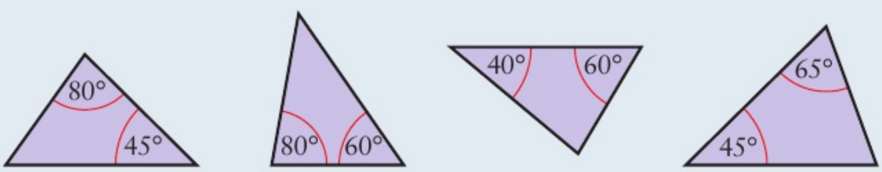
A B C D

d



A B C D

e

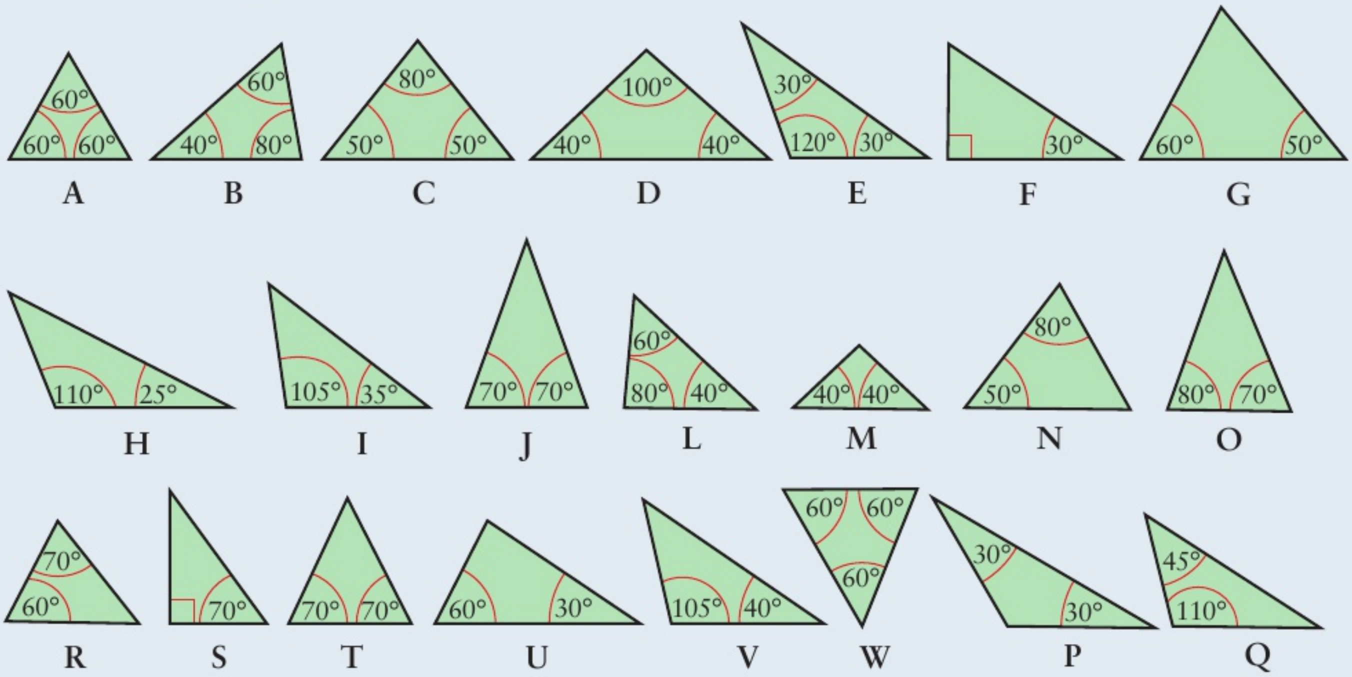


A B C D

3 There are ten pairs of similar triangles in the triangles shown below. Each of the triangles A to J is similar to one of the triangles L to W. Select the triangle that is similar to triangle:

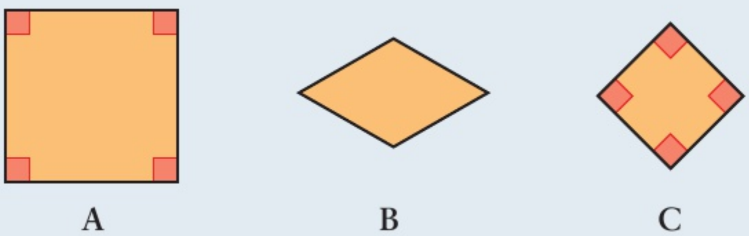
a A b B c C d D e E

f F g G h H i I j J

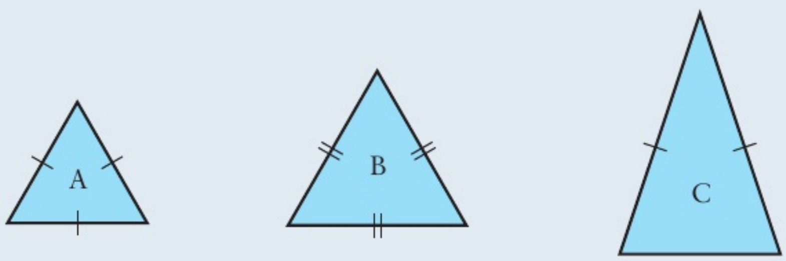


4 Which of the shapes, in each trio, cannot be similar to the other two? Justify your answers.

a



b



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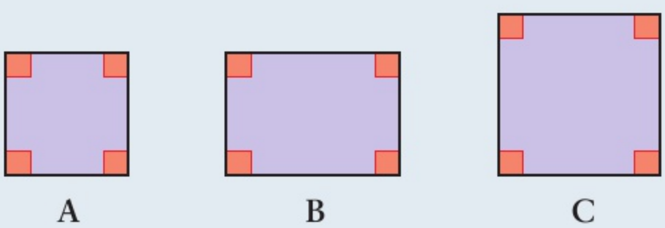

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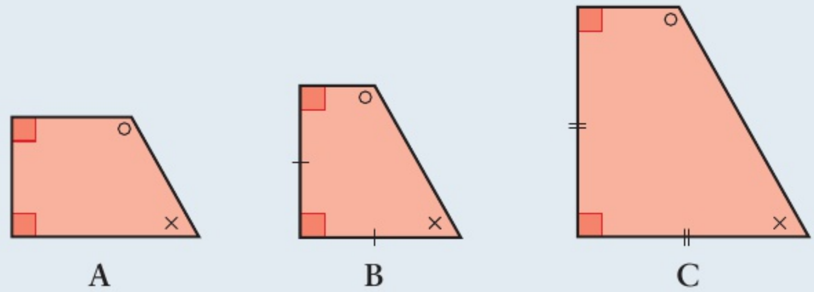
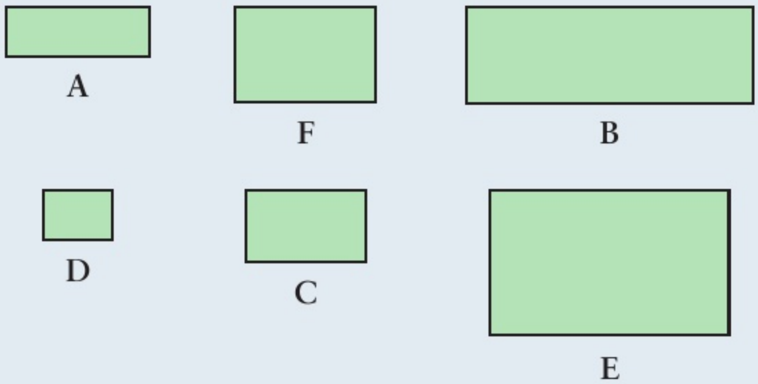


5 Measure the lengths and widths of each of the rectangles below and select the rectangles that are similar. Justify your answers.

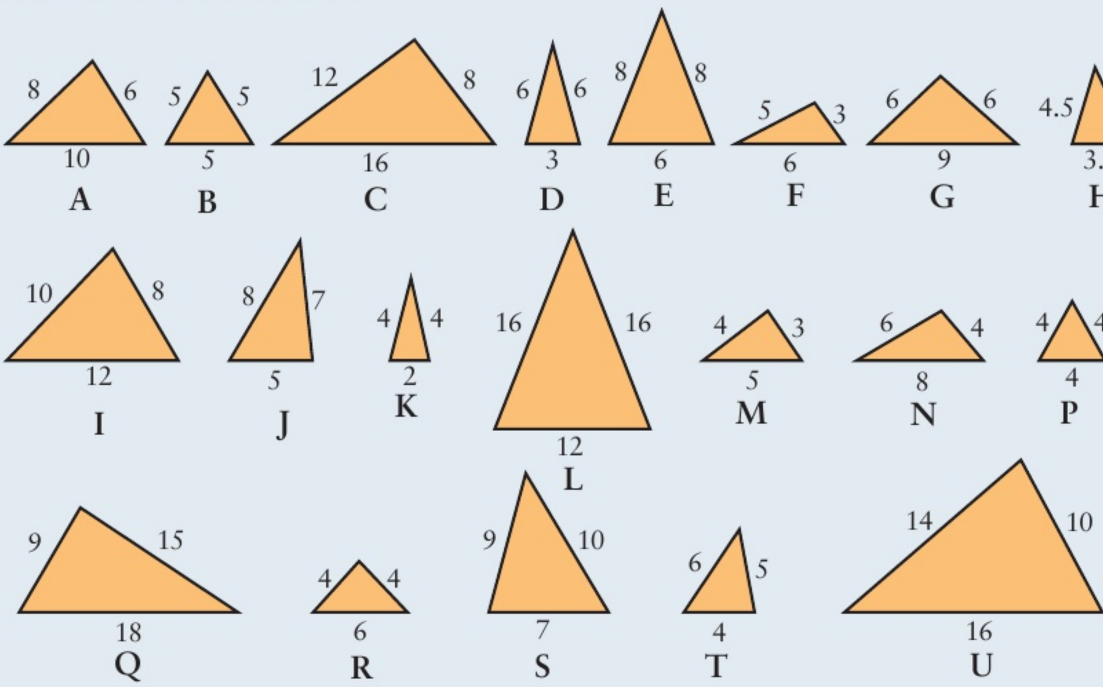


6 Below are ten pairs of similar triangles. Each of the triangles A to J is similar to one of the triangles K to U. Find the triangle that is similar to:

a A **b** B **c** C **d** D **e** E

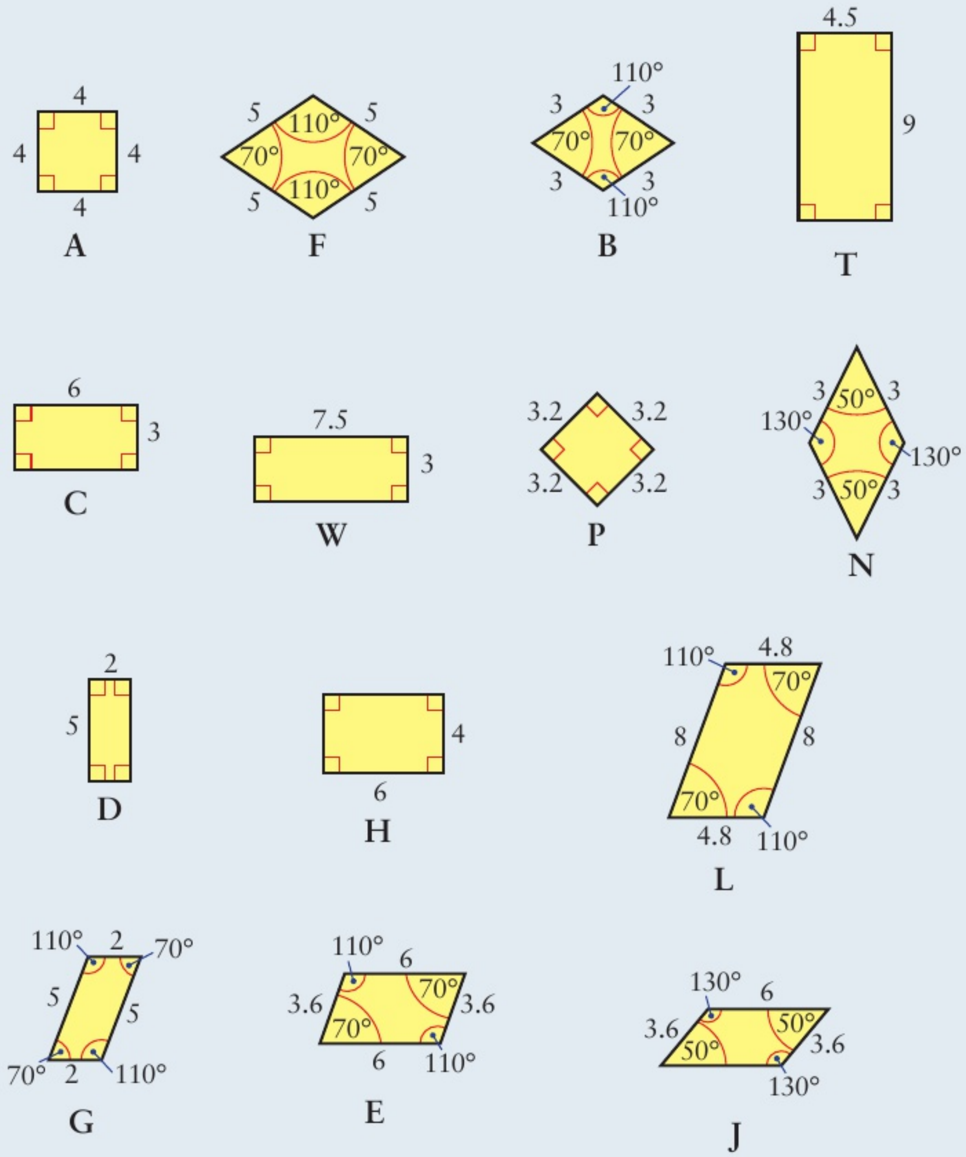
f F **g** G **h** H **i** I **j** J

All measurements are in cm.

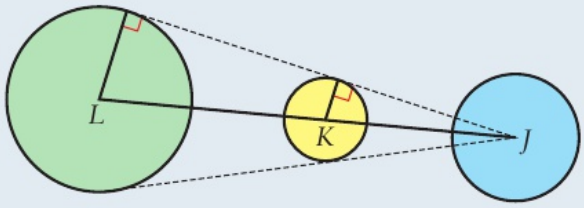


7 From the set of polygons below, select the polygon similar to:

a A **b** B **c** C **d** D **e** E

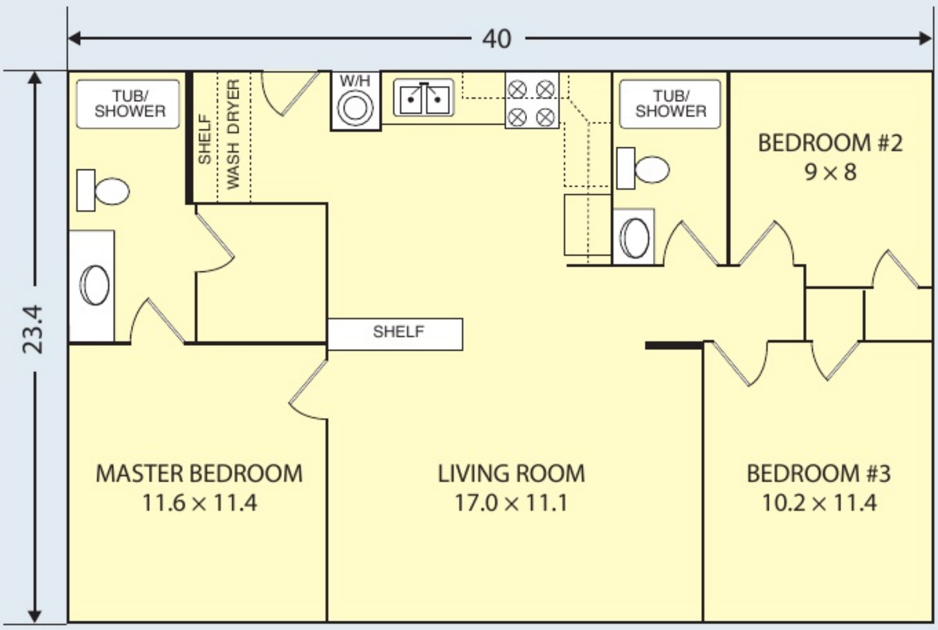


8 A total eclipse of the Sun occurs when the Moon is directly in line with the Sun and blocks the Sun's rays from reaching Earth, as shown in the diagram below. J, K and L are the centres of the Earth, Moon and Sun, respectively. The approximate distance between the centres of the Moon and the Earth is 384 400 km, and the distance between the centres of the Earth and the Sun is approximately 150 210 000 km. The radius of the Sun is approximately 696 340 km.



Use this information to estimate the diameter of the moon.

9 Below is an architectural drawing of an apartment. The architect uses a scale of 1 : 50. Measurements on the drawing are in cm.



a What are the dimensions of the whole apartment in metres?

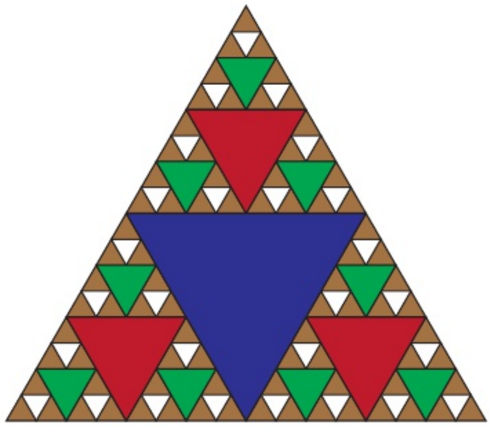
b What is the total area of the apartment?

c Approximately what proportion of the apartment is used for bedrooms?

Connections Q8

Mathematics can be used to help determine distances in space.

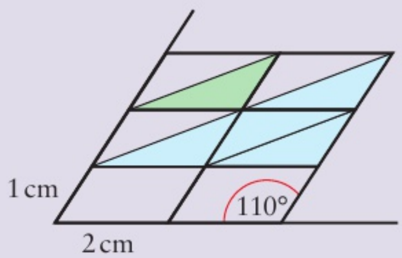
8.2 Similar triangles



Triangles are polygons, so we expect similar triangles to have the same properties as similar polygons: corresponding angles are congruent and corresponding sides are in the same ratio. However, in triangles we may not need to refer to the 'full' version of the definition. Think about the condition that all three angles in one triangle must be equal to their corresponding angles in another triangle. Do we really need to know three angles?

Explore 8.2

Consider the two shaded triangles drawn on a grid of parallel lines.

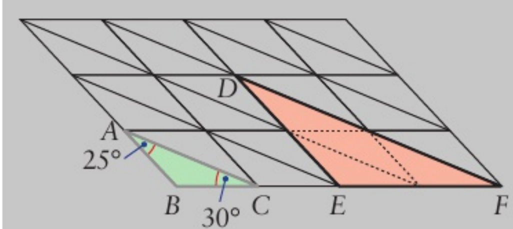


Can you justify why the two triangles are similar?

Can you think of the minimum number of conditions we need to have so that we can claim that two triangles are similar? For example, do we need to say all corresponding angles must be congruent? Give as many choices as you can.

Worked example 8.4

Triangle ABC has angles of 25° and 30° as shown. Copies of $\triangle ABC$ have been used to make the grid, which has three sets of parallel lines.



a Are $\triangle ABC$ and $\triangle DEF$ similar?

b List the measures of all angles involved and justify your answer.

c Are the matching sides of the two triangles in the same ratio?

Solution

a To show that the triangles are similar, we need to show that the matching (corresponding) angles are congruent, or that matching sides are in the same ratio.

Since $\triangle DEF$ is made of copies of $\triangle ABC$, then

$\sphericalangle D \cong \sphericalangle A$, $\sphericalangle E \cong \sphericalangle B$ and $\sphericalangle F \cong \sphericalangle C$

Thus, the two triangles are similar because all matching angles are congruent.

b Since two of the angles of $\triangle ABC$ are 25° and 30°, the third angle, $\sphericalangle B$, must be $180° - (25° + 30°) = 125°$

It follows that angles D, E and F have the same measures as their matching angles in $\triangle ABC$, so $D = 25°$, $E = 125°$ and $F = 30°$

c DE consists of two copies of AB, EF consists of two copies of BC, and DF consists of two copies of AC.

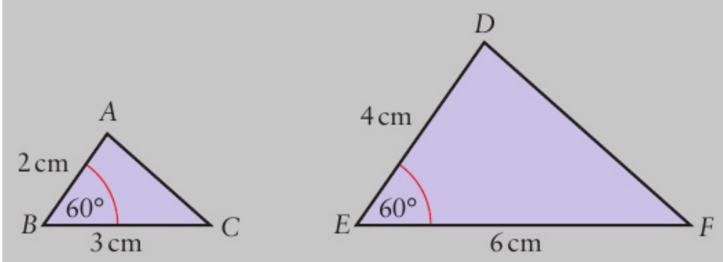
Therefore $\dfrac{DF}{AC} = \dfrac{DE}{AB} = \dfrac{EF}{BC} = 2$

Reflect

Do we really need to know that three angles in one triangle are congruent to three angles in another for the triangles to be similar?

Worked example 8.5

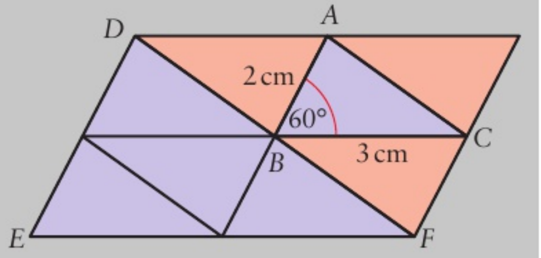
Consider triangles $\triangle ABC$ and $\triangle DEF$ with the given measures.



a Show that $\frac{DE}{AB} = \frac{EF}{BC}$

b What else could you use before saying that the triangles are similar?

c In the diagram, a grid based on parallelograms formed from $\triangle ABC$ is shown. $\triangle DEF$ is also shown on the grid.



i Why is $\angle A \cong \angle D$ and $\angle C \cong \angle F$?

ii Does $\frac{DF}{AC} = 2$?

Solution

a $\frac{DE}{AB} = \frac{4}{2} = 2$ and $\frac{EF}{BC} = \frac{6}{3} = 2 \Rightarrow \frac{DE}{AB} = \frac{EF}{BC}$

b For two shapes to be similar, all matching angles must be congruent, or all matching sides must be of the same ratio. Any of the following conditions will suffice to say the triangles are similar:

$\frac{DF}{AC} = 2$, or $\angle A \cong \angle D$, or $\angle C \cong \angle F$

c i $\angle A \cong \angle D$ and $\angle C \cong \angle F$ because they are corresponding (matching) angles of the small congruent triangles.

ii Since DF is made up of two copies of AC, then $\frac{DF}{AC} = 2$

Hint

Note that it is enough to have two matching angles in $\triangle ABC$ and $\triangle DEF$ congruent. The reason is that the third angles must also be congruent, since the three angles of a triangle add up to 180°.

Hint

All the small triangles are copies of the same triangle ABC.

Reflect

What conclusion about similar triangles can you draw from Worked example 8.5?

Thinking skills

In this section, you learned about the **postulates of similar triangles**. These are:

1 AA postulate: two triangles are similar if two angles in one triangle are congruent to two angles in the other.

2 SSS postulate: If three sides of one triangle are in the same ratio as three sides of another, then the triangles are similar.

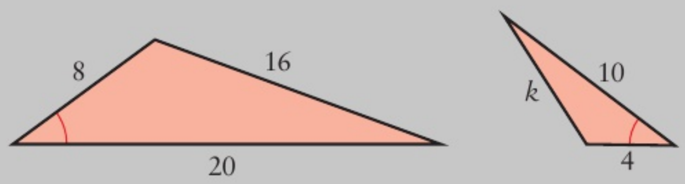
3 SAS postulate: Two triangles are similar if an angle of one triangle is congruent to an angle of the other and the lengths of the corresponding sides that form the angles are in the same ratio.

Fact

The AA postulate is also called the AAA postulate.

Worked example 8.6

Find the value of k in the diagram below.



Solution

The two triangles appear to be similar.

Examining the possible matching sides, it is clear that the two triangles are similar by SAS.

The measures of corresponding sides have the same ratio: $\frac{4}{8} = \frac{10}{20} = \frac{1}{2}$, and the included angles in both triangles are congruent.

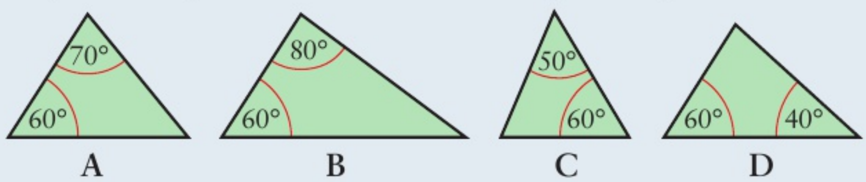
Thus, the third sides must have the same ratio.

$\frac{k}{16} = \frac{1}{2} \Rightarrow k = 8$

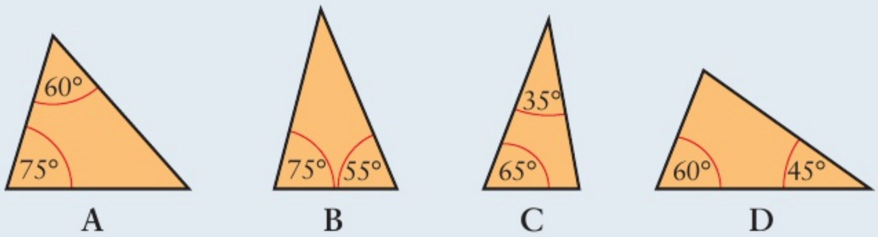
Practice questions 8.2

1 Identify the triangles that are similar in each question part.

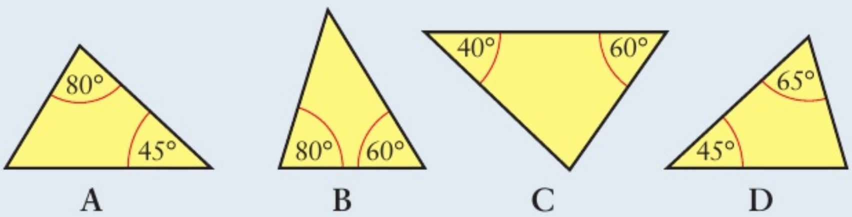
a



b



c

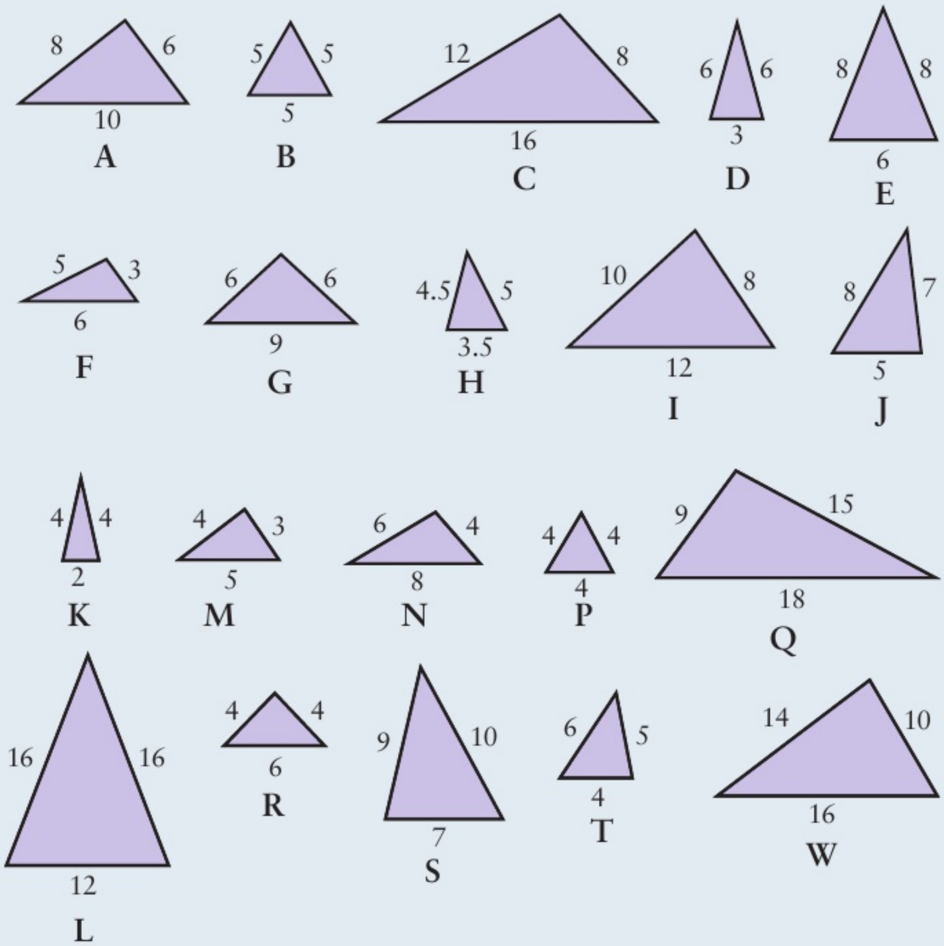


2 Ten pairs of similar triangles are shown. Each of the triangles A to J is similar to one of the triangles K to W. Find the triangle that is similar to:

a A b B c C d D e E

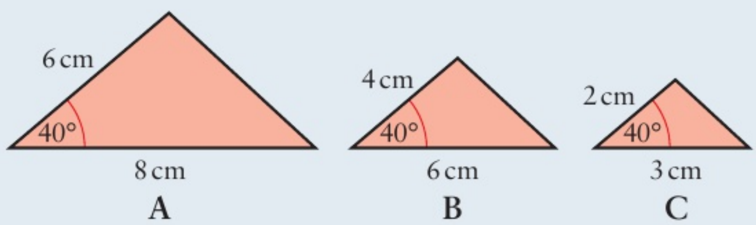
f F g G h H i I j J

Give reasons for your answers. All measurements are in cm.

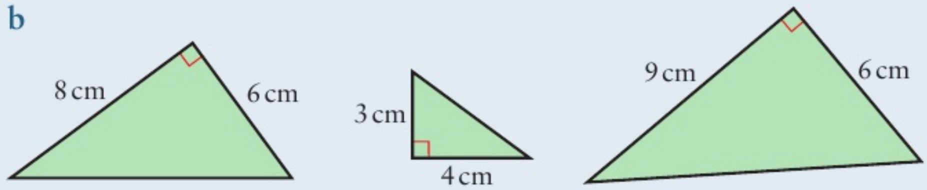


3 Identify the pair of similar triangles in each of the following. Give reasons for your answers.

a

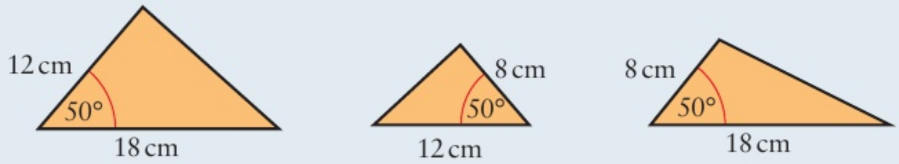


b



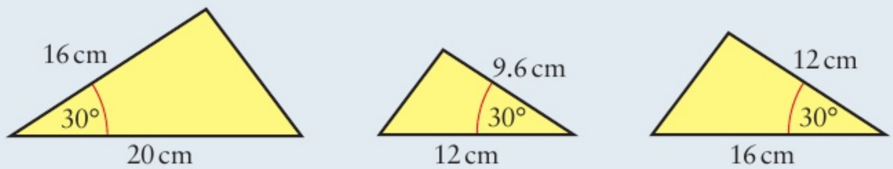
A **B** **C**

c



A **B** **C**

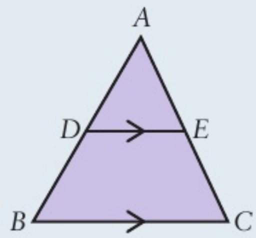
d



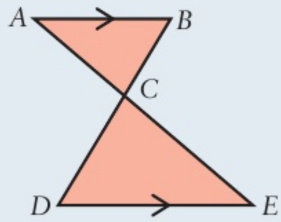
A **B** **C**

4 Find two similar triangles in each of the following. State which condition could be used to show that the triangles are similar.

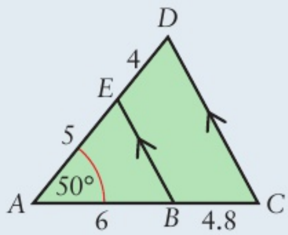
a



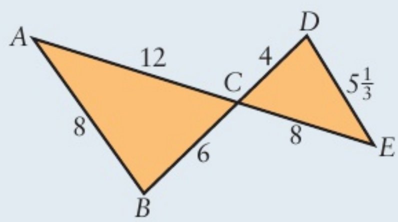
b



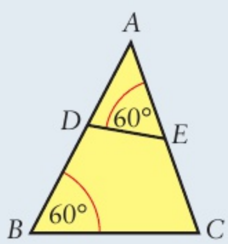
c



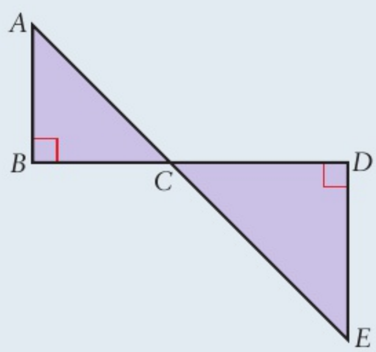
d



e



f



8.3 Applications of similar triangles

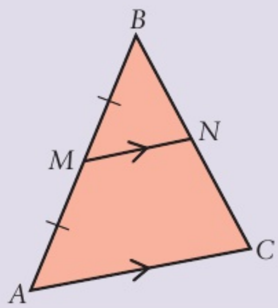
In mathematics, as well as in real life, many situations can be modelled using similar triangles. For example, in mathematics, several theorems concerning lines that divide sides of triangles in equal proportions, or segments intercepted by angle bisectors on their opposite sides and others, can be proved using similar triangles. In real life, examples include estimating the heights of inaccessible objects (such as cliffs or trees) using smaller models.

Explore 8.3

Consider triangle ABC shown.
M is the midpoint of AB and MN is parallel to AC.

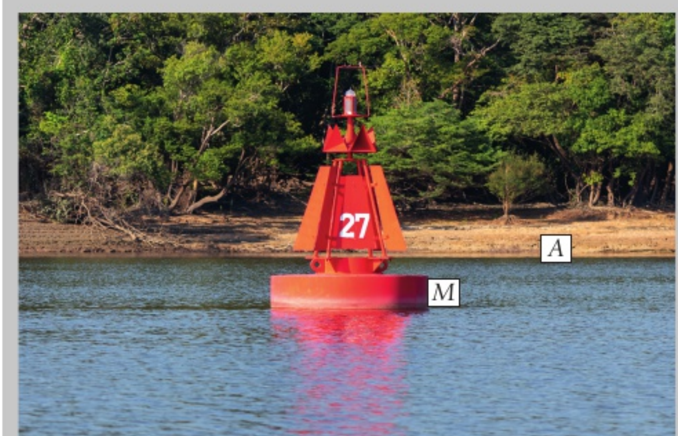
What can you observe about point N?

What can you say about the length of MN?



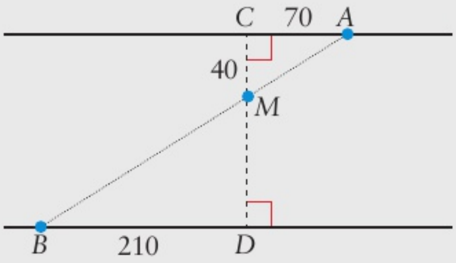
Worked example 8.7

A channel marker, M, in a river is located 40 m away from the opposite shore as shown in the picture. The point marked A is 70 m down shore. The line of sight from A to a point on the nearer shore is 210 m up shore from the marker. Work out how wide the river is near the marker. Assume the two shores to be parallel.



Solution

In order to understand the situation, we need to draw a picture first.



From the diagram, we can deduce that our task is to find the distance CD.

The diagram helps us see that there are two triangles that can be proved similar, enabling us to find MD, which in turn will help us find CD.

The first task is to show that triangles MCA and MDB are similar.

$\measuredangle C \cong \measuredangle D$ because they are right angles.

$\measuredangle BMD \cong \measuredangle AMC$ because they are vertically opposite angles.

Thus the triangles are similar by the AA postulate.

Now, $\frac{MD}{MC} = \frac{BD}{AC} \Rightarrow \frac{MD}{40} = \frac{210}{70} = 3$

By solving the last equation, we have $MD = 120\,\text{m}$

The width of the river at this point is $DC = 40 + 120 = 160\,\text{m}$

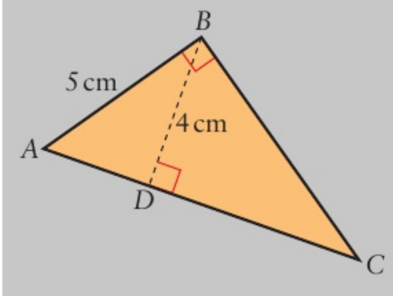
Fact

A line drawn through the midpoint of a side of a triangle and parallel to another side bisects the third side.

The segment joining the midpoints of two sides of a triangle has half the measure of the third side.

Worked example 8.8

Triangle ABC is right-angled at B. DB is the height from D.
AB and BD are 5 cm and 4 cm as shown.



Work out the lengths of the unknown sides.

Solution

We need to find the lengths of BC, DC and AD.

There are three triangles that contain these sides: ABC, ABD, and DBC

These triangles are similar. We will use similarity proportion to find the missing lengths.

AD can be found directly because it is a leg in the right-angled triangle ABD where the hypotenuse and one leg are known.

Use Pythagoras' theorem:

$AD^2 + DB^2 = AB^2 \Rightarrow AD^2 + 4^2 = 5^2 \Rightarrow AD = 3\,\text{cm}$

$\triangle ABC$ and $\triangle ABD$ are similar by the AA postulate:

$\sphericalangle ABC \cong \sphericalangle BDA$ since they are both right angles.

$\sphericalangle A$ is common to both triangles.

The corresponding sides are:

AC in $\triangle ABC$ corresponds to AB in $\triangle ABD$ as they are both opposite to the right angle.

AB in $\triangle ABC$ corresponds to AD in $\triangle ABD$ as they are opposite congruent angles.

This is so, because:

In $\triangle ABD$, $\sphericalangle ABD$ is complementary to $\sphericalangle A$.

In $\triangle ABC$, $\sphericalangle C$ is complementary to $\sphericalangle A$.

Thus, $\sphericalangle ABD \cong \sphericalangle C$

BC in $\triangle ABC$ corresponds to BD in $\triangle ABD$ since they are opposite to $\sphericalangle A$.

Thus, $\frac{AC}{AB} = \frac{AB}{AD} = \frac{BC}{BD} \Rightarrow \frac{AC}{5} = \frac{5}{3} = \frac{BC}{4}$

Solving the two equations above gives $AC = \frac{25}{3}$

Thus $DC = \frac{25}{3} - 3 = \frac{16}{3}$ and $BC = \frac{20}{3}$

To look back at the solution and examine whether it makes sense, we can use the right-angled triangle BDC.

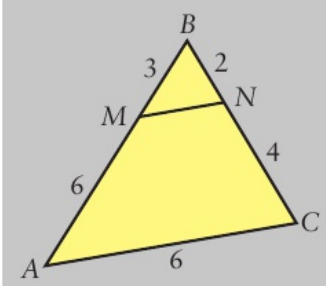
$BC^2 = BD^2 + DC^2 \Rightarrow \left(\frac{20}{3}\right)^2 = 4^2 + \left(\frac{16}{3}\right)^2 \Rightarrow \frac{400}{9} = \frac{144 + 256}{9}$,

which is a true statement.

Worked example 8.9

Consider triangle ABC with points M and N chosen on sides AB and AC.

All measurements are in centimetres.



a Show that $MN \parallel AC$

b Find the length of MN

Solution

a We need to show that the line segments MN and AC are parallel. We will use what we know about parallel lines. We can consider AB and BC as transversals and show that corresponding angles are congruent.

Consider triangles MBN and ABC.

$\sphericalangle B$ is common to both triangles.

Additionally, $\dfrac{MB}{AB} = \dfrac{3}{9} = \dfrac{1}{3}$ and $\dfrac{BN}{BC} = \dfrac{2}{6} = \dfrac{1}{3}$

so the corresponding sides have the same ratio (they are proportional).

Thus, the two triangles are similar by the SAS postulate.

Hence, $\sphericalangle A \cong \sphericalangle M$ as they are matching angles in two similar triangles. They are also corresponding angles for the two line segments MN and AC. Therefore the two lines are parallel.

b Since the triangles are similar,

then $\dfrac{MN}{AC} = \dfrac{1}{3} \Rightarrow MN = \dfrac{1}{3}AC = \dfrac{1}{3} \times 6 = 2\,\text{cm}$

Fact

When the measures of all corresponding parts of two shapes or objects are equal, we call the shapes or objects **congruent**. The word equal means that they are made of the same elements. That is why when the measures of $\sphericalangle A$ and $\sphericalangle B$, for example, are equal, we do not say that the angles themselves are equal, because they are not the same angle.

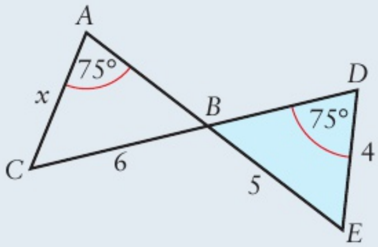
Reflect

Looking at Worked example 8.9, can you generalise what you observed?

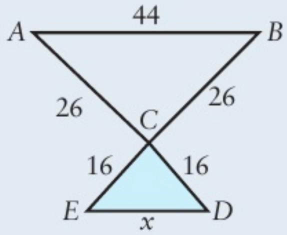
Practice questions 8.3

1 For each shape, prove that the triangles are similar and then find the value of the variable.

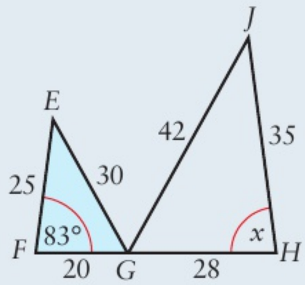
a



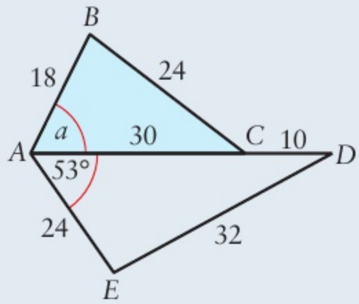
b



c



d



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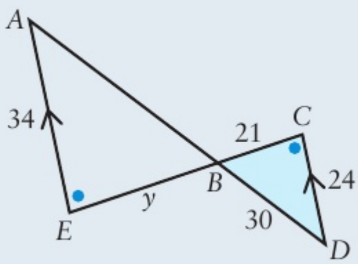
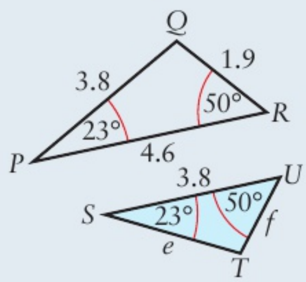
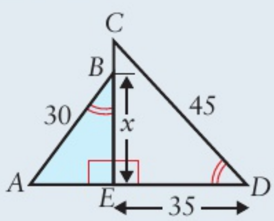


2 State which condition could be used to prove that the triangles are similar and then find the value of the variables.

a



b



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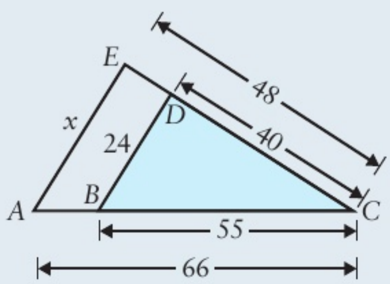

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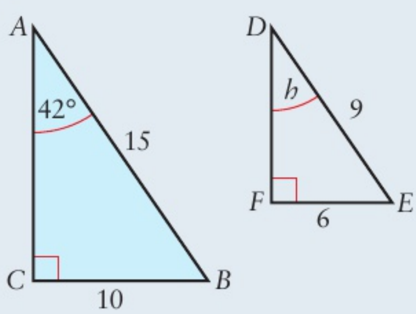
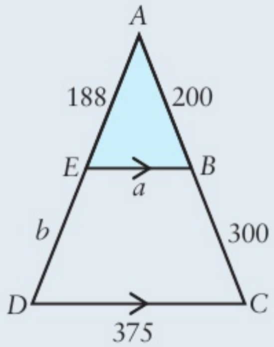


e



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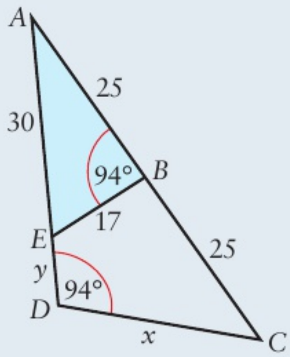
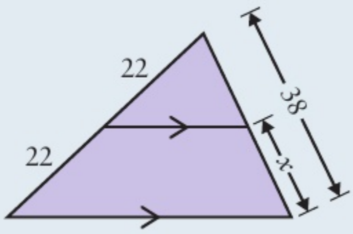
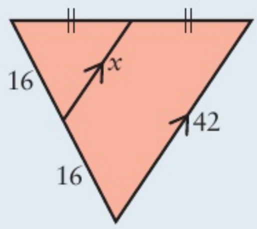


3 Find the value of the variable in each shape. Remember to justify each step.

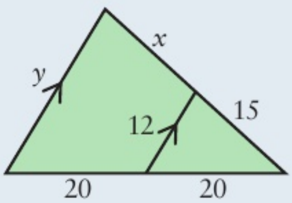
a



b

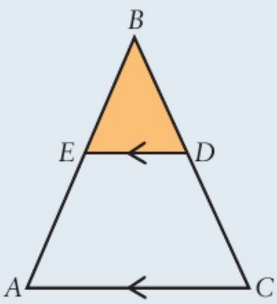


c



4 Answer all parts of the following questions. Give reasons for each step.

a



D divides BC in the ratio $4:5$

$AC \parallel ED$, $BA = 35\,\text{m}$

i Prove that $\triangle EBD \sim \triangle ABC$

ii Find the ratio of BD to BC

iii Find the lengths of BE and EA

iv Find the ratio of BE to EA

v Show that ED divides BA and BC in the same ratio.

In other words, show that $\dfrac{BE}{EA} = \dfrac{BD}{DC}$

Challenge Q4

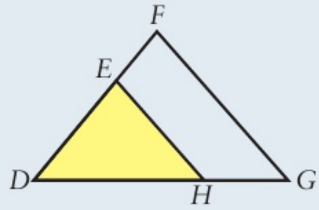
Hint Q4

Question 4a proves a theorem: A line parallel to one side of a triangle divides the other two sides in the same ratio. Question 4b proves the converse of the theorem.

Fact

The symbol '~' can be used to mean 'is similar to'.

b



$DE:EF = 3:2$, $DH:HG = 3:2$

i Find $DE:DF$

ii Find $DH:DG$

iii Prove that $\triangle EDH \sim \triangle FDG$

iv Is $\sphericalangle DEH \cong \sphericalangle DFG$? Why?

v Is $\sphericalangle DHE \cong \sphericalangle DGF$? Why?

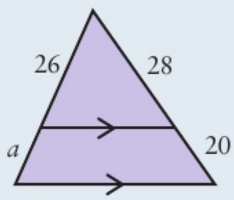
vi Is $EH \parallel FG$? Why?

Thinking skills

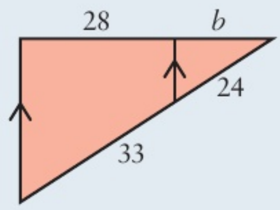
vii You have shown that if a line divides ______ of a triangle in the ______ it must be _____ to the third side.

5 Find the value of the variable in each of the following. Lengths are in metres.

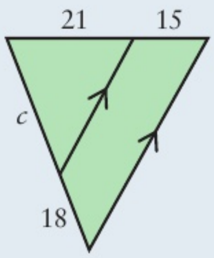
a



b

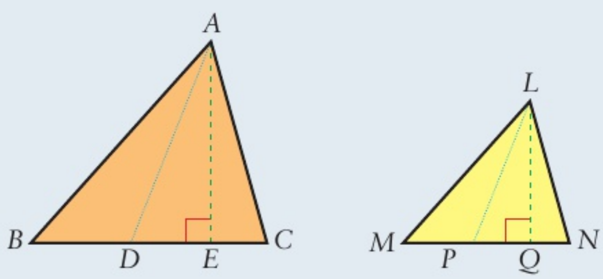


c



6 $\triangle ABC \sim \triangle LMN$, AD is a median of $\triangle ABC$, and LP is a median of $\triangle LMN$.

AE and LQ are the vertical height of $\triangle ABC$ and $\triangle LMN$ respectively.



a Show that $\dfrac{AD}{LP} = \dfrac{AC}{LN}$

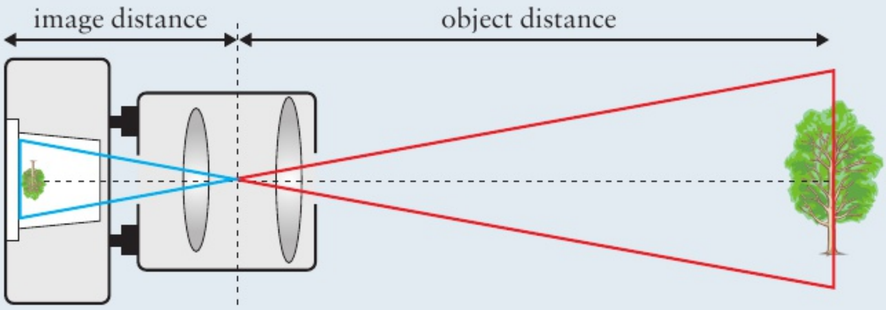
b Show that $\dfrac{AE}{LQ} = \dfrac{AC}{LN}$

Challenge Q6b

7 A picture of a modern 35 mm SLR camera is shown.
The simple mathematics behind how images form in your camera depends on similar triangles. The diagram shows a very simplified version of what happens. The image size at the back of your camera measures 35 mm. The image distance is 80 mm. The object, a tree in this case is 12 m away, so the object distance is 12 m.

Find the height of the tree.

Challenge Q7



8 Legend claims that the Greek mathematician Thales was the first to determine the height of the Egyptian Great Pyramid of Giza.

Challenge Q8

He considered that rays of light from the sun are parallel at a specific place. So he stood next to the pyramid at midday and measured the pyramid's shadow (115 m) and his own (1.51 m). He also knew his own height (1.82 m). He then had an estimate of the pyramid's height. What was his estimate?

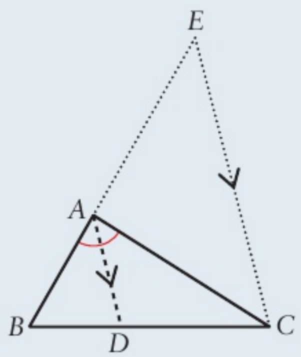
Draw your own sketch of the situation and make clear what assumptions you make and then work out the solution.

Challenge Q9

Connections

One of Thales' famous sayings: *The most difficult thing in life is to know yourself.*

9 Consider triangle ABC with AD as the bisector of angle A.



a Through C, draw a line parallel to AD and extend line BA so that it intersects this line at E.

Show that $AE \cong AC$

b Show that $\frac{BD}{DC} = \frac{BA}{AE}$

c Hence, show that $\frac{BD}{DC} = \frac{BA}{AC}$

Hint Q9c

Triangle AEC is isosceles.

Additional enrichment material for this chapter can be accessed via the link on this page of your eBook.

Self assessment

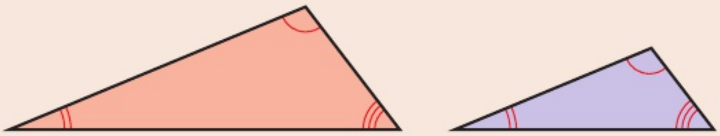
- I can find the scale factor and use it to solve problems.
- I can show and prove that triangles are similar using the AA postulate, the SAS postulate or the SSS postulate.
- I can use proportional relationships to find side lengths in similar shapes.
- When parallel lines intersect two or more lines, I know how the segments formed are related to each other.
- I know the different sets of conditions necessary to show or prove that two triangles are similar to each other, and how to find missing side lengths or angle measures.

Check your knowledge questions

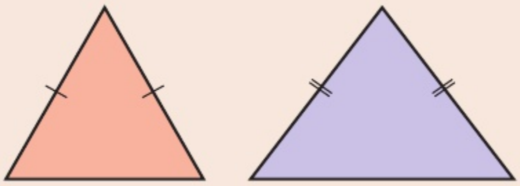
Communication skills

1 Are the following pairs of triangles similar?
Give reasons for your answers.

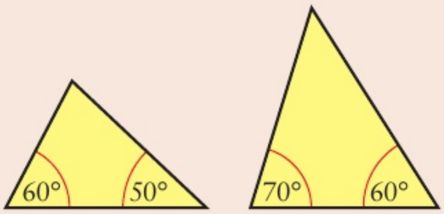
a



b

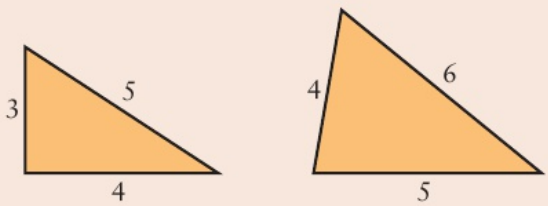


c

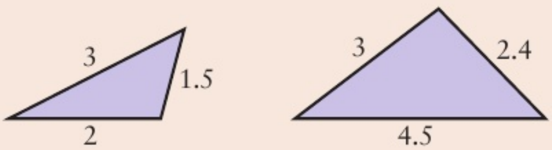


2 Are the following pairs of triangles similar?
Give reasons for your answers.

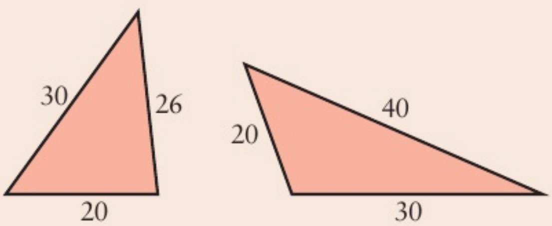
a



b

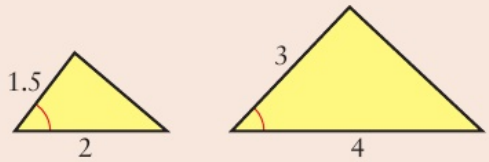


c

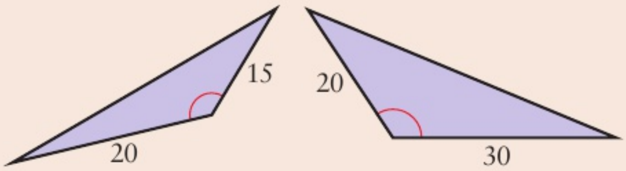


3 Are the following pairs of triangles similar?
Give reasons for your answers.

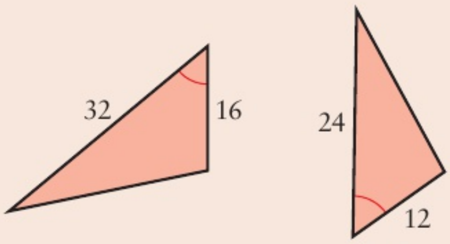
a



b

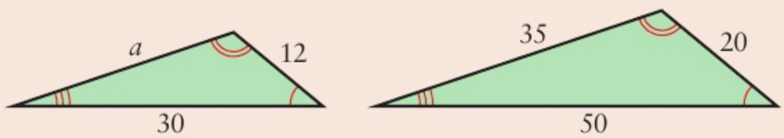


c

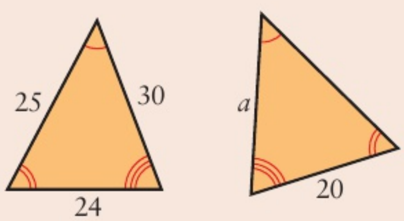


4 Find the value of the variable in each pair of similar triangles.

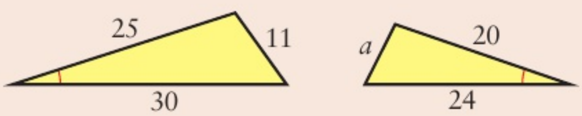
a



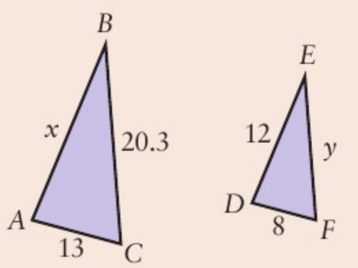
b



c

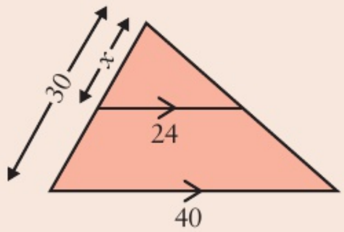


5 Given that $\triangle ABC \sim \triangle DEF$, find the values of x and y.

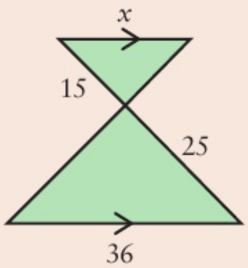


6 Find the value of the variable in each of the following.

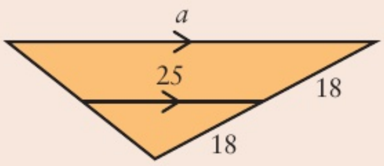
a



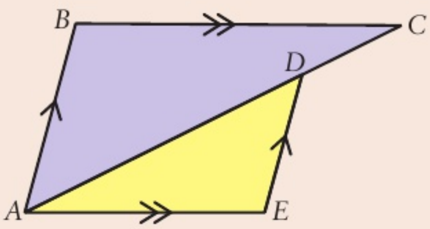
b



c

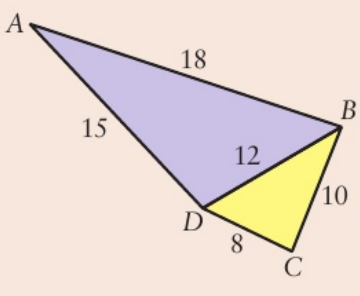


7 **a**



Show that $\triangle ABC \sim \triangle DEA$

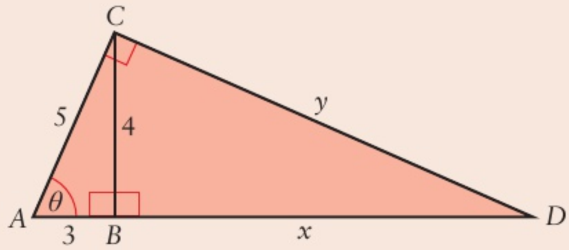
b



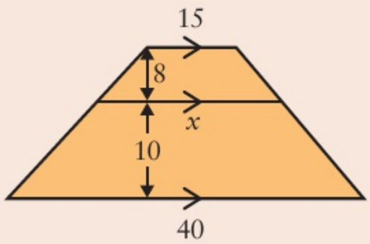
Show that $\triangle ABD \sim \triangle BDC$

8 You are trying to reduce a drawing that is 80 cm by 100 cm onto paper that is 18 cm by 22 cm. What are the maximum dimensions that the new drawing can have in order to fit on the smaller paper?

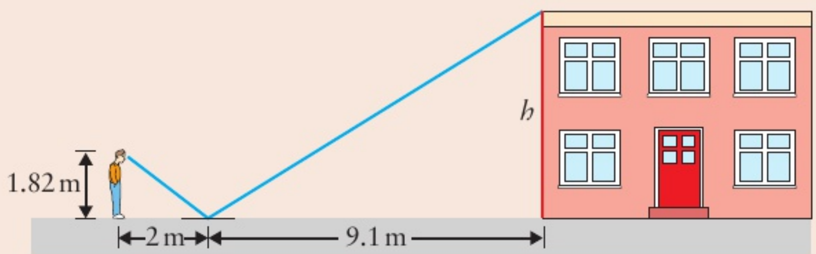
9 Find the value of x and y in the following diagram.



10 Find the value of x.



11 One way of measuring heights of tall objects such as buildings is with the use of mirrors. The principle is that when light hits a surface it reflects at an angle equal to its incident angle. A young man places a mirror on the floor, as shown, and walks back until he sees the top of the building. The young man' s eyes are at a height of 1.82 m from the horizontal ground.
The distances are shown. Find the height of the building, h.



9 Congruency

9 Congruency

KEY CONCEPT

Form

RELATED CONCEPTS

Creativity, Patterns

GLOBAL CONTEXT

Personal and cultural expression

Statement of inquiry

Personal and cultural beliefs can be expressed through geometric form in creative ways.

Factual

- What does congruent mean?
- What are the SSS, SAS, AAS theorems?
- What are the properties of congruent triangles?

Conceptual

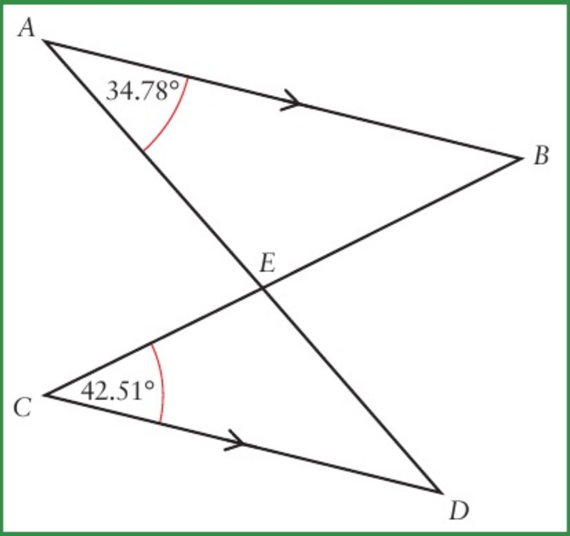
- What is the difference between equal and congruent?
- How can we prove that two shapes are congruent?
- Can congruent shapes be found within other shapes or are they always separate?

Debatable

- What is the difference between demonstrating and proving?
- What does it mean to represent something in abstract form?
- Can mathematics be used to analyse artwork without detracting from the appreciation of the art?

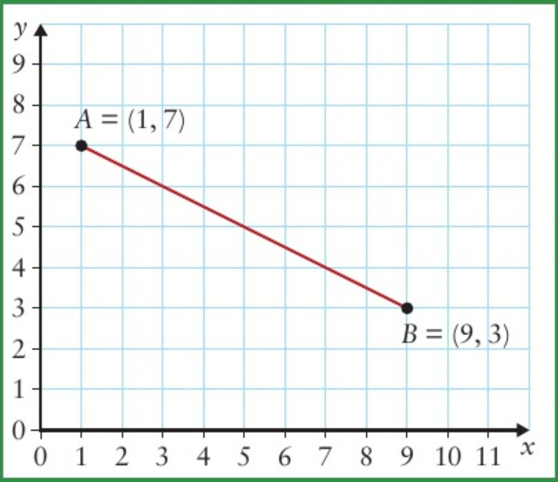
Do you recall?

1 Determine the size of all the angles in the triangles *AEB* and *CED*.



2 a Determine the midpoint of $\overline{AB}$.

b Determine the length of $\overline{AB}$.



Fact

A bar directly above two points, *A* and *B*, is shorthand notation for a line segment *AB*, which is a *finite* measurement distinct from the *infinite* line *AB*. It should not be mistaken for the vector notation of an arrow above two points.

9.1 Equality and congruency

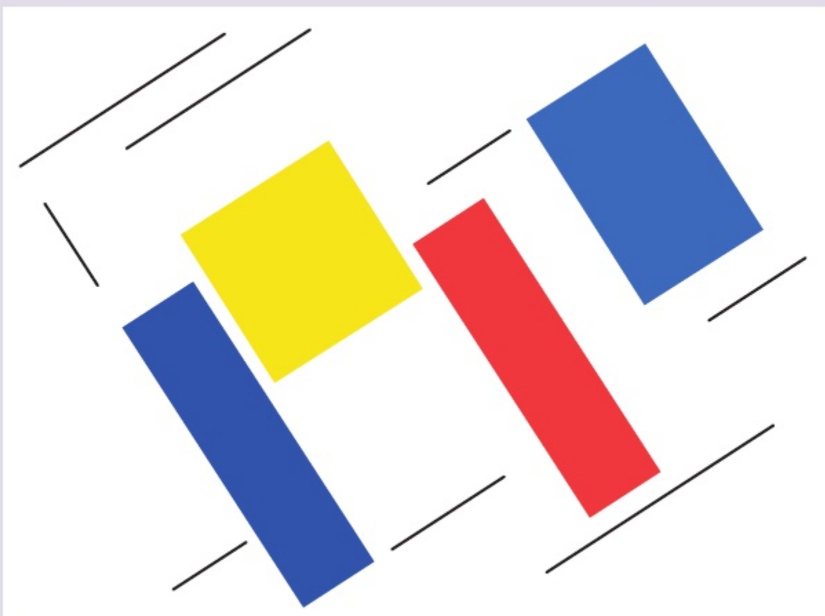
The De Stijl movement was formed in the Netherlands in 1917 by two Dutch artists, Theo van Doesburg and Piet Mondrian. Their work expresses their idea of the very basic structure of reality with primary colours and basic geometric shapes. The art style is geometric abstraction and it was represented in paintings, design and architecture.

Look at the image on the chapter opener page. What do you notice first, the colour, the lines or the shapes?

For further information see here:

Explore 9.1

The painting below is *Roche Corneille*, 1995, by the French artist Geneviève Claisse.



Are there any line segments with the same length?

Are there rectangles with the same dimensions?

A collection of Claisse's work can be found here:

What do you notice about her pieces?

Reflect

The painting in Explore 9.1 is classed as a geometric abstraction painting. The style originated as artists tried to represent their understanding of the underlying structure of reality in terms of pure geometric shapes.

Do these lines and shapes represent the simplest mathematical geometry?

Thinking skills

In geometry, if two line segments have equal length, they are said to be **congruent**.

Fact

If two angles have equal measure, they are said to be congruent.

Explore 9.2

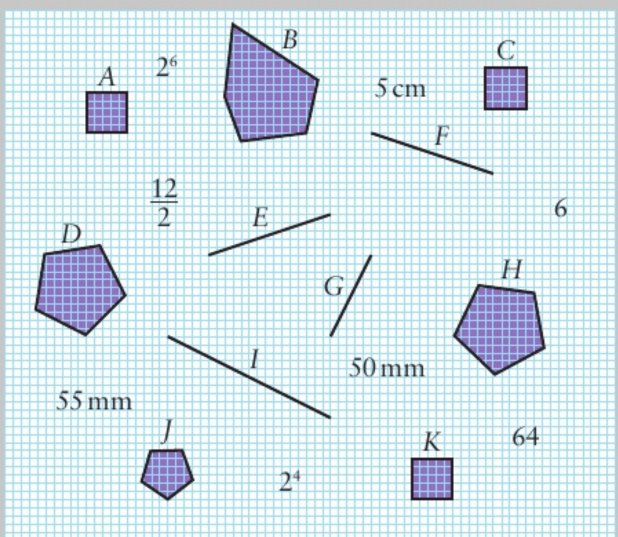
Two line segments can be:

- congruent but not parallel
- congruent and parallel
- parallel and not congruent.

Can you show the three different relationships using a diagram?

Worked example 9.1

Sort the following into sets of equal value, sets of congruent shapes or line segments, and elements that are not a member of any set.



Solution

Understand the problem

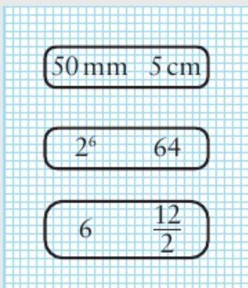
The diagram consists of shapes, lines, numbers and quantities.
We need to identify the individual elements that are equal or congruent.

Make a plan

We will evaluate the quantities and numbers and compare them to find out whether any are equal. We will measure the lines and identify any that are equal in length. We will look at shapes with the same number of sides and measure the side lengths to identify congruent shapes.

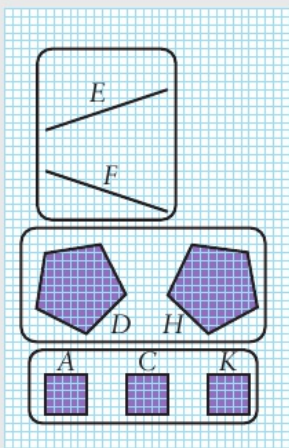
Carry out the plan

Equal quantities



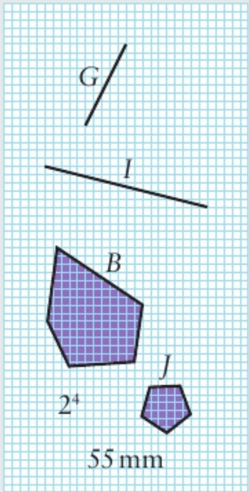
Each of these paired numbers or quantities represents the same amount. They are equal.

Congruent shapes



Each of these grouped shapes or line segments have equal lengths or side lengths. We can use the grid lines to see that the sides have the same length. They are congruent.

Neither



Each of these elements are unique. They are neither equal nor congruent.

Look back

We can review the sorting diagrams to check that the equal quantities and numbers are paired correctly, and that the congruent shapes and line segments are correctly grouped together.

In a **regular polygon**:

- all angles are congruent
- all sides are congruent.

Worked example 9.2

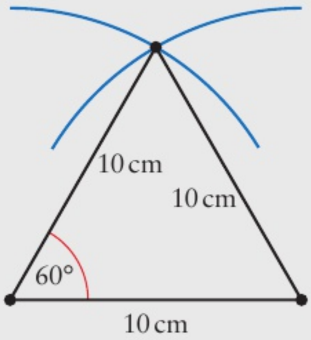
Draw a regular triangle of side 10 cm, and a regular quadrilateral of side 8 cm. Identify the congruent sides.

Solution

Regular triangles and regular quadrilaterals have congruent sides and congruent interior angles. To draw them accurately, sides should be the same length and angles should be of equal measure. A regular triangle has interior angles of 60° and a regular quadrilateral has interior angles of 90°. That is, we have an **equilateral triangle** and a **square**.

For both shapes, the base side can be drawn to a fixed length. The angles can be constructed with the required sizes.

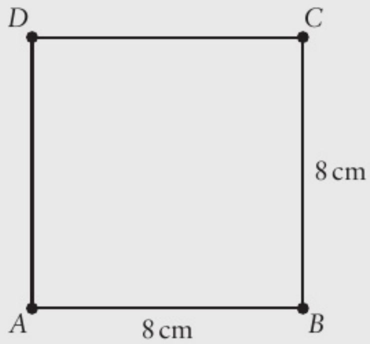
For the triangle



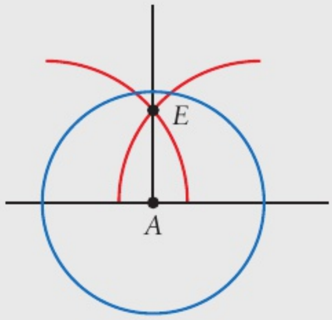
Open the compasses to a 10 cm radius, draw a line with a straight edge, and then mark a segment of length 10 cm.

Now, we have two vertices of the triangle. Since all sides are congruent, the third vertex will lie on two loci, each a circle with centre at one of the vertices and radius of 10 cm. So, with the same radius, and using each of the endpoints as centres, draw two arcs that will intersect at the third vertex of the triangle. In this case the base is 10 cm, therefore the other sides must be 10 cm, and all the angles congruent with a value of 60°.

For the quadrilateral



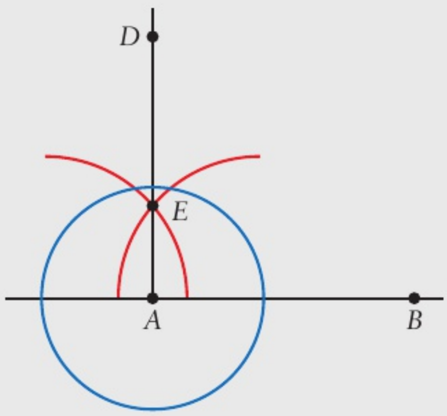
A rough sketch of what we are aiming for can help us decide on an approach. It does not have to be accurate. Let's start with the base $\overline{AB}$, which we can construct as we did for the base of the triangle. Then, at point A, construct a right angle,which is a skill that was covered in Year 3. Here is a reminder:



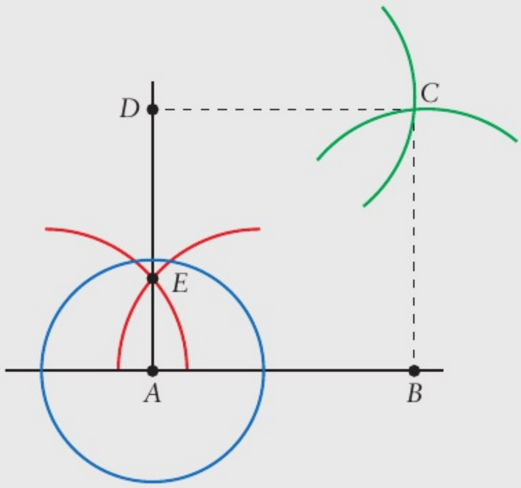
For any radius, and centre A, draw a circle (shown in blue) that intersects the line at two points. Now, with each point of intersection as centre, and a radius larger than before, draw two arcs that intersect at a point. Call this point E. Join AE. Now, the angle at A is a right angle.

Hint

This construction creates a segment and constructs a perpendicular bisector for that segment – Year 3 work.



Next, mark the point D on line AE. AD, measuring 8 cm, gives us the third vertex of the square.



The fourth vertex, C, must lie on two loci: one is a circle with centre D and radius of 8 cm and the other is another circle with centre B and radius 8 cm. The two loci intersect at the point C. The quadrilateral $ABCD$ is a square with a length of 8 cm, as required.

Fact

Equality is the concept of quantities and variables being the same.

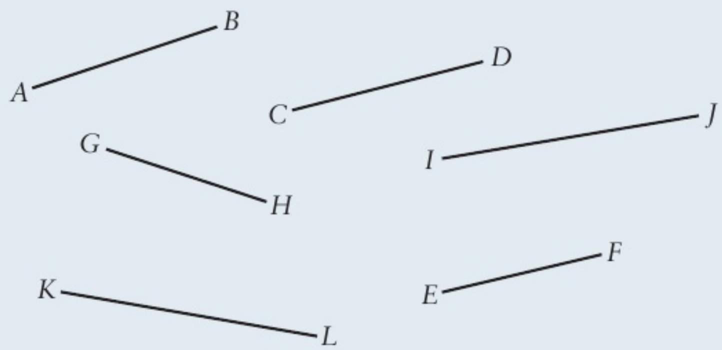
Congruency is the concept of shapes having equal corresponding parts.
If one shape can be overlaid on the other, then the two shapes are congruent.

Fact

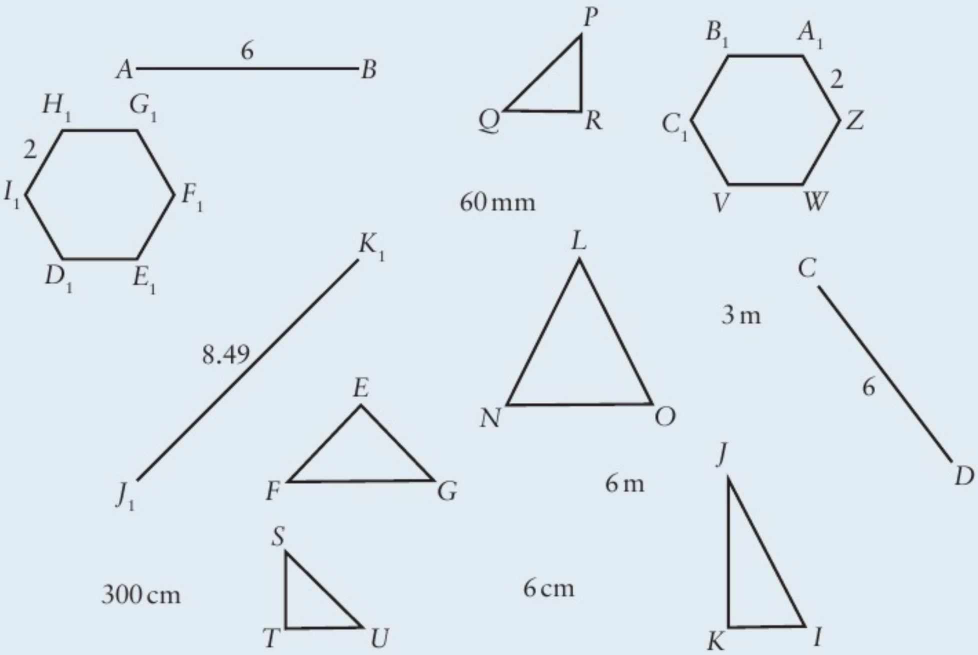
Regular triangles have equal interior angles measuring 60°. They are equilateral triangles. Regular quadrilaterals have equal interior angles measuring 90°. They are squares.

Practice questions 9.1

1 Identify the congruent line segments.



2 Sort the following into equal quantities, congruent shapes, congruent line segments, or neither.



3 Construct two congruent regular quadrilaterals.

4 Construct two congruent scalene triangles.

5

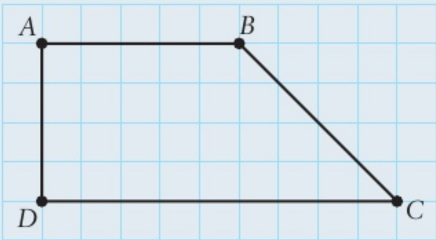
The picture above shows the Art Nouveau detail of the glass dome in the Koruna Palace in Prague.

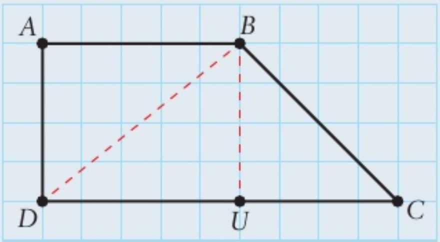
a Can you identify five sets of congruent shapes in the dome? Justify your answer.

b Do you think the use of congruent shapes enhances the design of the structure?

6 The irregular quadrilateral *ABCD* can be divided into three congruent, isosceles, right-angled triangles as shown with the red dotted lines *BD* and *BU*.

Challenge Q6





Can you divide the irregular quadrilateral *ABCD* into four congruent regions?

Reflect

The example in question 5 shows structures that are joined together to give the appearance of a curved surface. If shapes are on a curved surface, does this affect their congruency?

9.2 Congruent line segments

Fact

A line segment is a straight line that has a finite length and defined end points.

To show that two line segments have the same length we use the notation $MK = ML$

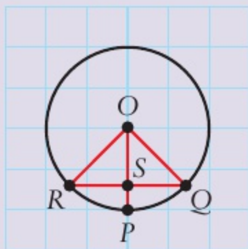
To show that these two lines are congruent, we use the notation $\overline{MK} \cong \overline{ML}$

Explore 9.3

For each of the diagrams, can you identify which line segments are congruent?

Give your answer using the correct mathematical notation.

a



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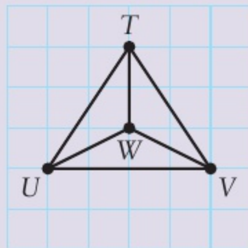


Reminder

Pythagoras' theorem states that $h^2 = a^2 + b^2$, where h is the length of the hypotenuse in a right-angled triangle and a and b are the lengths of the remaining two sides.

Hint

This Investigation could be done using geometry software: Plot a fixed point, then a movable point. Create the midpoint between the two, and then move the movable point around and observe the relationship between the coordinates.

Connections

The midpoint of a line is used in coordinate geometry.

Investigation 9.1

a Plot the points $A(1, 7)$ and $B(5, 3)$ on a coordinate grid and join the points with a straight line to form the line segment AB.

b Work out the length of the line segment.

c Verify your answer by measuring the length of the line segment.

d Work out the midpoint of the line segment and write down its coordinates.

e Verify your answer by measurement.

f What is the connection between the coordinate points of the end points of the line segment and the coordinate points of the midpoint?

Reflect

In the investigation you found the coordinates of the midpoint of a line, which formed two segments of equal length and hence congruent line segments.

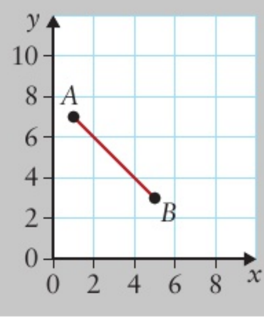
Is there a limit to the accuracy with which we can measure the midpoint of a line?

The midpoint of a line segment divides it into two line segments of equal length. The two line segments are congruent.

Worked example 9.3

The diagram shows the line segment AB drawn on a coordinate grid.

Copy the diagram. Use compasses to construct the perpendicular bisector of the line segment and verify the midpoint you found in Investigation 9.1.



> **Hint**
>
> You can also use GeoGebra or Desmos Geometry to construct perpendicular bisectors.

Solution

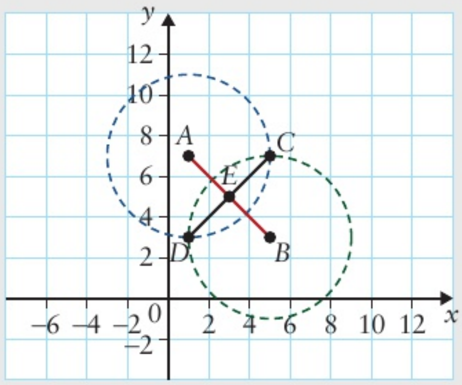
We need to construct the perpendicular bisector of $\overline{AB}$, which will intersect $\overline{AB}$ at the midpoint. The point of intersection can be determined from the diagram.

We can draw the loci of points that are equidistant from the point A by drawing a circle with centre at point A. The length of the radius should be more than half of the length of line segment AB. We can draw the loci of points that are equidistant from the point B in the same way. We must make sure that both circles have the same radius.

The perpendicular bisector is the line that represents the set of points, A and B, that are an equal distance from the two ends of the line segment. This line will go through the points where the two circle loci intersect. We can join these points with a straight line to draw the perpendicular bisector.

We can then read the point of intersection of $\overline{AB}$ with the perpendicular bisector from the diagram. This is the midpoint of $\overline{AB}$.

The diagram shows the two loci circles and a segment of the perpendicular bisector.



The perpendicular bisector intersects $\overline{AB}$ at the point E. This is the midpoint. We can read the coordinates of E from the diagram as (3, 5).

We can check our answer by measuring $\overline{AE}$ and $\overline{EB}$. The two line segments have equal length, so E is the midpoint.

Fact

We can identify congruent line segments by marking them with a small line, |

If there are two sets of congruent lines, the second set should be marked with ||

A third set should be marked with |||

Alternative symbols might also be used.

Reflect

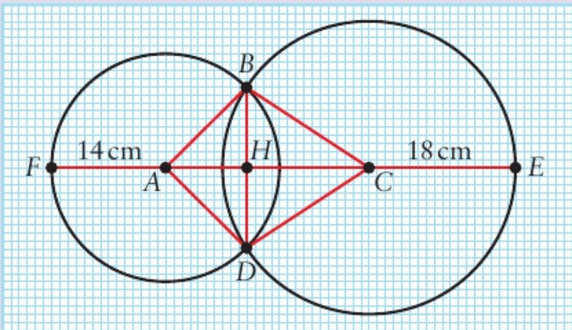
In Worked example 9.3, if the points A and C, and A and D, are joined with line segments, they form radii of the circle around A. These are therefore congruent line segments.

Can you identify any other line segments that are congruent?

Explore 9.4

The diagram shows two intersecting circles.

$\overline{AF}$, which measures 14 cm, is the radius of the smaller circle and $\overline{CE}$, which measures 18 cm is the radius of the larger circle.



Can you identify line segments on the diagram that are congruent? Justify your answers.

Can you use your construction skills to reproduce the diagram in your notebook?

Measure the line segments that you thought were congruent. Have you reproduced the diagram accurately?

Fact

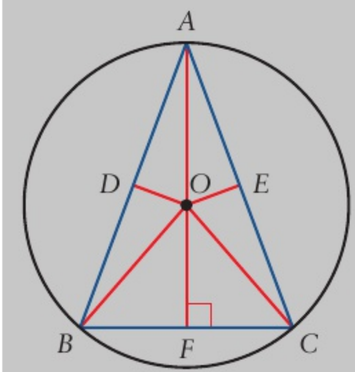
When we construct perpendicular bisectors, we do not need to draw the entire circle every time. We often draw arcs at the points we expect the circles to intersect.

We can find congruent line segments within complex and composite shapes. Some, such as the radius of a circle, are already familiar.

Worked example 9.4

In the diagram, the point O is the centre of the circle. The points A, B and C lie on the circumference and are joined with straight lines to form the triangle ABC.

Point D is the midpoint of $\overline{AB}$, point E is the midpoint of $\overline{AC}$ and point F is the midpoint of $\overline{BC}$.



$\overline{AB}$ and $\overline{AC}$ are of equal length. Identify any line segments on the diagram that are congruent. Justify your answers.

Solution

The diagram shows a triangle inscribed within a circle. We need to look for congruent line segments.

We can use the radii of the circle, the midpoints and the information given in the question to justify our answers.

$\overline{OA}$, $\overline{OC}$, and $\overline{OB}$ are all radii of the circle with centre O, so they are of equal length, therefore $\overline{OA} \cong \overline{OC} \cong \overline{OB}$

$\overline{AB}$ and $\overline{AC}$ are of equal length, so $\overline{AB} \cong \overline{AC}$

D is the midpoint of $\overline{AB}$, so $\overline{AD}$ and $\overline{DB}$ are of equal length. Similarly, E is the midpoint of $\overline{AC}$ and we already know that $\overline{AB} \cong \overline{AC}$.
So $\overline{AD} \cong \overline{DB} \cong \overline{AE} \cong \overline{EC}$

F is the midpoint of $\overline{BC}$, so $\overline{BF} \cong \overline{FC}$

Finally, since $\overline{AB} \cong \overline{AC}$ and $\overline{OA} \cong \overline{OC} \cong \overline{OB}$, we can see that $\overline{DO} \cong \overline{OE}$

Looking back, we have considered each of the lines and identified congruent line segments from the information given.

Fact

A line segment that joins two points on a circumference of a circle is called a **chord**.

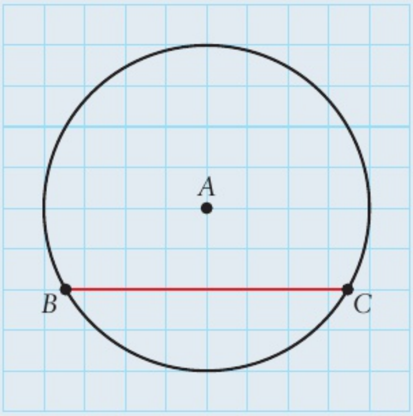
Connections

You will explore these ideas further when you learn about circle theorems.

Practice questions 9.2

1 Plot the points $A(1, 5)$ and $B(6, 2)$ on a coordinate grid and join the points with a line segment. Construct the perpendicular bisector and identify the congruent line segments that are formed.

2 Reproduce the diagram opposite on a sheet of squared paper by constructing a circle with radius 4 cm and centre, A. Place two points, B and C, on the circumference of the circle, both below the centre point, and join the two points with a line segment.



a Using compasses, construct the perpendicular bisector of $\overline{BC}$.

b Construct a triangle by joining the points B and C to point A.

c Measure the line segments that are formed and mark the congruent line segments on your diagram.

3 A circle of radius 3 cm and centre at $A(3, 3)$ intersects a second circle of radius 3 cm and centre at $B(6, 3)$ at two points, C and D. Determine the length of the vertical line segment CD that connects the two intersection points. Give your answer correct to one decimal place.

4 Plot the points $A(2, 6)$, $B(10, 0)$ and $C(10, 10)$ and join the points to form a triangle.

a Using compasses, construct the perpendicular bisector of each of the sides.

b What are the coordinates of the point of intersection, D, of the three perpendicular bisectors?

c Measure the distance from point D to each of the three vertices, A, B and C.

d What do you notice about the values?

5 a You are asked to design your own name badge. On a grid or graph paper, generate your name with congruent line segments. Be creative with your design!

b How many sets of congruent line segments do you have in your name?

Fact

The point of intersection of the three perpendicular bisectors of the sides of a triangle is called the **circumcentre**.

6 A school in Belgium is spread out over three separate sites. When placed on a coordinate grid, the junior school, site A, is at the point (10, 40); the middle school, site B, is at the point (30, 10); and the senior school, site C, is at the point (60, 40).

The school would like to build a swimming pool that can be used by the three sections of the school. Given that the school is built in an area without building restrictions, where would be a reasonable place to build it?

Challenge Q6

9.3 Congruent 2D shapes

9.3.1 Congruent triangles

Explore 9.5

David Bomberg was a British artist who started his artistic career around the time of the Cubist artistic movement, led by Picasso. He painted his abstract painting, Ju Jitsu, as a geometric representation of movement. It is a painting filled with geometric shapes, representing real life with what he classes as 'pure form':

> 'My object is the construction of pure form. I reject everything in painting that is not pure form.'

Do you observe congruency in the painting?

Which aspect of real life do you feel this painting represents?

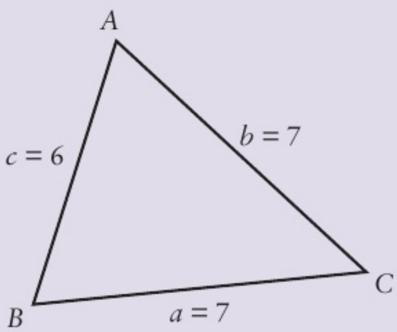
Reflect

If triangles have the same angles and the same side length but they are a different colour, are they congruent?

How can we use mathematical reasoning to determine congruency?

Explore 9.6

Triangle ABC is labelled with its side lengths.



Can a triangle with the same side lengths as triangle ABC have angles of a different size to triangle ABC? Can you draw one?

Try to create a new triangle from the sides of triangle ABC by cutting out each side separately. Can you use the three lines to form a triangle of different angles?

Hint

The diagrams can also be constructed with compasses and straight edges.

Hint

You can also use GeoGebra to explore this problem:

Reflect

If you were given the angles of the triangle instead of the side lengths, would you be able to form a different triangle?

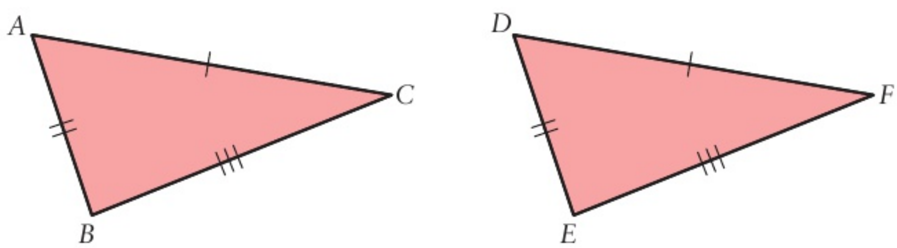
Fact

Strictly speaking, measurements of shapes, such as sides, radii or chords, are considered to be line segments, but when we refer to them as a side, radius or chord, we do not usually use the bar notation.

Fact

This is called the **SSS rule**.

If the length of the sides in a triangle are the same as the lengths of the sides in another triangle, the triangles are congruent.

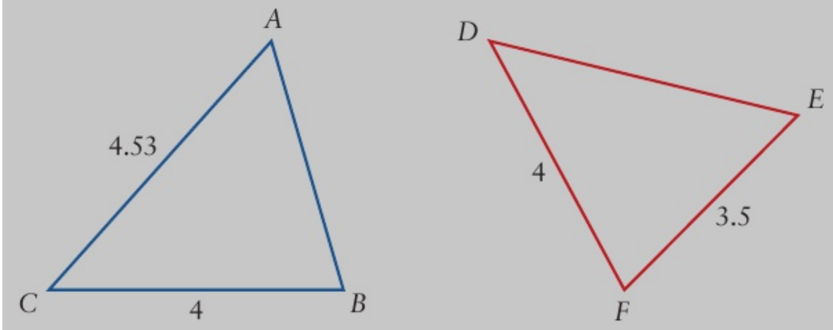


If we know the sides are congruent, we can reason that the triangles are also congruent:

If $AB \cong DE$, $AC \cong DF$, and $BC \cong EF$ then $\triangle ABC \cong \triangle DEF$

Worked example 9.5

The triangles below are congruent.



a Identify the congruent sides.

b Determine the length of the missing side for each triangle.

Solution

As the triangles are congruent, there must be corresponding and equal sides. We can use the information in the diagram to identify the corresponding congruent sides and the length of the missing side for each triangle.

a Both CB and DF have a length of 4 units, therefore $CB \cong DF$

As $CA \neq FE$, and $\triangle ABC \cong \triangle DEF$, we can deduce that $CA \cong DE$ and $AB \cong FE$

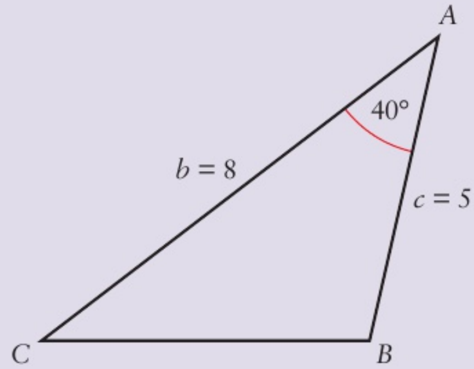
b Given $CA \cong DE$ then $DE = 4.53$

Given $AB \cong FE$ then $AB = 3.5$

Explore 9.7

The triangle ABC is labelled with two given side lengths and their angle of intersection.

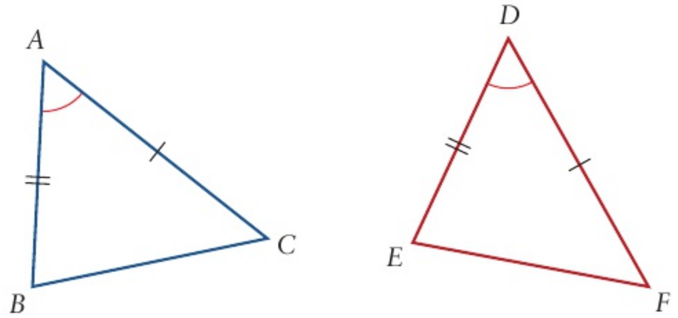
Can a triangle with the same two side lengths, and given angle, as triangle ABC have a different length to CB? Can you draw one?



Hint

You can also use GeoGebra to explore this problem:

If the lengths of two sides in a triangle are equal to the lengths of two sides in another triangle, and the sizes of the angles between the sides are also equal, then the triangles are congruent.



Fact

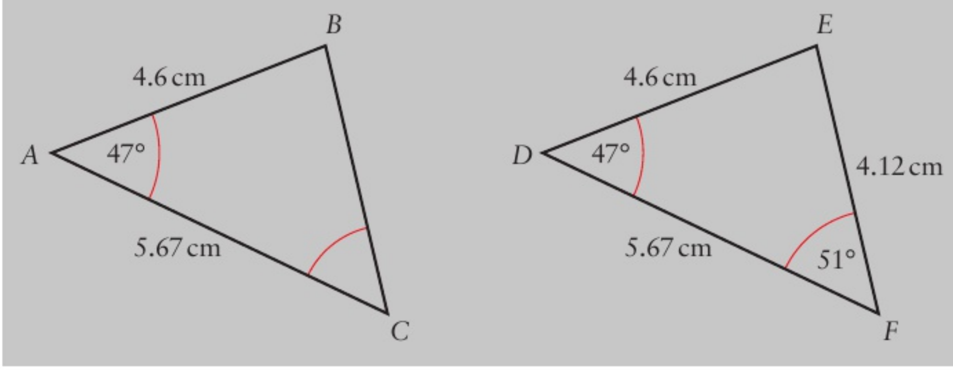
This is called the **SAS rule**.

If we know the sides are congruent, and we know the angles between them are also congruent, then the triangles are congruent.

If $AB \cong DE$, $AC \cong DF$, and $\sphericalangle BAC \cong \sphericalangle EDF$ then $\triangle ABC \cong \triangle DEF$

Worked example 9.6

Show that $\triangle ABC \cong \triangle DEF$ and hence find the length of BC and the size of angle ABC.



Solution

The diagrams have two congruent sides, and a congruent angle between them, so we can use the SAS rule to show that they are congruent. We can then use the congruency of the triangles to find the required angle.

$\sphericalangle BAC \cong \sphericalangle EDF$, $AC \cong DF$ and $AB \cong DE$, so $\triangle ABC \cong \triangle DEF$ (SAS)

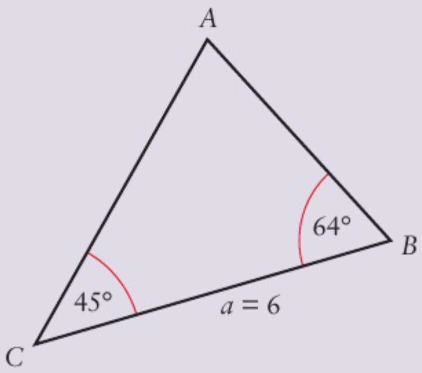
This means that $BC \cong EF$, therefore $BC = 4.12$ cm

$\sphericalangle ABC \cong \sphericalangle DEF$ and $\sphericalangle DEF = 180 - (47 + 51) = 82°$
Therefore $\sphericalangle ABC = 82°$

Looking back, we can see that the larger angle is opposite the longer side, which helps confirm that our answer is correct.

Explore 9.8

The triangle *ABC* is labelled with two angles and a side length between them.

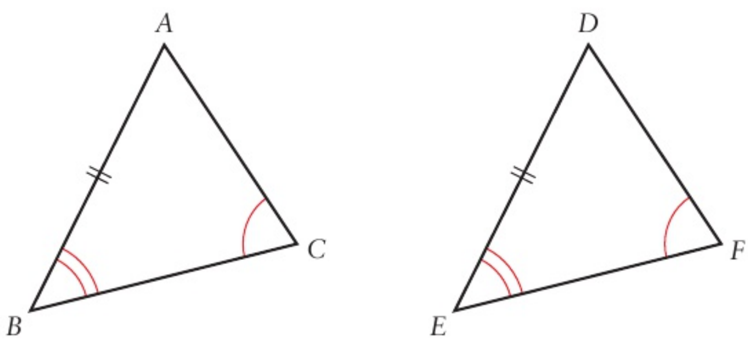


Can you draw a triangle with the same angles and side length, *a*, but with *AC* or *AB* a different length?

Hint

You can also use GeoGebra to explore this problem:

If the size of two angles in a triangle are equal to the size of the corresponding angles in another triangle, and a corresponding side is also equal in length, then the triangles are congruent.



If we know the angles are congruent, and we know two corresponding sides are also congruent, then the triangles are congruent.

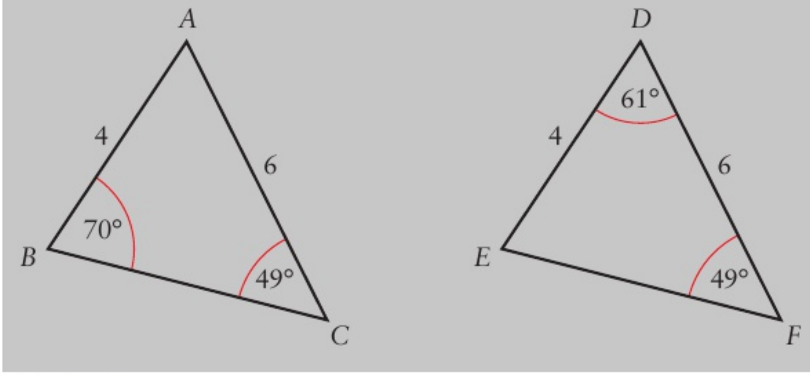
If $\measuredangle ABC \cong \measuredangle DEF$, $\measuredangle ACB \cong \measuredangle DFE$ and $AB \cong DE$, then $\triangle ABC \cong \triangle DEF$

Fact

This is called the **AAS rule**. This is also known as ASA because if two angles in one triangle are congruent to two angles in another triangle, the remaining angles are also congruent.

Worked example 9.7

Are these two triangles congruent? Justify your answer with mathematical reasoning.



Solution

The diagrams show two triangles but not all the sides are labelled. There are two known angles in each triangle, but one of them is not a corresponding angle. If we can determine whether all the angles are the same, then we can assess for a corresponding side.

For triangle ABC, the missing angle is angle BAC.

$$\sphericalangle BAC = 180 - (70 + 49) = 61°$$

For triangle DEF, the missing angle is angle DEF.

$$\sphericalangle DEF = 180 - (61 + 49) = 70°$$

The two triangles therefore have the same three angles. This identifies AB and DE as corresponding sides, AC and DF as corresponding sides, and BC and EF as corresponding sides.

We can use the following mathematical reasoning to confirm that the two triangles are congruent.

$\sphericalangle ABC \cong \sphericalangle DEF$, $\sphericalangle ACB \cong \sphericalangle DFE$ and $AB \cong DE$, then $\triangle ABC \cong \triangle DEF$ (AAS)

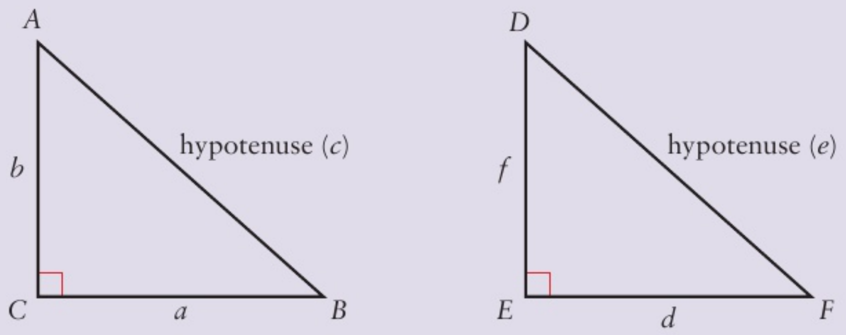
The only missing value is the length BC. However, this is not required to determine congruency. It is clear from the AAS rule that the value will be the same for both triangles.

Reflect

If the side lengths that have equal values are not corresponding sides, are the triangles congruent?

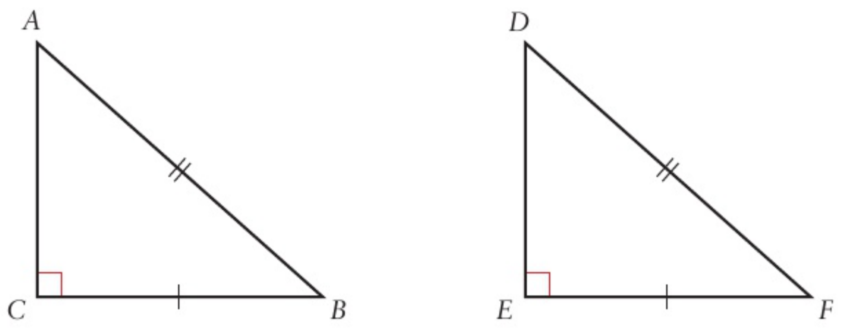
Explore 9.9

The diagram shows two right-angled triangles.



Can you justify why two right-angled triangles with equal hypotenuse values and a second equal corresponding side will always be congruent?

If the hypotenuses in two right-angled triangles are congruent, and a pair of corresponding sides are congruent, then the triangles are congruent.



Fact

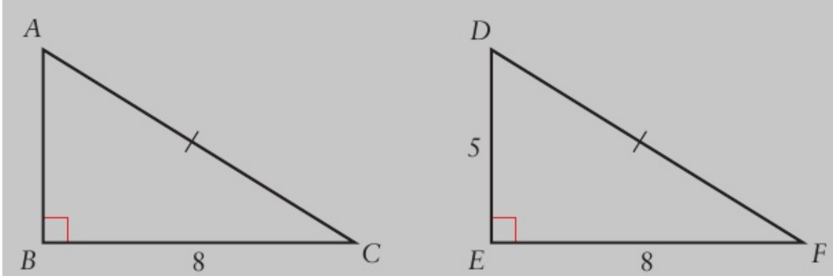
This is called the **RHS rule.** It is also known as the **HL** (hypotenuse–leg) **rule.**

Reflect

Why is this rule only true for right-angled triangles?

Worked example 9.8

$\sphericalangle ABC$ and $\sphericalangle DEF$ are right angles. Show that the two triangles are congruent and determine the lengths of sides AB and AC. Give your answer correct to two decimal places where necessary.



Solution

The two triangles are right-angled and there are two sides that are also of equal length. If the triangles are congruent, we can use corresponding sides to determine the length of side AB. We can find the hypotenuse, AC, using Pythagoras' theorem.

The two triangles are right-angled, the hypotenuses are of equal length and $BC \cong EF$. Therefore, by the RHS rule, the two triangles are congruent.

As the two triangles are congruent, $AB \cong DE$. Therefore $AB = 5$

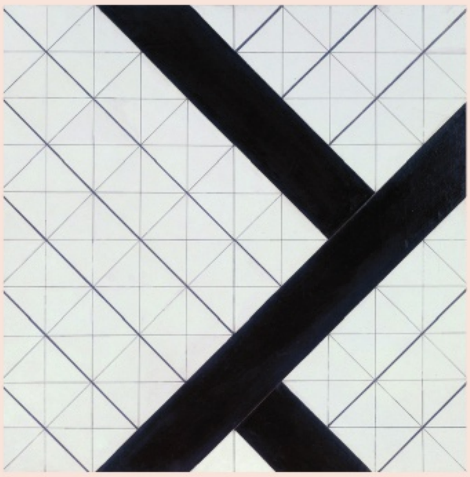
By Pythagoras's theorem:

$$\begin{aligned} AC &= \sqrt{AB^2 + BC^2} \\ &= \sqrt{5^2 + 8^2} \\ &= 9.43 \text{ (2 d.p.)} \end{aligned}$$

Looking back, we see that the length of AC is greater than the length of the other sides of the triangles. This is consistent with the side being the hypotenuse.

Investigation 9.2

The image is a print of the artwork *Counter Composition VI* by Theo van Doesburg. The background is a grid of congruent right-angled triangles.



A background of congruent equilateral triangles, called isometric paper, is often used in mathematics. Using isometric paper, or GeoGebra, create your own geometric abstraction design incorporating congruent line segments and triangles.

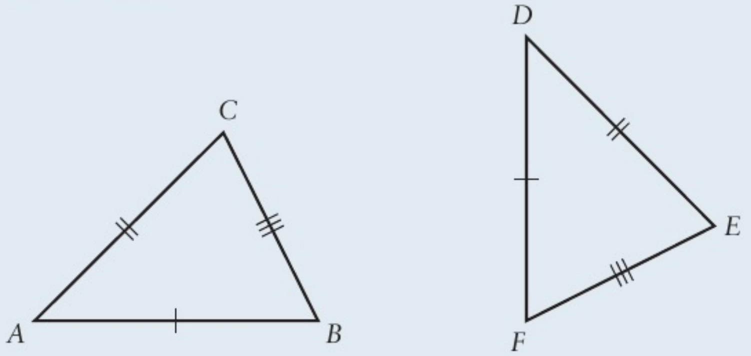
Hint

Isometric paper can be found as a background on GeoGebra. A paper version can be downloaded here:

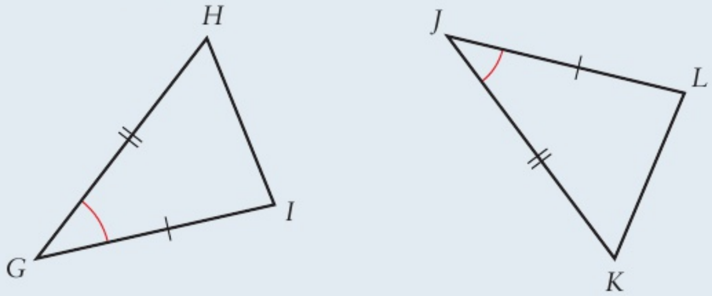
Practice questions 9.3.1

1 Use mathematical reasoning to show that:

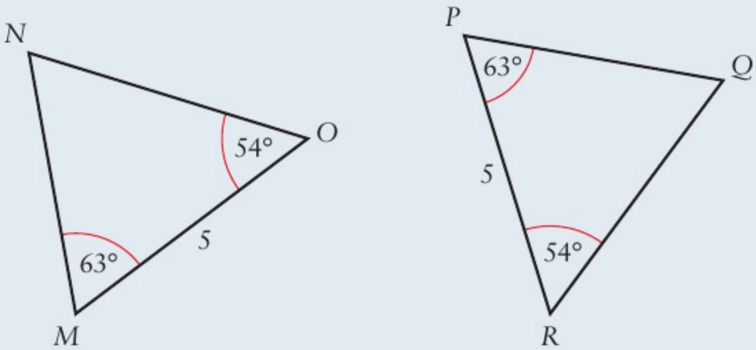
a $\triangle ABC \cong \triangle DEF$



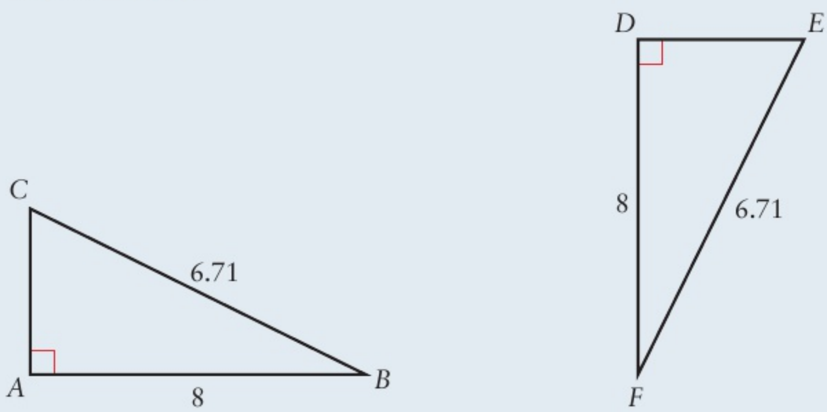
b $\triangle GHI \cong \triangle JKL$



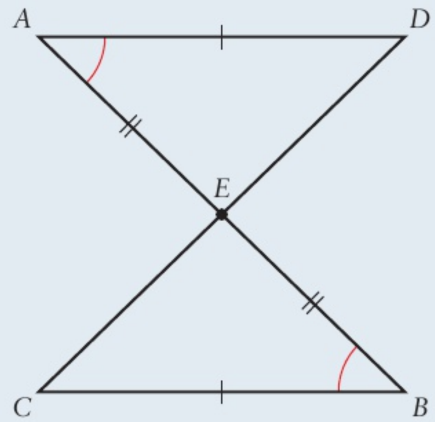
c $\triangle MNO \cong \triangle PQR$



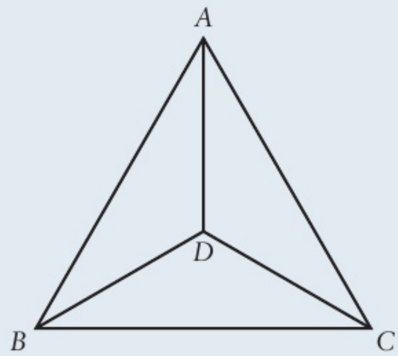
d $\triangle ACB \cong \triangle DEF$



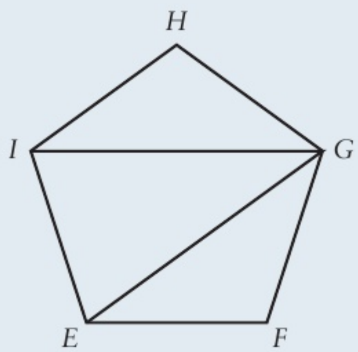
2 Use mathematical reasoning to show that triangles AED and BEC are congruent.



3 Triangle ABC is an equilateral triangle and point D is the circumcentre of $\triangle ABC$. Show that $\triangle ABD \cong \triangle ADC \cong \triangle BDC$



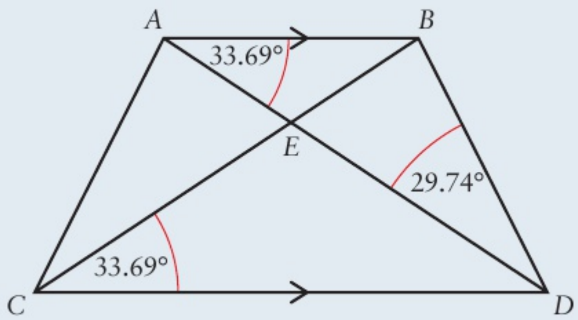
4 $EFGHI$ is a regular pentagon.



a Show that $\triangle IHG \cong \triangle GFE$

b Is $\triangle IGE \cong \triangle GFE$? Explain your reasoning.

5



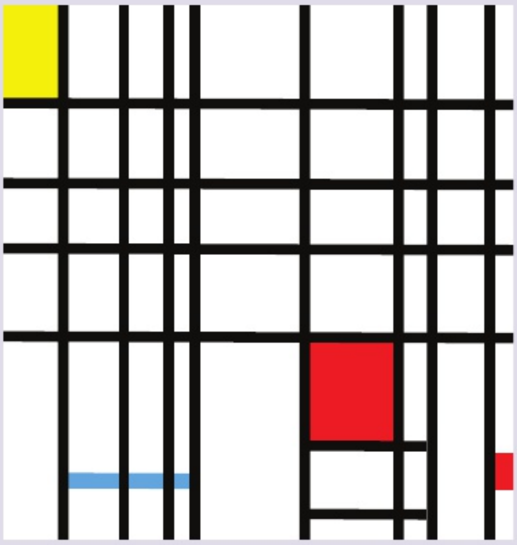
a Show that $\triangle AEC \cong \triangle BED$

b Find the size of angle AEC

9.3.2 Congruent quadrilaterals

Explore 9.10

At the start of this chapter there is a picture of the work by Piet Mondrian, Composition with Red, Blue and Yellow, 1930. Below is another work by the same artist:



Can you identify any differences between the work, *Composition with Red, Blue and Yellow*, 1930 and this work, *Composition with Yellow, Blue and Red* 1937–42?

What do you think Mondrian is trying to express with this work?

Does the painting contain congruent rectangles?

Reflect

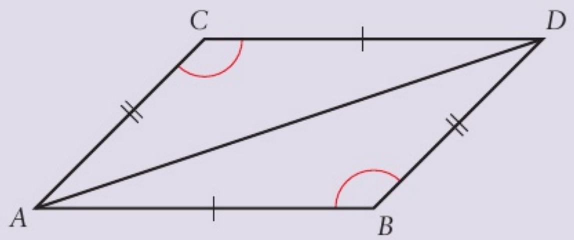
The artwork in Explore 9.10 represents part of the process of Mondrian's move away from representing the world as 'real' to 'abstract' form.
The progression of his ideas through his art can be seen here:

Why do the De Stijl movement artists class mathematical geometry as the fundamental form?

Explore 9.11

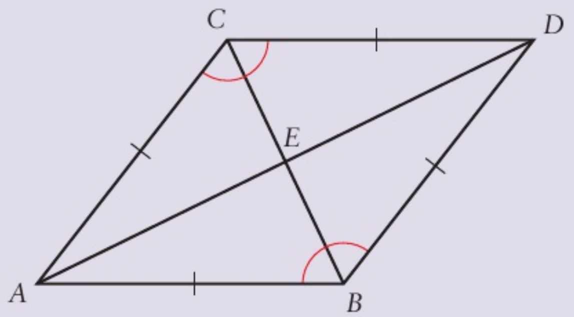
a The diagram below represents a parallelogram divided into two triangles.

Can you use the congruency laws for triangles to summarise its properties?



b The diagram below represents a rhombus divided into four triangles.

Can you use the congruency laws for triangles to summarise its unique properties?



Worked example 9.9

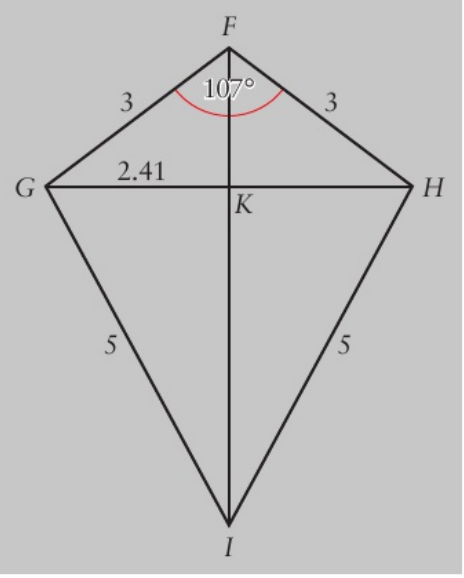
In the diagram opposite, $\measuredangle GFH = 107°$, $GF \cong FH$, $GI \cong HI$ and $GK = 2.41$

By using the properties of congruent triangles find:

a the length of FI and GH

b the size of angle GIH.

Give your answers correct to two decimal places.



Solution

The diagram shows a quadrilateral divided into four triangles. We can use the properties of congruent triangles to prove that the angles around the point K are right angles. If the triangles are right-angled, then we can use trigonometric functions to determine the size of the angles.

Use the congruent triangle properties to reason that the angles around the point K are right angles.

Use the proved congruency to find the length of GH.

Use Pythagoras's theorem to determine the lengths of FK and KI, and hence FI.

Use the trigonometric ratios to find the size of angle $\sphericalangle GIH$

a $GI \cong HI$, $GF \cong FH$ and FI forms a side in both triangles, so $\triangle FGI \cong \triangle FHI$ by SSS.

GK and KH go from corresponding points on congruent triangles to a common point on both triangles. Therefore $GK \cong KH$ and $\triangle FKG \cong \triangle FKH$ by SSS. Hence $\sphericalangle FKG \cong \sphericalangle FKH$ and, as these angles lie on a straight line, $\sphericalangle FKG = \sphericalangle FKH = 90°$

By the same argument on the lower triangles, $\triangle GKI \cong \triangle HKI$ and $\sphericalangle GKI = \sphericalangle HKI = 90°$

We can therefore calculate the length of GH as:

$$\begin{aligned} GH &= 2 \times 2.41 \\ &= 4.82 \end{aligned}$$

$FI = FK + KI$ Using Pythagoras' theorem:

$FK = \sqrt{3^2 - 2.41^2}$ and $KI = \sqrt{5^2 - 2.41^2}$

So:

$$\begin{aligned} FI &= \sqrt{3^2 - 2.41^2} + \sqrt{5^2 - 2.41^2} \\ &= 6.17 \text{ (2 d.p.)} \end{aligned}$$

b $\sphericalangle GIH = \sphericalangle GIK + \sphericalangle KIH$ and $\sphericalangle GIK \cong \sphericalangle KIH$, so $\sphericalangle GIH = \sphericalangle 2GIK$

Using trigonometric ratios:

$$\begin{aligned} \sin\sphericalangle GIK &= \frac{\text{length of the opposite side}}{\text{length of the hypotenuse}} \\ &= \frac{2.41}{5} \\ \sphericalangle GIK &= \arcsin\left(\frac{2.41}{5}\right) \\ &= 28.82...° \\ \sphericalangle GIH &= 2 \times 28.82... = 57.63° \text{ (2 d.p.)} \end{aligned}$$

Investigation 9.3

The Mondrian Squares Riddle

Can you cover a 4 cm × 4 cm square with non-overlapping rectangles?

The rectangles must not be congruent.

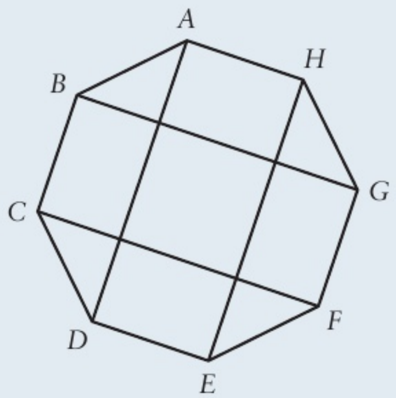
Your score is the area of the largest rectangle minus the area of the smallest.

The challenge is to gain as low a score as possible.

Does the score depend on the size of the square?

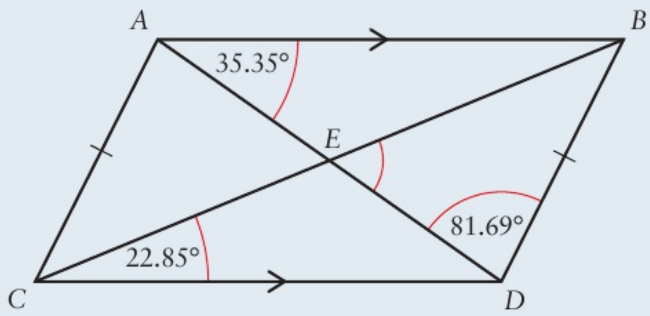
Practice questions 9.3.2

1 The regular octagon *ABCDEFGH* has two rectangles formed by joining points *AHED* and *BGFC*.



Show that $AHED \cong BGFC$

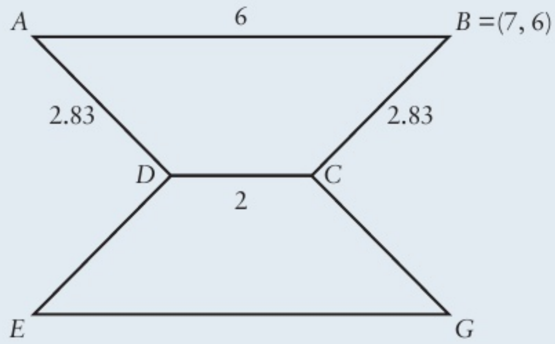
2 The quadrilateral *ABDC* has parallel sides *AB* and *CD*.



a Determine the size of angle *BED*

b Determine the size of angle *ACE*

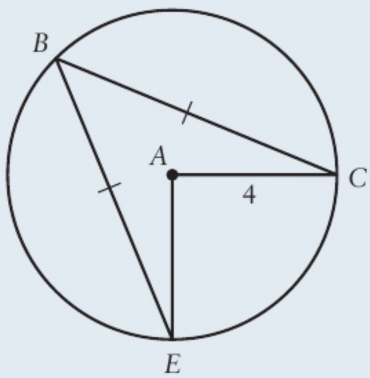
3 The trapezium $ABCD$ is reflected in DC to form the image $EGCD$.
$AB = 6$, $CB = DA = 2.83$ and $DC = 2$



a Determine the coordinates of point E.

b Calculate the angle ADC.

4 In the quadrilateral $BCAE$, the point A is the centre of a circle with radius 4. The points C, B and E lie on the circumference of the circle.



a Show that $\sphericalangle BEA \cong \sphericalangle BCA$

b Given that $\sphericalangle EAC$ measures 93.16° calculate $\sphericalangle BEA$

Connections

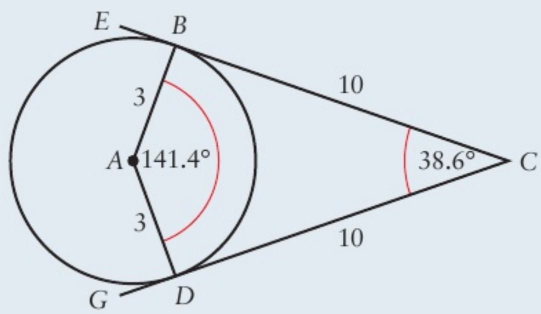
You will explore these ideas further when you study circle theorems.

5 When a line touches the circumference of a circle at exactly one point, it is called a tangent.

In the diagram, the line EC is a tangent to the curve at point B.
$BC = 10$

The line GC is a tangent to the curve at the point D. $DC = 10$

Both tangents meet at the point C.



By drawing the line segment joining A and C, show that angles ABC and ADC, formed between the tangents and the radius, are 90°.

6 The example in question 5 shows that the angle formed at the point where the tangent meets the radius is 90°. Is this always true?

Construct two other circles with different radii. Draw the tangent to the curve at two points and measure the angle formed between the radii and the tangents.

Challenge Q7

7 For the diagram given in question 5, find the area enclosed by the quadrilateral *ABCD* that lies outside of the circle.

Hint Q8

Here is an animation of the proof:

8 Pythagoras' theorem can be proved in a number of ways, one of which is a geometric proof by Henry Perigal.

Can you repeat the proof using your understanding of congruent quadrilaterals?

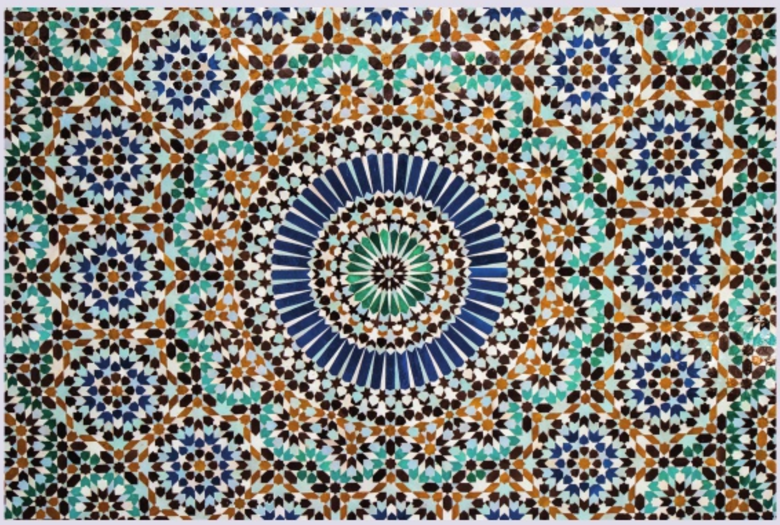
Connections

Pythagoras' theorem and geometric proof.

9.3.3 Congruent circles

Explore 9.12

Many cultures incorporate congruent circles in their designs, and many use a mixture of congruent shapes to generate beautiful symmetries.



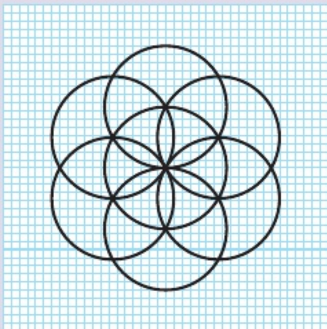
Hint

For a GeoGebra animation of how to draw the Seed of Life follow the link here and press the play button at the bottom of the screen:

Islamic art has many examples which are based on the simple design of the Seed of Life, a pattern generated from seven congruent circles.

The Seed of Life can be constructed by using compasses.

Can you explain the steps required to draw this basic template?



By generating congruent shapes using the points of intersection created, design your own version of the Seed of Life.

How many congruent shapes can you generate?

Are there congruent shapes other than triangles, rectangles and circles?

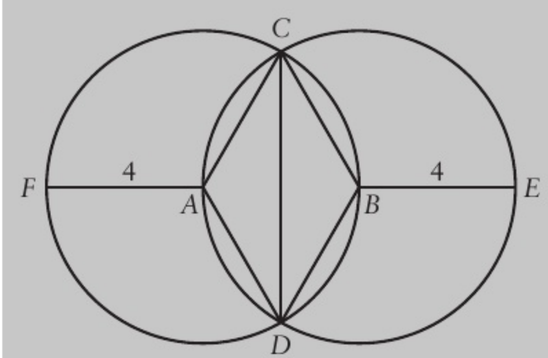
Hint

For a GeoGebra animation of an example, follow the link here and press the play button at the bottom of the screen:

Thinking skills

Worked example 9.10

In the diagram below, the vesica piscis is formed by two congruent circles with radii of 4 cm.



By using the congruent properties of the circles and triangles:

a calculate the length of $\overline{CD}$

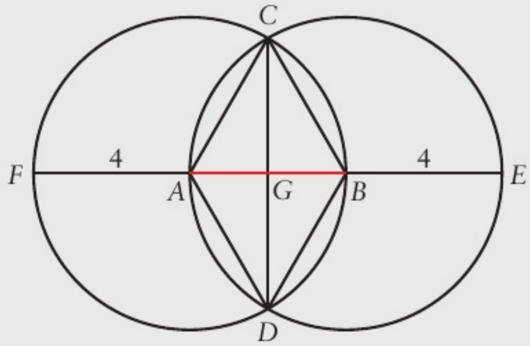
b determine the size of angle CAD.

Fact

The vesica piscis is a shape formed by the intersection of two circles with the same radius, where the centre of each circle lies on the perimeter of the other.

Solution

a By adding $\overline{AB}$ to the diagram, we form an additional point, G.



$\overline{CD}$ is the perpendicular bisector of $\overline{AB}$. We can prove this by noting that points C and D are an equal distance from the respective endpoints of $\overline{AB}$, since they lie on the intersections of the congruent circles.

As $\overline{CD}$ is the perpendicular bisector, by definition the angles formed at G are all right angles and $\overline{AG} \cong \overline{GB}$

$\overline{AC}$, $\overline{CB}$, $\overline{AD}$ and $\overline{DB}$ are all equal to the length of the radii, so the four triangles formed in the vesica piscis are congruent.

The points A and B are the centre points of the two circles and each lies on the circumference of the other circle. This means the length of $\overline{AB}$ is also equal to the radii of both circles, and the length of $\overline{AG}$ is equal to half the radius, so $AG = 2$

We can now calculate the length of $\overline{CD}$ using Pythagoras' theorem, and realising that $CD = 2CG$

$$CG^2 = AC^2 - AG^2$$

$$= 4^2 - 2^2$$

$$CG = \sqrt{12}$$

$$= 2\sqrt{3}$$

Therefore,

$$CD = 2(2\sqrt{3})$$

$$= 4\sqrt{3}$$

b By congruency reasoning:

The triangle ACB is an equilateral triangle since $\overline{AC} \cong \overline{CB} \cong \overline{AB}$
Therefore $\sphericalangle CAB \cong \sphericalangle ABC \cong \sphericalangle BCA$ and hence $\sphericalangle CAG = 60°$ and $\sphericalangle CAD = 120°$

By using trigonometric ratios:

Given that $\sphericalangle CAG \cong \sphericalangle DAG$, then $\sphericalangle CAD = 2\sphericalangle CAG$

we can use the cosine ratio:

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

So,

$$\sphericalangle CAG = \arccos\left(\frac{2}{4}\right)$$
$$= 60°$$

hence

$$\sphericalangle CAD = 120°$$

Looking back, the question asked for the length of the perpendicular bisector CD. The calculated length is within a reasonable range for the given lengths. The justification for the congruency of the triangles is complete and reasoned. The size of the angle has been found in two ways: by using congruency reasoning and verified by trigonometry.

Investigation 9.4

Thinking skills

In Worked example 9.10, the length of the perpendicular bisector CD that joins the points of intersection of the circles forming the vesica piscis was found to be:

$$CD = 4\sqrt{3}$$

Repeat the calculation for congruent circles with the following radii. Leave your answer in the form $a\sqrt{b}$, where $a \in \mathbb{Q}$ and $b \in \mathbb{Z}^+$

a $r = 6$

b $r = 8$

c $r = 10$

What do you notice about the answers?

Does the pattern also work with odd numbers?

Can you generalise the result for congruent circles with a radius length of r?

Can you prove the result for all values of r?

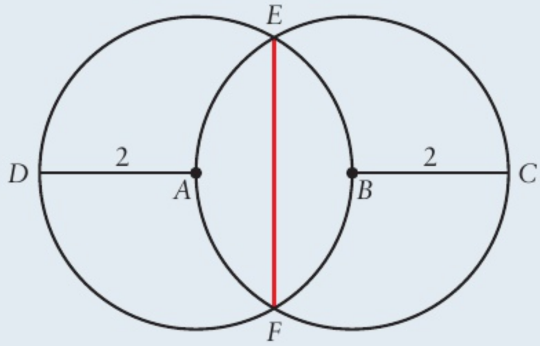
Reflect

The pattern found in Investigation 9.4 works for congruent circles.
Will there be a similar generalisation for non-congruent circles?

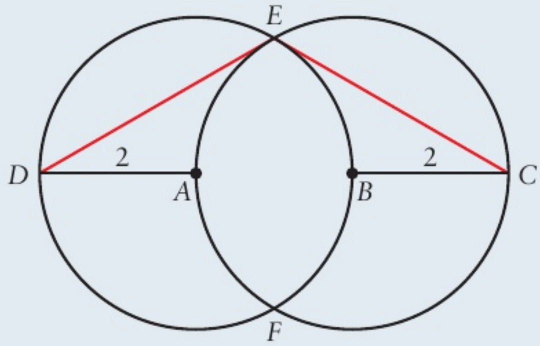
Practice questions 9.3.3

1 The diagram shows two congruent circles with centre points A and B and radii measuring 2 cm. Points E and F are the points of intersection of the two circles.

Find the length of $\overline{EF}$.



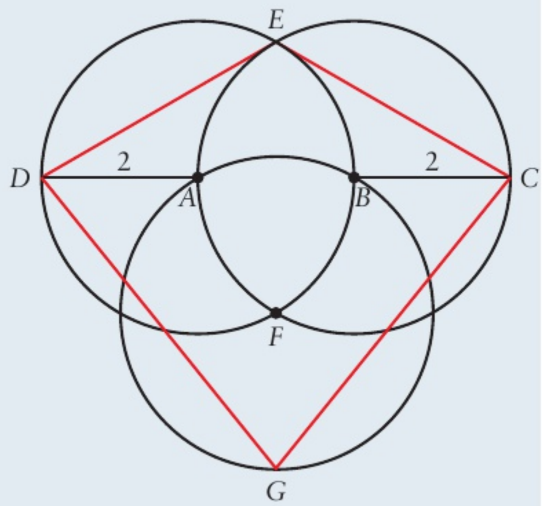
2 The circles from question 1 are shown again here. $\overline{DA}$ and $\overline{BC}$ are horizontal. Find the lengths of $\overline{ED}$ and $\overline{EC}$.



3 A third congruent circle is added to the diagram from questions 1–2 and is shown here.

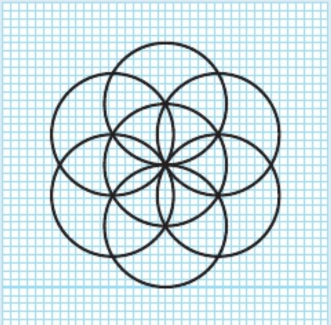
$\overline{FG}$ is a vertical radius and is an extension of $\overline{EF}$ from question 1.

Determine the perimeter of the quadrilateral, $ECGD$.

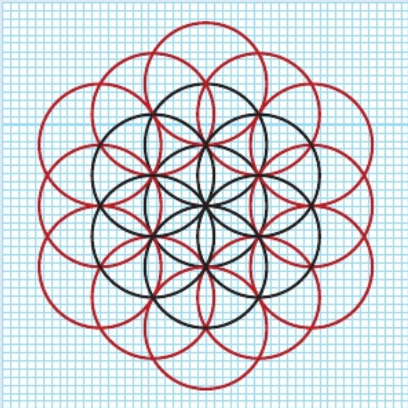


4 Explore 9.12 showed The Seed of Life as a common pattern found within many cultural drawings. The basic pattern has one central circle and a ring of six congruent circles.

The pattern can be extended by repeating the process.



Seed of Life



Seed of Life extended with another ring

We can see this as a sequence where:

- the first circle is pattern 1
- the addition of the first ring of 6 congruent circles is pattern 2
- the pattern created with the additional ring of circles is pattern 3.

and explore the total number of circles in the pattern as rings of congruent circles are added.

a How many circles are in the outer ring of pattern 4?

b How many circles are in the outer ring of pattern 5?

c What is the general formula for the number of circles in the outer ring of the nth pattern?

d Verify that your general formula works by predicting the number of circles in the outer ring for pattern 6 and checking by drawing the pattern.

5 In question 4 you found a general formula for the number of circles added to the outer ring of the nth pattern of the Seed of Life. Can you determine the general formula for the total number of circles within the nth pattern?

Hint Q4

Using GeoGebra will help you to generate the patterns for question 4 much more efficiently.

Thinking skills

Connections

Sequences.

Challenge Q5

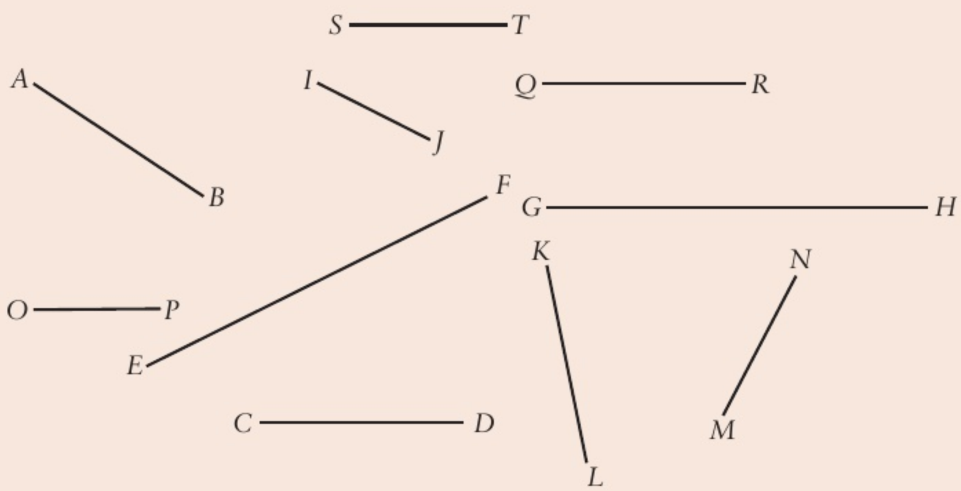
9 Congruency

Self assessment

- I can understand the difference between equality and congruency.
- I can identify lines that are congruent.
- I can identify shapes that are congruent.
- I can use the correct mathematical notation to describe congruent line segments, shapes and angles.
- I can identify congruent shapes within complex diagrams.
- I understand that the perpendicular bisector of a line segment generates two congruent line segments with equal length.
- I understand that two or more radii in the same circle are congruent line segments.
- I can find the circumcentre of a triangle.
- I understand that the circumcentre of a triangle is the point that is equidistant from each of the vertex points.
- I understand that the line segments joining the circumcentre to the vertex points are congruent.
- I can identify the four conditions that can be used to confirm triangle congruency.
- I can use the congruency rules to reason that triangles are congruent.
- I can use the concept of congruency to identify missing values in complex diagrams.
- I can identify patterns within congruent geometric structures.

Check your knowledge questions

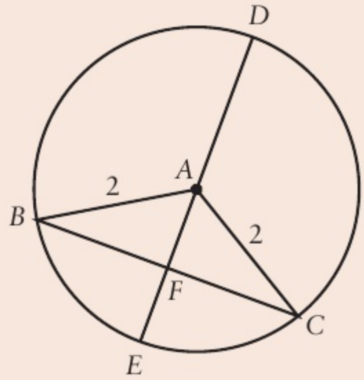
1 For the following examples, identify the congruent line segments.



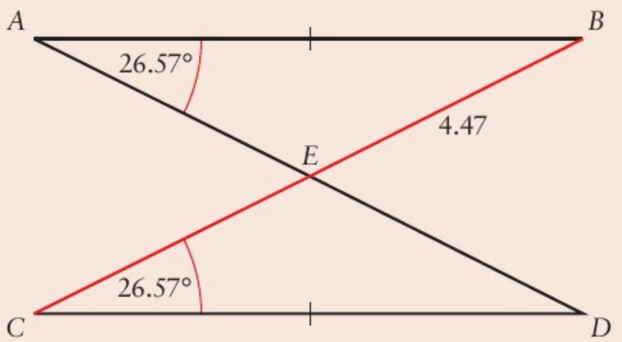
2 Organise the shapes in the diagram into congruent groups.

3 The diagram below is a circle with centre point A, radius 2, and two points on the circumference at B and C. $\overline{DE}$ is the perpendicular bisector of $\overline{BC}$.

Show that triangles ABF and ACF are congruent.



4 In the diagram below, $\overline{AD}$ and $\overline{BC}$ intersect.
$CB = 4.47$ and $\overline{AB} \cong \overline{CD}$



a Show that the triangles ABE and CDE are congruent.

b Find the length of $\overline{AE}$.

c Find the size of angle CED.

5 Plot the points $A(3, 8)$, $B(2, 3)$ and $C(8, 3)$ on a coordinate grid and join the coordinate points with line segments to form a triangle.

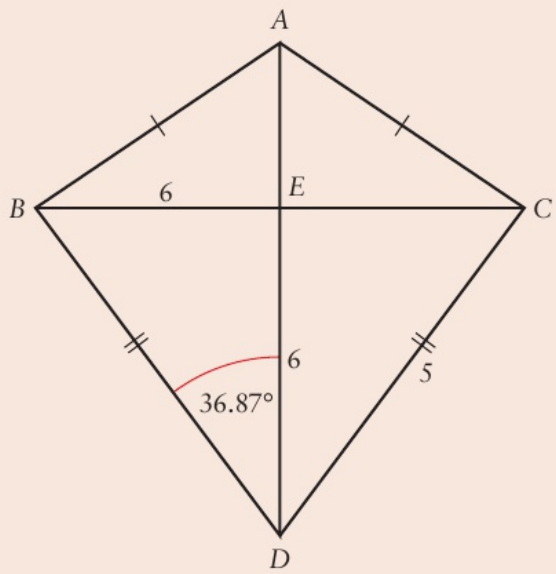
a Construct the perpendicular bisectors for the three sides on the same diagram.

b Identify the circumcentre of the triangle. Label it D.

c Measure the length of the congruent sides, AD, DC and DB.

d Verify that AD, DC and DB are congruent by constructing a circle through the vertices of the triangle with centre point D.

6 In the quadrilateral $ACDB$, $AB \cong AC$, $BD \cong CD$, $BC = AD = 6$ and $\measuredangle BDA = 36.87°$



a State the length of BE.

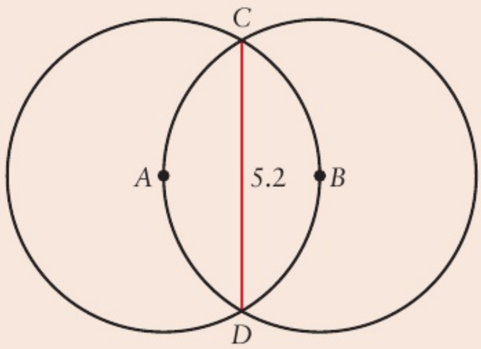
b State the length of BD.

c Calculate the length of AE.

d Calculate the length of AB.

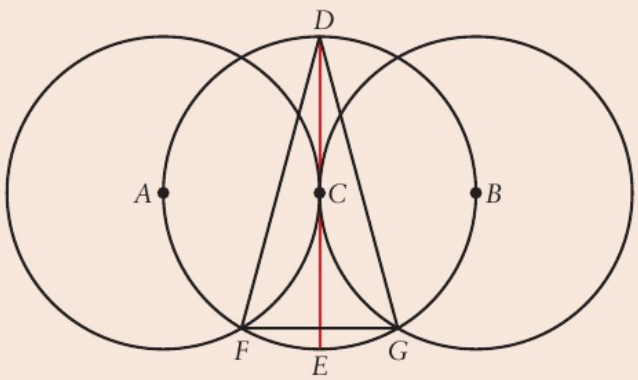
e Find the size of angle ECD.

7 Two congruent circles with radius r intersect at the points C and D. The centre point of the first circle is A, the centre point of the second circle is B, and B lies on the circumference of the first circle. The length of CD is 5.2.



What is the length of the radii of the two circles, r?

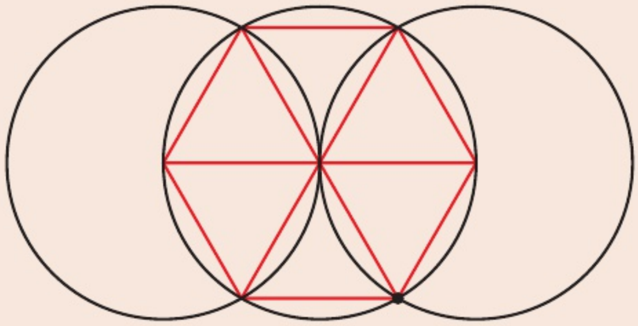
8 The diagram shows three congruent circles with radius length 3.5. The centre points of the three circles lie along a horizontal line. The centre of the third circle, C, lies at the point where the other two circles touch.



a Find the length of FG.

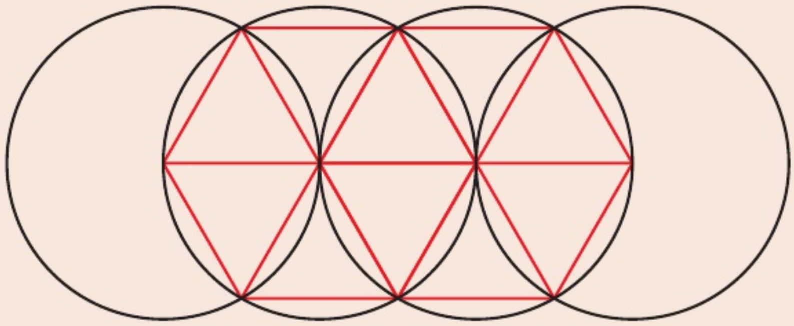
b Find the length of DF.

9 When three congruent circles are drawn as shown in Pattern below, four points of intersection are formed and six congruent equilateral triangles are formed.



Pattern 1

a When an additional circle is drawn, as in Pattern 2, what is the number of congruent equilateral triangles?

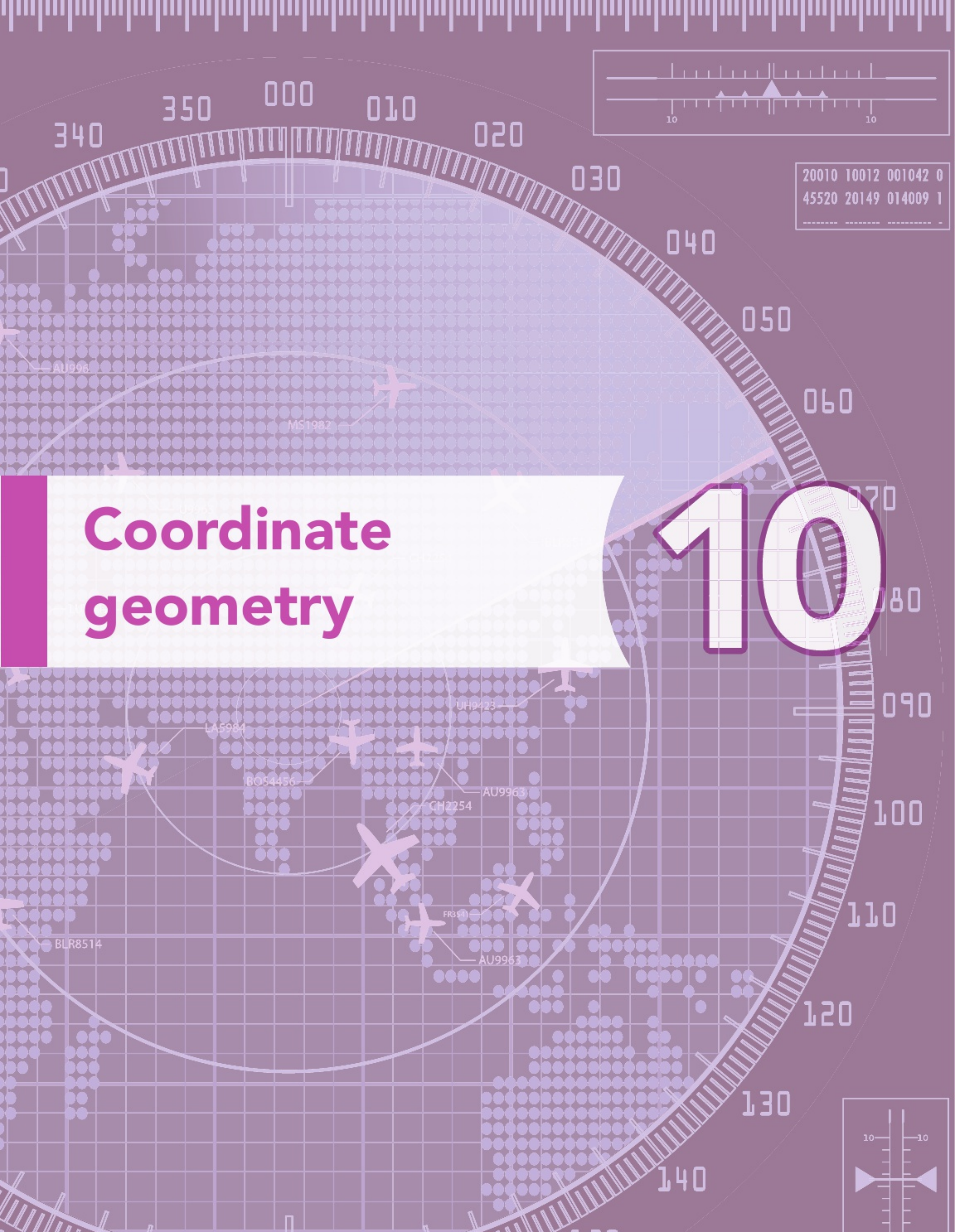


Pattern 2

b How many congruent equilateral triangles will be in the next pattern in the sequence?

c How many congruent equilateral triangles will be in Pattern n?

d How many congruent equilateral triangles will there be with n circles?



10 Coordinate geometry

10 Coordinate geometry

KEY CONCEPT

Logic

RELATED CONCEPTS

Change, Patterns, Representation, Systems

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Logical algebraic and geometric systems can be used to represent changes to the orientation of objects in space.

Factual

- How can you determine distance, midpoint and length in the coordinate plane?
- What is the connection between gradient and perpendicular lines?

Conceptual

- How are properties used to classify shapes?
- How are the patterns for determining perimeter and area of shapes in the coordinate plane related to their formulae?

Debatable

- Is there a best order for performing multiple transformations in the plane?

Do you recall?

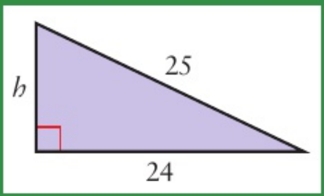
1 What is the average of 5 and −3?

2 What is the gradient of the line that connects (3, 2) and (7, 3)?

3 What is the gradient of a line that is perpendicular to $y = -3x + 6$?

4 What is the area of a rectangle with length 8 cm and width 12 cm?

5 What is the height h of the triangle shown? What is its area?



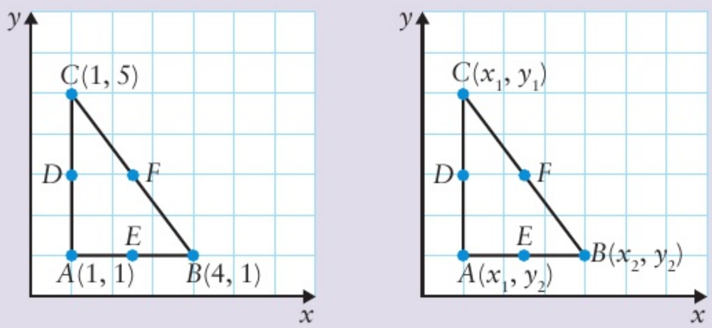
10.1 Distance and midpoint formulae

10.1.1 Distance and midpoint formulae in the coordinate plane

Coordinate geometry provides a connection between algebra and geometry through graphs of lines and curves. This enables geometric problems to be solved algebraically and provides geometric insights into algebra. The global positioning system (GPS) is a space-based satellite navigation system that provides, among other things, location information. In a GPS, the longitude and the latitude of a place are its coordinates. Smart phones, digital mapping services and many exercise trackers use GPS.

Explore 10.1

Consider triangle ABC shown in the diagram on the left-hand side.



Can you work out the lengths of sides AC, AB and CB?

D, E and F are the midpoints of their respective sides. Can you work out the coordinates of each?

Now consider a new triangle as shown in the diagram on the right-hand side. Can you answer the same questions for this triangle?

When answering the questions, show the steps you followed.

Worked example 10.1

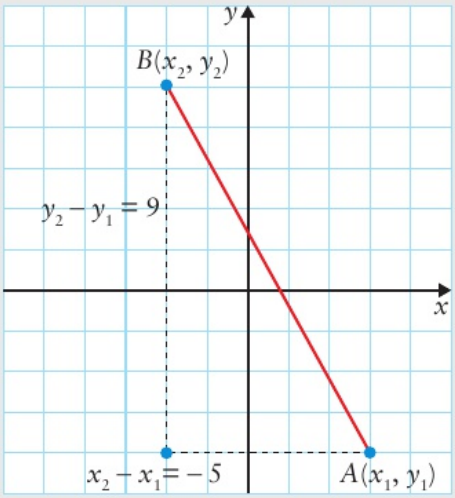
Consider points $A(3, -4)$ and $B(-2, 5)$ in a coordinate plane.

a Find the length of $\overline{AB}$.

b Work out the coordinates of the midpoint of $\overline{AB}$.

Solution

We can gain better understanding by plotting the points on a coordinate plane.



We can use the distance formula to find the length of $\overline{AB}$ and the midpoint formula to find the coordinates of the midpoint. We will first identify which coordinate will be (x_1, y_1) and which will be (x_2, y_2), and then substitute into the formulae.

a $x_1 = 3$ and $y_1 = -4$
$x_2 = -2$ and $y_2 = 5$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-2 - 3)^2 + (5 - (-4))^2}$$
$$= \sqrt{106}$$

Identify the coordinate points to be substituted into the distance formula.
Substitute the values into the formula and simplify.

Hence, the distance between $A(3, -4)$ and $B(-2, 5)$ is $\sqrt{106}$

Fact

The distance formula is:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The midpoint formula is:

$(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Hint

The **distance formula** uses Pythagoras' theorem. It doesn't matter which coordinate pair you label as (x_1, y_1) or (x_2, y_2), as long as you are consistent with your labels.

Hint

The x-coordinate of the midpoint is the average of the x-coordinates of the endpoints. The y-coordinate of the midpoint is the average of the y-coordinates of the endpoints.

b To find the midpoint between the two given points, we substitute the coordinates into the midpoint formula and simplify:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{3 + (-2)}{2}, \frac{-4 + 5}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Hence, the midpoint is located at $\left(\frac{1}{2}, \frac{1}{2}\right)$

Looking back, we can check our answer is sensible by looking at the diagram. Does the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ look halfway between A and B?

Explore 10.2

Draw a rectangular coordinate grid and sketch triangle ABC with $A(-2, -5)$, $B(2, 3)$, and $C(4, -3)$. Use your sketch to answer the following questions.

Can you say that this triangle is right-angled? Can you justify your answer?

Can you show that the midpoint of the longest side is equidistant from A, B and C?

Worked example 10.2

If $P(12, 1)$ is one endpoint of line segment PQ, and $M(7, -2)$ is the midpoint of line segment PQ, what are the coordinates of Q?

Solution

This question is asking us to find the second endpoint of a line segment. There are two approaches we can take. We could use the midpoint formula to solve algebraically for the other endpoint, or we could take a graphical approach.

Method 1: Algebraic

$x_1 = 12$ and $y_1 = 1$ x_2 and y_2 are the unknowns	Identify the coordinate points to be substituted into the midpoint formula.
$x_m = 7$ and $y_m = -2$	These are the coordinates of the midpoint.
$x_m = \dfrac{x_1 + x_2}{2}$ $y_m = \dfrac{y_1 + y_2}{2}$	We can break down the midpoint formula into one equation for each coordinate.

Now we can substitute the values in for each coordinate and simplify.

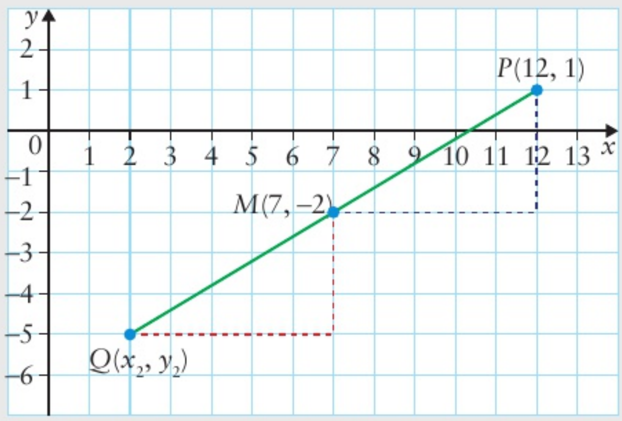
$$x_m = \frac{x_1 + x_2}{2}$$
$$7 = \frac{12 + x_2}{2}$$
$$14 = 12 + x_2$$
$$2 = x_2$$

$$y_m = \frac{y_1 + y_2}{2}$$
$$-2 = \frac{1 + y_2}{2}$$
$$-4 = 1 + y_2$$
$$-5 = y_2$$

Hence, the coordinates of Q are $(2, -5)$

We can use a graphical approach to check our answer.

Method 2: Graphical



We can think about the segment connecting P to M as describing the gradient to move from P to M.

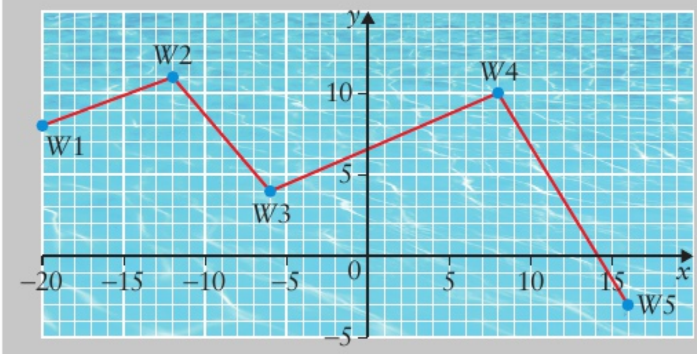
To move from P to M, we move vertically down 3 units and horizontally left 5 units.

Hence, to move from M to Q, we start at M and move down 3 units and left 5 units.

Therefore, the coordinates of the other endpoint are $Q(2, -5)$.

Worked example 10.3

An offshore oil field has 5 wells. The wells need to be connected by pipes as shown. If each unit on the grid represents 2 km, find the total length of the pipe.



Solution

Understand the problem

This question is asking us to determine the distance from $W1$ to $W5$ passing through $W2$, $W3$ and $W4$ as shown.

Make a plan

To determine this distance, we first need to identify the coordinates of where the wells are located. Then we can use the distance formula to find the distance. However, as each unit on the grid represents 2 km, we will need to multiply our answer by 2 to know the actual distance.

Carry out the plan

Start by identifying the coordinates of the two wells:

From $W1(-20, 8)$ to $W2(-12, 11)$

$x_1 = -20$ and $y_1 = 8$

$x_2 = -12$ and $y_2 = 11$

Then substitute the coordinates into the distance formula and solve:

$$d_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-12 - (-20))^2 + (11 - 8)^2}$$
$$= \sqrt{8^2 + 3^2} = \sqrt{73}$$

Repeat the process for each other pair of wells:

From $W2\ (-12, 11)$ to $W3(-6, 4)$:

$$d_2 = \sqrt{(-6 - (-12))^2 + (4 - 11)^2} = \sqrt{6^2 + (-7)^2} = \sqrt{85}$$

From $W3(-6, 4)$ to $W4(8, 10)$:

$d_3 = \sqrt{(8-(-6))^2 + (10-4)^2} = \sqrt{14^2 + 6^2} = \sqrt{232}$

From $W4(8, 10)$ to $W5(16, -2)$:

$d_4 = \sqrt{(16-8)^2 + (-2-10)^2} = \sqrt{8^2 + (-12)^2} = \sqrt{208}$

So total distance $= \sqrt{73} + \sqrt{85} + \sqrt{232} + \sqrt{208} = 47.4$ map units (3 s.f.)

Look back

We need to make sure that we have answered the question. As each unit on the grid represents 2 km, we multiply the distance by 2. The total length of the pipe is 94.8 km (3 s.f.)

Reflect

How can you estimate the length of the pipe needed in Worked example 10.3, using only the graph shown? Is your estimate reasonably close?

Practice questions 10.1.1

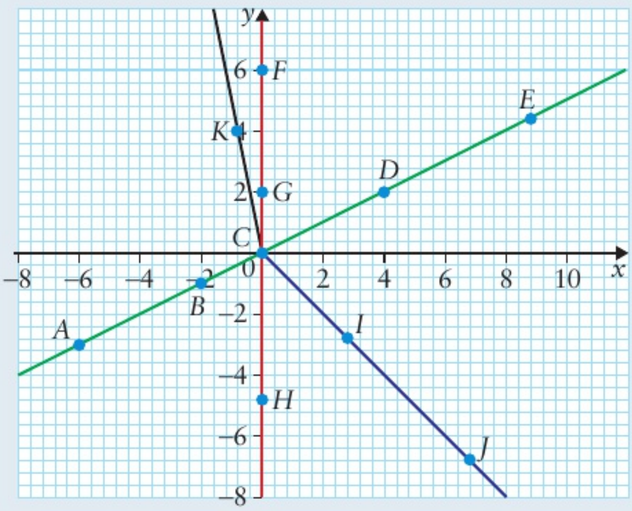
1 For the given pairs of points:
 i find the distance between them, giving your answer to 1 decimal place
 ii find the coordinates of the midpoints.

a $(-16, -2)$ and $(-5, 15)$
b $(-9, -5)$ and $(-7, -10)$
c $(1, 20)$ and $(5, -7)$
d $(7, 2)$ and $(8, 9)$
e $(6, -16)$ and $(-1, -3)$
f $(-12, -14)$ and $(4, -9)$
g $(-13, 3)$ and $(11, -15)$

2 In each part you are given one endpoint and the midpoint M of line segment AB. Find the coordinates of the other endpoint.

a $A(-1, 10)$, $M(4, 3)$
b $A(4, 10)$, $M(-2, -4)$
c $A(-6, 7)$, $M(6, -3)$
d $B(-3, 8)$, $M(3, 1)$
e $B(7, 3)$, $M(4, -3)$
f $B(1, 12)$, $M(-5, -5)$

3 A subway map is shown with units in km. Assume the routes between stations are straight lines.



Key: A: Ash, B: Birch, C: Cedar, D: Dogwood, E: Elm, F: Forest, G: Grove, H: Holly, I: Ivy, J: Juniper, K: Kentia

Find the distance you would travel between the following stations to the nearest tenth of 1 km.

a Ash to Dogwood

b Elm to Birch

c Kentia to Ivy

d Birch to Forest

e Grove to Elm

4 A triangle has vertices at $(-2, 2)$, $(1, 6)$ and $(3, 3)$. What is the perimeter of the triangle?

5 A rectangle has three vertices with coordinates $(-3, 0)$, $(-1, 6)$, and $(8, 3)$.

a Find the coordinates of the fourth vertex.

b Find the perimeter and area of the rectangle.

c Find the coordinates of the centre of the rectangle.

Hint Q5c

The centre is the intersection of the diagonals.

6 The vertices of a triangle are $A(-3, -3)$, $B(1, 5)$ and $C(-11, 6)$. Classify the triangle as scalene, isosceles or equilateral.

Challenge Q7

Reminder

Collinear means the points are on the same line.

7 Points $C(-6, 4)$, $G(4, 2)$ and S are collinear. One of the points is the midpoint of the line segment formed by the other two points.

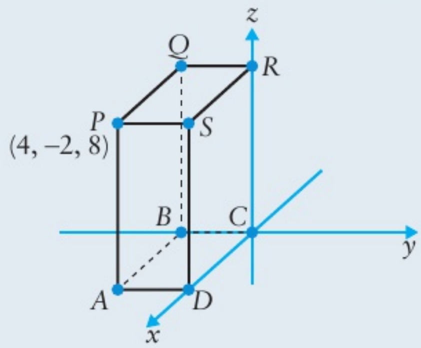
a What are the possible coordinates of S?

b If $SG = \sqrt{416}$, how does this affect your answer to part a?

Challenge Q8

8 You can use three coordinates (x, y, z) to locate points in three dimensions.

Point P has coordinates $(4, -2, 8)$ as shown in the diagram.



a Give the coordinates for A, B, C, D, Q, R and S.

In three dimensions, the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) can be calculated with an extension of the distance formula:
$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

b Find the distance between C and P.

c Find the distance between D and Q.

d Find the distance between A and S.

10.1.2 Applications of distance and midpoint formulae

Explore 10.3

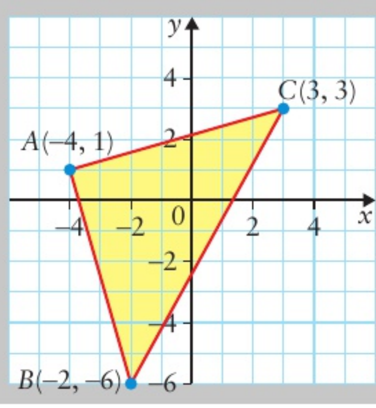
Consider the following points as vertices of a quadrilateral:
$A(-3, 4)$, $B(2, 4)$, and $C(4, 1)$

Can you find where the 4th vertex D could be placed to make a quadrilateral whose area and perimeter you can calculate?

Worked example 10.4

a Classify triangle ABC, shown on the grid here, as scalene, isosceles, or equilateral.

b Is it a right-angled triangle?



Solution

This question is asking us to classify the triangle. In order to do this, we need to determine the length of each side and whether any of the sides intersect at a right angle.

a We can find the length of each side of the triangle by using the distance formula:

Starting with $A(-4, 1)$ and $B(-2, -6)$, let $x_1 = -4$ and $y_1 = 1$, $x_2 = -2$ and $y_2 = -6$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2 - (-4))^2 + (-6 - 1)^2}$$
$$= \sqrt{53}$$

We can follow the same procedure to find the lengths of BC and AC:

For BC, $x_1 = -2$ and $y_1 = -6$, $x_2 = 3$ and $y_2 = 3$

$$BC = \sqrt{(3 - (-2))^2 + (3 - (-6))^2} = \sqrt{106}$$

For AC, $x_1 = -4$ and $y_1 = 1$, $x_2 = 3$ and $y_2 = 3$

$$AC = \sqrt{(3 - (-4))^2 + (3 - 1)^2} = \sqrt{53}$$

Therefore, since $AB = AC$, ABC is an isosceles triangle.

b There are two methods we can use to determine whether triangle ABC is right-angled: we can use the converse of Pythagoras' theorem or we can use the gradient method.

Method 1: Converse of Pythagoras' theorem

a is clearly the longest side, $a^2 = (\sqrt{106})^2 = 106$

$b^2 + c^2 = (\sqrt{53})^2 + (\sqrt{53})^2 = 53 + 53 = 106$

so $a^2 = b^2 + c^2$ and by the converse of Pythagoras' theorem there must be a right angle at A.

Reminder

For a triangle, it is conventional to use a to label the side opposite vertex A, b opposite B and c opposite C.

Method 2: Gradient

Using the gradients of the sides:

Gradient of AB	Gradient of BC	Gradient of AC
$x_1 = -4$ and $y_1 = 1$	$x_1 = -2$ and $y_1 = -6$	$x_1 = -4$ and $y_1 = 1$
$x_2 = -2$ and $y_2 = -6$	$x_2 = 3$ and $y_2 = 3$	$x_2 = 3$ and $y_2 = 3$
$m = \dfrac{-6-1}{-2-(-4)} = -\dfrac{7}{2}$	$m = \dfrac{3-(-6)}{3-(-2)} = \dfrac{9}{5}$	$m = \dfrac{3-1}{3-(-4)} = \dfrac{2}{7}$

The gradients of AB and AC are reciprocals of each other with opposite signs, therefore they are perpendicular. Hence, triangle ABC is a right-angled isosceles triangle.

Reminder

The formula for the gradient is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Practice questions 10.1.2

1 Plot the points and label each triangle with the given vertices. Determine whether the triangle is scalene, isosceles or equilateral and if any triangle is a right-angled triangle.

a $A(2, 3)$, $B(4, 10)$, $C(7, 6)$

b $P(-7, 1)$, $Q(3, 9)$, $R(2, 0)$

c $X(1, 3)$, $Y(3, 11)$, $Z(-15, 7)$

2 Plot the points of each triangle with the given vertices in the coordinate plane, then find its perimeter and area.

a $A(-4, 11)$, $B(-4, 3)$, $C(9, 3)$

b $A(5, 1)$, $B(-3, 1)$, $C(-1, 3)$

c $A(-1, 5)$, $B(-8, 8)$, $C(-5, 1)$

3 Plot the points and label each quadrilateral with the given vertices. Give the most precise name for each quadrilateral.

a $W(-10, 4)$, $X(-8, 6)$, $Y(-2, 6)$, $Z(0, 4)$

b $E(3, -3)$, $F(7, -3)$, $G(6, -1)$, $H(2, -1)$

c $J(-2, 5)$, $K(3, 10)$, $L(6, 9)$, $M(5, 6)$

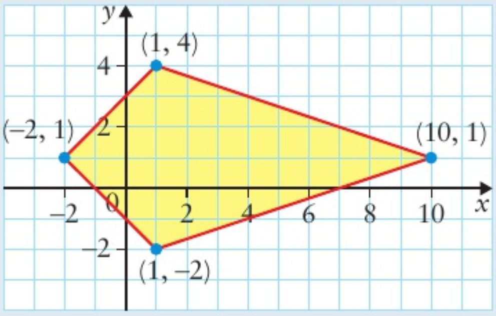
d $P(-8, 8)$, $Q(7, 10)$, $R(9, -5)$, $S(-6, -7)$

4 A kite is shown below.

a Calculate the coordinates of the midpoint of each side.

b Connect the midpoints to form a quadrilateral.

c Classify the quadrilateral that is formed when the midpoints are connected.



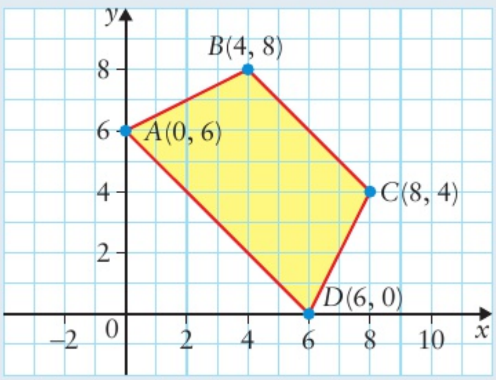
Thinking skills

5 Given the quadrilateral shown below:

a Show that it is an isosceles trapezium.

b Calculate the coordinates of the midpoint of each side, and then connect the midpoints to form a quadrilateral.

c Determine the shape that is formed when the midpoints of the sides are connected.



Thinking skills

6 An aeroplane is located 3 km north and 5 km east of an air traffic control tower. A second aeroplane is located 5 km north and 8 km west of the same air traffic control tower.

a How far is each aeroplane from the air traffic control tower?

b How far are the aeroplane from each other?

c Classify the triangle formed by the two aeroplanes and the air traffic control tower.

Investigation 10.1

Part 1

1 On a piece of square graph paper, plot the points $A(0, 0)$, $B(10, 0)$, $C(6, 8)$, $D(2, 8)$ and connect them in order to form quadrilateral $ABCD$.

2 Find the midpoint of AD and label it S. Find the midpoint of BC and label it T. Draw a line connecting S and T.

3 What is the name of quadrilateral $ABCD$?

4 What is the height of quadrilateral $ABCD$? Draw in a line segment that shows the height of $ABCD$.

5 Cut out $ABCD$ and then cut it along the midsegment, ST. Rearrange the pieces to create a parallelogram.

6 Use the formula for the area of a rectangle or parallelogram to calculate the area of $ABCD$.

7 What would be the general formula for the area of this figure?

Part 2

1 Now plot the points $E(-4, 0)$, $F(0, 2)$, $G(4, 0)$, and $H(0, -6)$ and connect them in order. What type of shape is quadrilateral $EFGH$?

2 Connect F and H to create a diagonal, and then connect E and G to create a second diagonal. What is the length of each diagonal?

3 Cut out $EFGH$ along its edges. Then cut $EFGH$ along both diagonals to create four triangles.

4 Use all four rectangles to create one rectangle or parallelogram.

5 Use the formula for the area of a rectangle or parallelogram to calculate the area of $EFGH$.

6 What would be a general formula for the area of this figure?

Part 3

1 Plot the points $J(0, 0)$, $K(2, 8)$, $L(10, 10)$, $M(8, 2)$ and connect them in order. What type of shape is quadrilateral $JKLM$?

2 Connect the diagonals as in step 2 of Part 2 above. Then repeat steps 3–6 above for quadrilateral $JKLM$.

Thinking skills

Communication skills

Fact

The line segment that connects two midpoints of a shape is called the **midsegment**.

Reminder

Height is always perpendicular to the base of a shape.

10.2 Transformations in the coordinate plane

In mathematics, a transformation is an operation that can be performed on a two-dimensional shape on a plane or coordinate system. A reflection, such as the mountain landscape in the mirror-like lake, is one such transformation. You have seen many transformations before, but you may have used different names for them such as shifts, slides and turns. Even when you play a jigsaw puzzle on your computer or your smart phone, you are working with transformations. You have also worked with transformations when you studied functions and relationships.

10.2.1 Isometric transformations

Translation

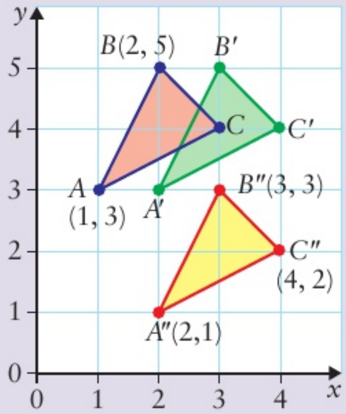
Hint

The original object or shape is called the **preimage**, and the resulting shape is called the **image**. So, ABC is the preimage and $A'B'C'$ and $A''B''C''$ are images.

Explore 10.4

Consider the three triangles in the diagram below. The original triangle is ABC. The other two triangles are 'translations' of that triangle.

Consider and describe all possible ways of getting from one triangle to the other two triangles.



In this section we will discuss some geometric transformations called **isometric transformations**. An isometric transformation (or **isometry**) is a movement in the plane or in space that preserves shape and size. The isometric transformations are reflection, rotation and translation and combinations of these such as the glide, which is the combination of a translation and a reflection.

Hint

Isometric transformations are also known as **congruence** or **rigid** transformations.

Reflect

What type of transformation is explored in Explore 10.4?

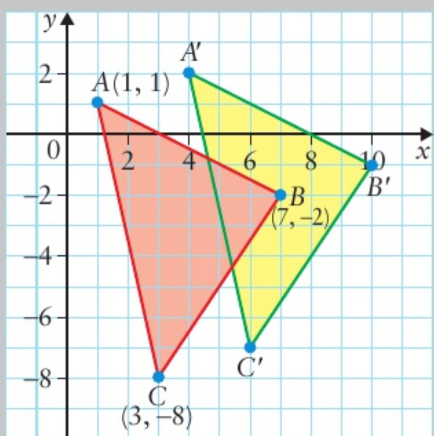
For any point with coordinates (x, y) a change of coordinates to $(x + h, y + k)$ is a **translation** of h units horizontally and k units vertically. The translation is written as $(x, y) \rightarrow (x + h, y + k)$. If $h > 0$, then it is a movement to the right and if $h < 0$, then it is to the left. Similarly, if $k > 0$, then the movement is upwards and if $k < 0$ then it is downwards.

Worked example 10.5

The diagram shows triangle ABC and its translated triangle $A'B'C'$

a State the translation used from triangle ABC to triangle $A'B'C'$

b If triangle ABC is translated 2 units left and 4 units up, what will the coordinates of the translated triangle $A''B''C''$ be?



Solution

This question is asking you to first identify how triangle ABC has been translated to triangle $A'B'C'$ and then identify the coordinates of a different translation.

a We look for a pattern in how the x- and y-coordinates have changed in order to identify the translation:

$A(1, 1) \rightarrow A'(4, 2)$

$B(7, -2) \rightarrow B'(10, -1)$

$C(3, -8) \rightarrow C'(6, -7)$

Begin by comparing the original coordinates to the translated coordinates.

$A(x = 1) \rightarrow A'(x = 4)$

$B(x = 7) \rightarrow B'(x = 10)$

$C(x = 3) \rightarrow C'(x = 6)$

Compare the change in the x-coordinates. To move from the original x-coordinate to the translated x-coordinate, we always add 3.

$A(y = 1) \rightarrow A'(y = 2)$

$B(y = -2) \rightarrow B'(y = -1)$

$C(y = -8) \rightarrow C'(y = -7)$

Compare the change in the y-coordinates. To move from the original y-coordinate to the translated y-coordinate, we always add 1.

Therefore, the translation is $(x, y) \rightarrow (x + 3, y + 1)$

b To find the coordinates of triangle $A''B''C''$ after triangle ABC is translated 2 units left and 4 units up:

$(x, y) \rightarrow (x - 2, y + 4)$

$A(1, 1) \rightarrow A''(-1, 5)$

$B(7, -2) \rightarrow B''(5, 2)$

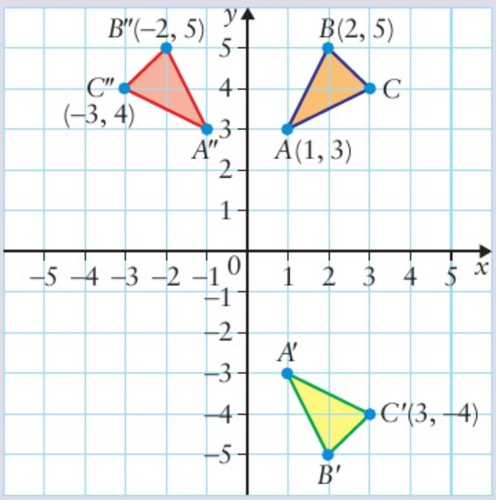
$C(3, -8) \rightarrow C''(1, -4)$

Reflection in a line

Explore 10.5

Consider the three triangles in the diagram. The original triangle is ABC. The other two triangles are **reflections** of that triangle.

Consider and describe all possible ways of getting from one triangle to the other two triangles.



For any point with coordinates (x, y) a change of coordinates to $(-x, y)$ is a **reflection in the *y*-axis**. This is written as $(x, y) \rightarrow (-x, y)$. A change of coordinates to $(x, -y)$ is a **reflection in the *x*-axis**. This is written as $(x, y) \rightarrow (x, -y)$. The distance from any point to the line of reflection is the same as the distance from the line of reflection to the corresponding reflected point.

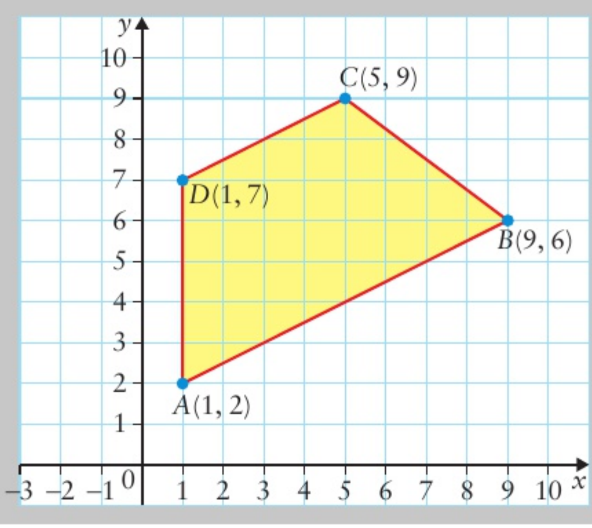
Reflect

What do you think a reflection $(x, y) \rightarrow (-y, -x)$ would be?

Worked example 10.6

Quadrilateral $ABCD$ has vertices $A(1, 2)$, $B(9, 6)$, $C(5, 9)$, and $D(1, 7)$

The quadrilateral is reflected in the y-axis. Give the coordinates of the resulting image $A'B'C'D'$.



Solution

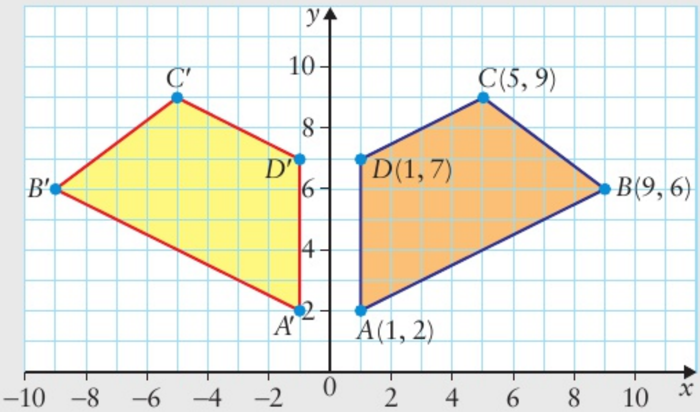
This is a direct application of the y-axis reflection. This means that the new coordinates will take the form $(-x, y)$

$A(1, 2) \rightarrow A'(-1, 2)$

$B(9, 6) \rightarrow B'(-9, 6)$

$C(5, 9) \rightarrow C'(-5, 9)$

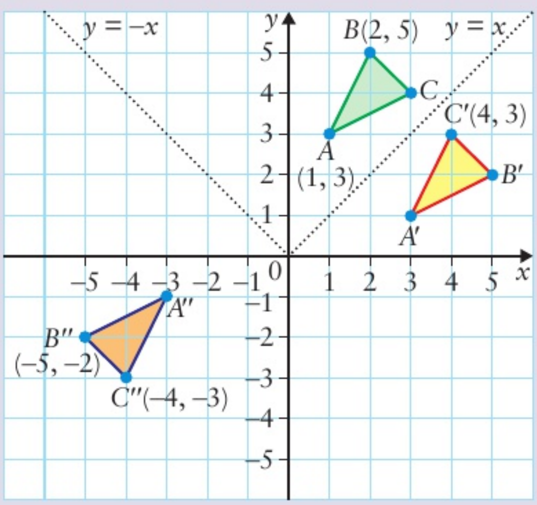
$D(1, 7) \rightarrow D'(-1, 7)$



Explore 10.6

Consider the three triangles in the diagram. The original triangle is ABC. The other two triangles are reflections of that triangle.

Consider and describe all possible ways of getting from one triangle to the other two triangles.



For any point with coordinates (x, y) a change of coordinates to (y, x) is a **reflection in the line $y = x$**. This is written as $(x, y) \rightarrow (y, x)$. A change of coordinates to $(-y, -x)$ is a reflection in the line $y = -x$. This is written as $(x, y) \rightarrow (-y, -x)$

Worked example 10.7

Consider quadrilateral $ABCD$ from Worked example 10.6. If $ABCD$ is reflected in the line $y = -x$, describe the new image, giving the coordinates of the vertices and drawing both $ABCD$ and $A'B'C'D'$.

Solution

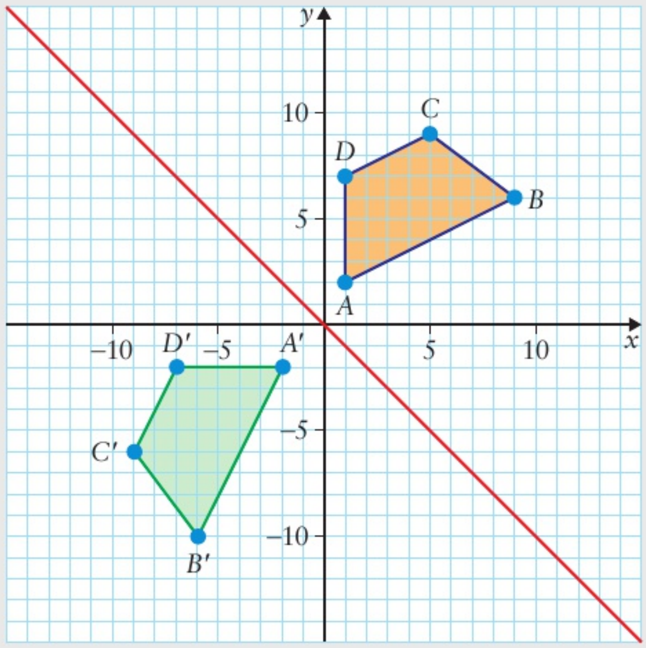
When points are reflected in the line $y = -x$, the x- and y-coordinates are swapped and both change sign. In other words, $(x, y) \rightarrow (-y, -x)$.
Hence the coordinates of the new image will be:

$A(1, 2) \rightarrow A'(-2, -1)$

$B(9, 6) \rightarrow B'(-6, -9)$

$C(5, 9) \rightarrow C'(-9, -5)$

$D(1, 7) \rightarrow D'(-7, -1)$

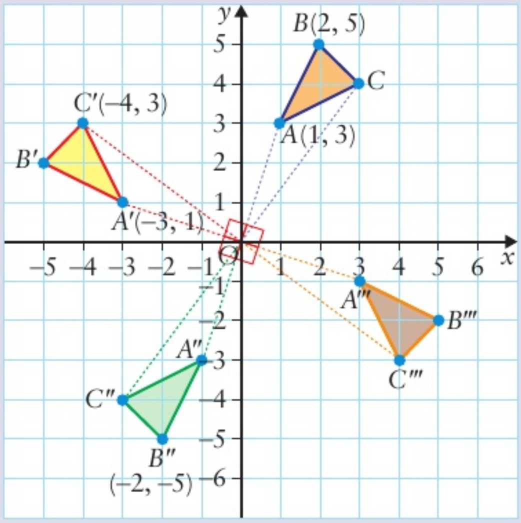


Rotation about a point

Explore 10.7

Consider the four triangles in the diagram below. The original triangle is ABC. The other triangles are **rotations** of that triangle around the origin.

Describe how to get from one triangle to the others.



Geometric rotation of a shape involves turning it around a point, called the **centre of rotation**. We rotate the shape through an angle, which is called the **angle of rotation**. In the two-dimensional plane, angles are measured **anti-clockwise as positive**. A rotation of 90° is therefore understood to be through a right-angle anti-clockwise. A rotation of 270° produces an image which is an exact replica of one from a rotation of –90° (negative being clockwise) around the same point.

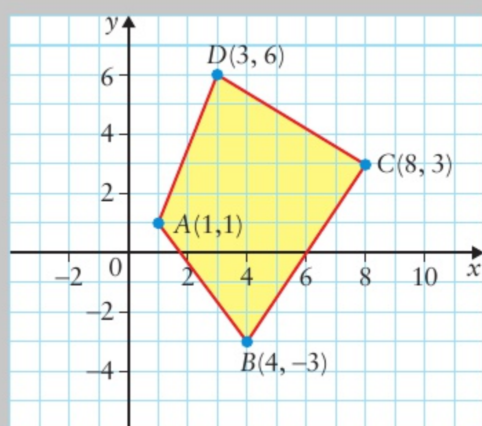
When describing a rotation fully, the angle and the centre of rotation must both be specified.

Fact

For any point with coordinates (x, y) a change of coordinates to $(-y, x)$ is a rotation of 90° about the origin. A change of coordinates to $(-x, -y)$ is a rotation of 180° about the origin. A change of coordinates to $(y, -x)$ is a rotation of 270° about the origin. Of course, a rotation of 360° will bring the point back to its original position.

Worked example 10.8

Quadrilateral $ABCD$ is shown. Draw its image $A'B'C'D'$ after a rotation of 270° centred on the origin.



Solution

Understand the problem

This question is asking us to draw the image of $ABCD$ after it is rotated 270°, centred on the origin. This is a rotation moving each point through three quadrants.

Make a plan

We know that in a rotation of 270°, the x- and y-coordinates swap and the x-coordinate will change sign, so $(x, y) \rightarrow (y, -x)$

Carry out the plan

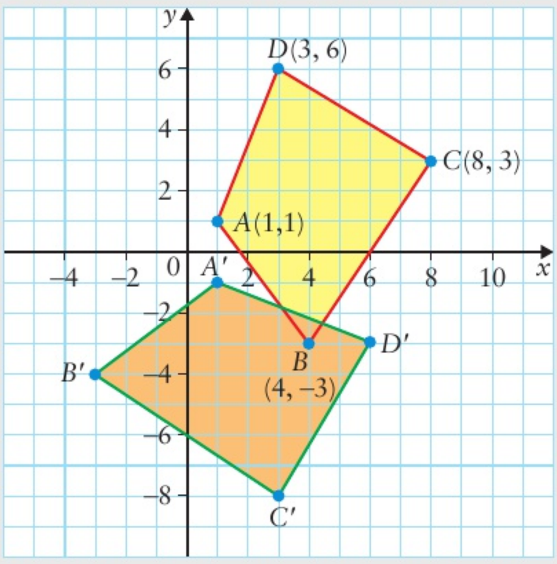
The coordinates of the rotated quadrilateral are:

$A(1, 1) \rightarrow A'(1, -1)$

$B(4, -3) \rightarrow B'(-3, -4)$

$C(8, 3) \rightarrow C'(3 -8)$

$D(3, 6) \rightarrow D'(6, -3)$



Look back

Looking at the image, how can we confirm that this is a rotation of 270°? Is there another method we could use to confirm our answer?

Hint

Remember that in the two-dimensional plane angles are measured anti-clockwise as positive and therefore clockwise as negative.

10.2.2 Other transformations

There are transformations other than isometric transformations. This section discusses another type of transformation: dilations. A **dilation** produces **similar** shapes to the original shape (preimage) that are not necessarily congruent to the preimage.

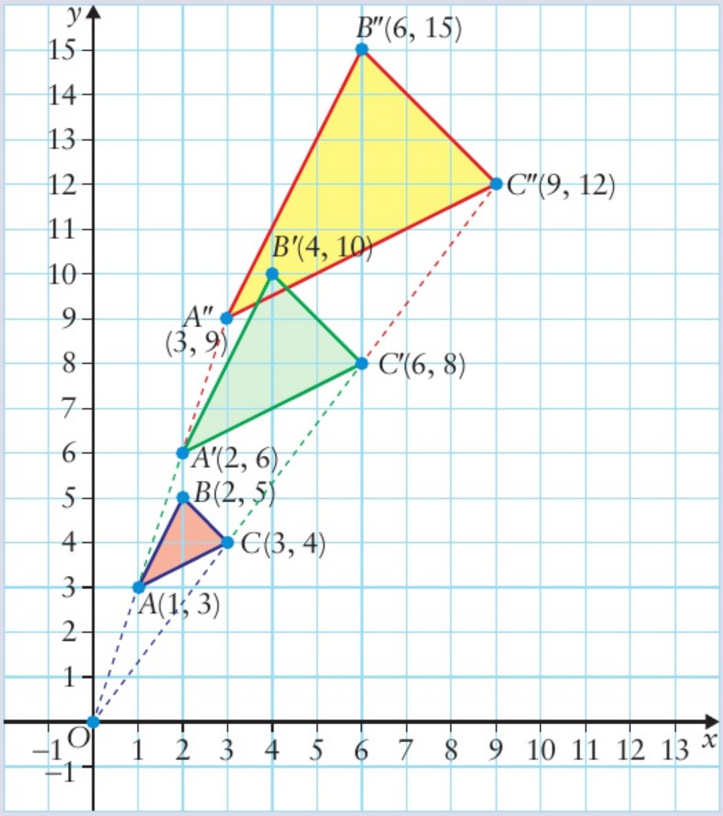
Reminder

Similar figures have the same shape, but may be different sizes. Congruent figures have the same shape and the same size.

Explore 10.8

Consider the three triangles in the diagram below. The original triangle is ABC. The other two triangles are dilations of that triangle.

Can you describe how you can get from the preimage to each of the other two triangles?



A **dilation** is a transformation that produces an image that is the same shape as the original but may be a different size. A dilation stretches or shrinks the original figure.

A description of a dilation includes the **scale factor** (or ratio) and the **centre of dilation**.

The centre of dilation is a fixed point in the plane. In this section, the centre of dilation will be the origin (0, 0).

- If the scale factor is greater than 1, the image is increased in size.
- If the scale factor is between 0 and 1, the image is decreased in size.
- If the scale factor is 1, the image and the preimage are congruent.

For example, the dilation in the Worked example 10.9 has centre O and scale factor k. Each point (x, y) has an image (kx, ky)

Fact

A dilation is also called an enlargement.

In everyday language "dilation" and "enlargement" refer to an increase, but in mathematics they are used for both increase and decrease. Compare this with the effect of multiplying by two and multiplying by a half.

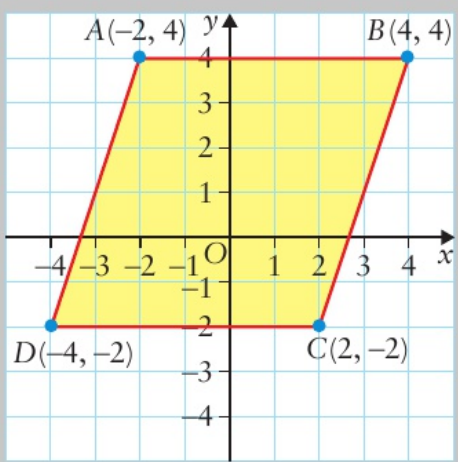
Worked example 10.9

Consider the quadrilateral $ABCD$ shown in the diagram.

a Work out and sketch the image after a dilation with O as centre and a scale factor of $\frac{1}{2}$

b Compare the sides and areas of the preimage and the image.

c Can you generalise the results in part b?



Solution

a Each vertex of the quadrilateral is transformed into its image by using the scale factor of $\frac{1}{2}$

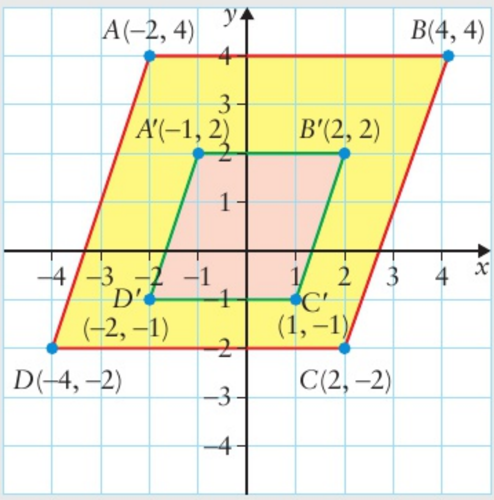
$(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$

$A(-2, 4) \rightarrow A'(-1, 2)$

$B(4, 4) \rightarrow B'(2, 2)$

$C(2, -2) \rightarrow C'(1, -1)$

$D(-4, -2) \rightarrow D'(-2 -1)$



b Quadrilateral $ABCD$ is a parallelogram with

$AB = DC = 6, AD = BC = \sqrt{(4-2)^2 + (4-(-2))^2} = \sqrt{40} = 2\sqrt{10}$

Image $A'B'C'D'$ is also a parallelogram with

$A'B' = D'C' = 3,\ A'D' = B'C' = \sqrt{(-2-(-1))^2 + (-1-2)^2} = \sqrt{10}$

Since $3 = \frac{1}{2} \times 6$ and $\sqrt{10} = \frac{1}{2} \times 2\sqrt{10}$, we conclude that each side of the image is half the length of the corresponding side of the preimage.

To find and compare the area of each, we need the heights of the parallelograms.

For $ABCD$ the height is the difference of the y-coordinates of A and D. Thus, $h = 6$

Area of $ABCD = CD \times h = 6 \times 6 = 36$

For $A'B'C'D'$ the height is the difference of the y-coordinates of A' and D'

Thus, $h' = 3$

Area of $A'B'C'D' = C'D' \times h' = 3 \times 3 = 9$

That is, area of $A'B'C'D' = 9 = \frac{1}{4} \times 36 = \frac{1}{4}$ area of $ABCD$.

c The scale factor is k, so the ratio of the image side to the preimage side is k.

To work out the area: the new side $= k \times$ old side, and the new height $= k \times$ old height, thus the area of the image $=$ $(k \times \text{old side}) \times (k \times \text{old height}) = k^2 \times \text{old area}$

Reflect

What if the centre of dilation is not the origin?

What if we combine an isometric transformation with a dilation?

Practice questions 10.2

In questions 1–6, work out the coordinates of the image after each transformation.

1 Point $M(9, 3)$ is translated 7 units left and 3 units down.

2 Point $R(-3, 6)$ is reflected in the x-axis.

3 Point $Q(-3, -4)$ is rotated 270° about the origin.

4 Point $B(1, 0)$ is translated 5 units right.

5 Point $C(3, 10)$ is reflected in the y-axis.

6 Point $D(2, 8)$ is rotated 90° about the origin.

7 Parallelogram $CDEF$ has vertices $C(-2, -2)$, $D(0, 3)$, $E(-4, 3)$ and $F(-6, -2)$. Draw the image of the parallelogram after the following transformations.

a Reflection in the y-axis

b Reflection in the line $y = 4$

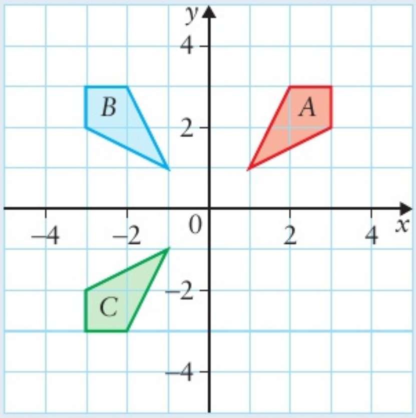
c Reflection in the line $y = x$

d Translation 4 units right and 3 units down e Translation 6 units up

f Rotation 180° about the origin

g Rotation –90° about (0, 0)

8 Kite A was reflected to get kites B and C. Give the equations of the two lines of reflection.



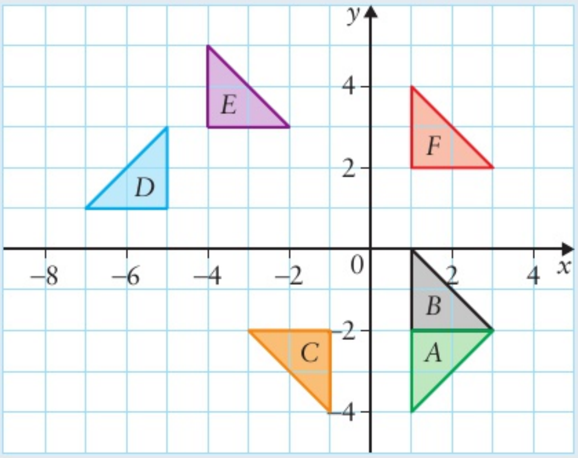
Challenge Q9a

9 a Describe completely how to transform triangle F to get the other triangles in the diagram below. Some may involve more than one transformation.

b Which triangles can be created by reflecting triangle A? (Not all triangles can be created by a single reflection of the green triangle.) Give the equations of the lines of reflection for each.

Hint Q9b

Not all the triangles can be created by reflecting triangle A.



Challenge Q10

10 Use the coordinates of the triangles in question 9 to illustrate that rotations, reflections and translations are rigid transformations. In other words, that the preimage and images are congruent.

Reminder

Congruent means the figures have the same shape and are the same size.

11 $\triangle ABC$ has vertices $A(5, 7)$, $B(-3, 5)$ and $C(0, 4)$ and is translated 4 units right and 6 units down.

a Give the coordinates of the new vertices $A'B'C'$.

b Find the midpoints of the three sides of $\triangle ABC$.

c Show that the images of the midpoints of $\triangle ABC$ are the midpoints of $\triangle A'B'C'$.

12 Plot $EFGH$ and its image $E'F'G'H'$ for a dilation with centre $(0, 0)$ and the given scale factor:

a $E(2, 3)$, $F(4, -2)$, $G(-3, -3)$, $H(-1, 2)$; scale factor $= 4$

b $E(12, 6)$, $F(-3, 3)$, $G(-6, -12)$, $H(9, -3)$; scale factor $= \frac{1}{3}$

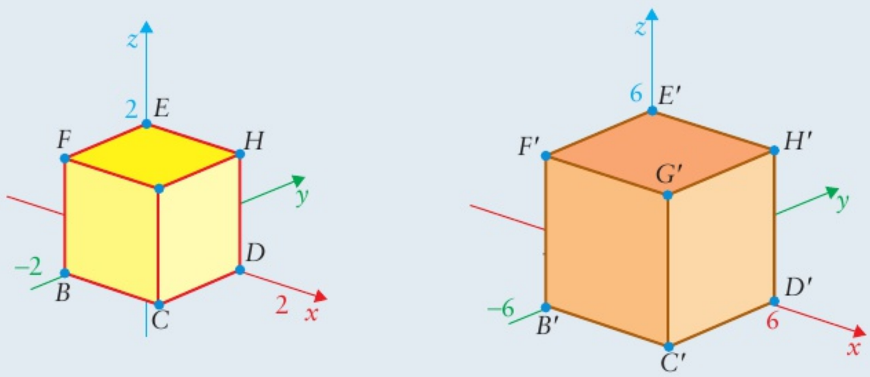
13 A dilation maps JSL onto $J'S'L'$. Find the missing values:

$JS = 8\,\text{cm}$ $\quad J'S' = 5\,\text{cm}$

$SL = 10\,\text{cm}$ $\quad S'L' = x$

$JL = y$ $\quad J'L' = 8.75\,\text{cm}$

14 A cube and its image are shown in the diagram.



Preimage **Image**

a What are the coordinates of the vertices of the preimage?

b What are the coordinates of the vertices of the image?

c How do the total surface areas compare?

d How do the volumes of the cubes compare?

Challenge 14

Reminder

Each coordinate point in three dimensions takes the form of (x, y, z).

Challenge Q15

15 $\triangle MNP$ has vertices $M(2, -2)$, $N(6, -2)$ and $P(4, -3)$. The triangle is reflected and the new vertices are $M'(0, 0)$, $N'(0, 4)$ and $P'(-1, 2)$

Find the equation of the line of reflection by following these steps.

a Draw line segments connecting corresponding points on the preimage and image.

b Work out the gradient of these line segments.

c Work out the gradient of the line perpendicular to these lines (this is the gradient of the line of reflection).

d Find the midpoint of each line segment; the midpoints are on the line of reflection.

e Use your answers to parts c and d to derive the equation of the line of reflection.

Challenge Q16

16 $\triangle XYZ$ has vertices $X(2, -2)$, $Y(6, -2)$ and $Z(4, -3)$. $\triangle XYZ$ is rotated 180° and the coordinates of the image are $X'(0, 2)$, $Y'(-4, 2)$ and $Z'(-2, 3)$. Use an algebraic approach to find the coordinates of the centre of rotation.

10.3 Tessellations

Connections

Mathematics and art are uniquely and beautifully connected through tessellation designs.

Tessellation, also known as **tiles** or **tiling**, is when repeated geometric shapes cover an entire plane without any gaps or overlaps between shapes. A regular tessellation uses copies of a single regular polygon to fill a space. A semi-regular tessellation uses copies of more than one regular polygon, but all with the same side length, to create a tessellation with the same pattern repeated at each vertex.

Examples of tessellations can be seen all around us and often connect mathematics with art. Dutch artist M.C. Escher and others made mathematically inspired artworks that feature intricate tessellations on the plane.

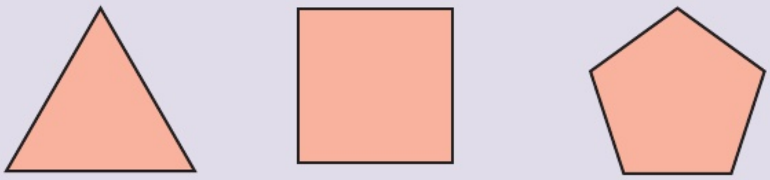
Explore 10.9

Consider the following regular shapes.

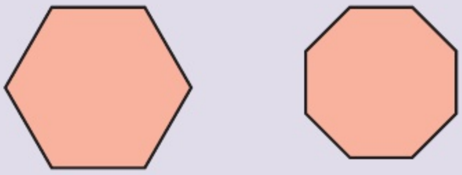
Which of them do you think can tessellate (cover an area without any gaps or overlaps)?

How can you tell whether there are no gaps or overlaps?

regular triangle   square   regular pentagon



regular hexagon   regular octagon



What do you think this means when creating semi-regular tessellations?

There are three different types of tessellation symmetry: translational reflectional, and rotational.

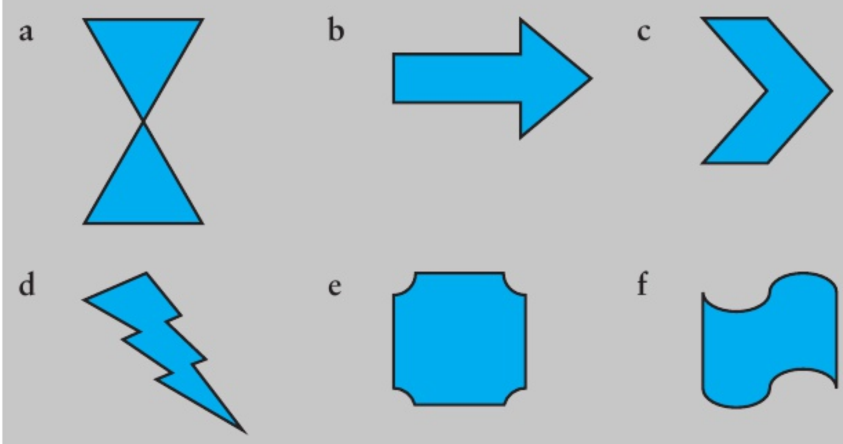
Translational symmetry in a tessellation is when the shape is moved up or down or left or right. This can be seen in the M.C. Escher tessellation of *Fish and Boats*. The fish and boats are shifted, but always face the same direction.

Reflection symmetry in a tessellation is when a shape is reflected over a line. This can be seen in the pattern of markings on a butterfly.

Rotational symmetry in a tessellation is when a shape rotates a specific number of degrees around a fixed point. This rotational symmetry is illustrated in M.C. Escher's work of *Starfish and Seashells*, which are spun, or rotated, around a central point.

Worked example 10.10

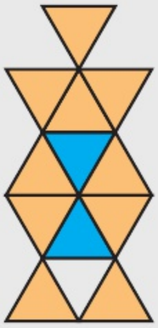
Determine which of the shapes below tessellate.



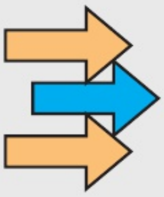
Solution

Remember that in order for a shape to tessellate, it must repeat its original shape to completely cover a space without overlapping or leaving any gaps.

a Yes: as the shape is repeated, it completely fills in the space with no gaps or overlaps.



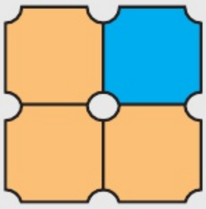
b No: when the shape is repeated, there are gaps that cannot be filled in by the original shape.



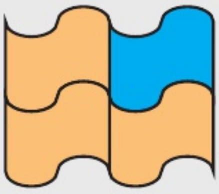
c Yes: as the shape is repeated, it completely fills in the space with no gaps or overlaps.

d No: when the shape is repeated, there are gaps that cannot be filled in by the original shape.

e No: when the shape is repeated, there are gaps that cannot be filled in by the original shape.



f Yes: as the shape is repeated, it completely fills in the space with no gaps or overlaps.



From Explore 10.9, you know there are three regular polygons that tessellate: the triangle, square and hexagon. From Worked example 10.10, you can see non-regular shapes can also tessellate.

Worked example 10.11

Remember that semi-regular tessellations involve tessellating more than one regular polygon. However, there are only eight different semi-regular tessellations. Two of them involve triangles and squares. What are the two different semi-regular tessellations that use only triangles and squares?

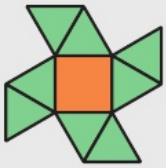
Solution

At each common vertex, the measure of the interior angles that come together must add up to exactly 360° in order to tessellate. Equilateral triangles have an interior angle measure of 60° and squares have an interior angle of 90°.

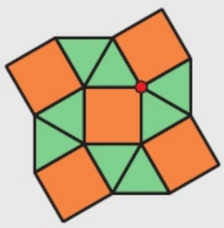
To determine how many of each shape needs to be used, we can test different combinations. If one of each shape is used, that will give 150°, which is not 360°. If two of each shape is used, that will give 300°, which is closer to 360°. Hence, if three triangles (180°) and two squares (180°) are used, that will yield 360°.

To determine the first pattern, we can start with a square and place a triangle on each side.

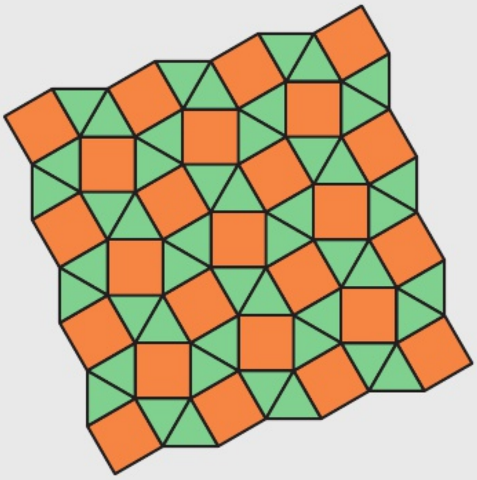
Next, we know that each vertex needs three triangles. Currently, each vertex has two triangles, so we can add a triangle to each vertex.



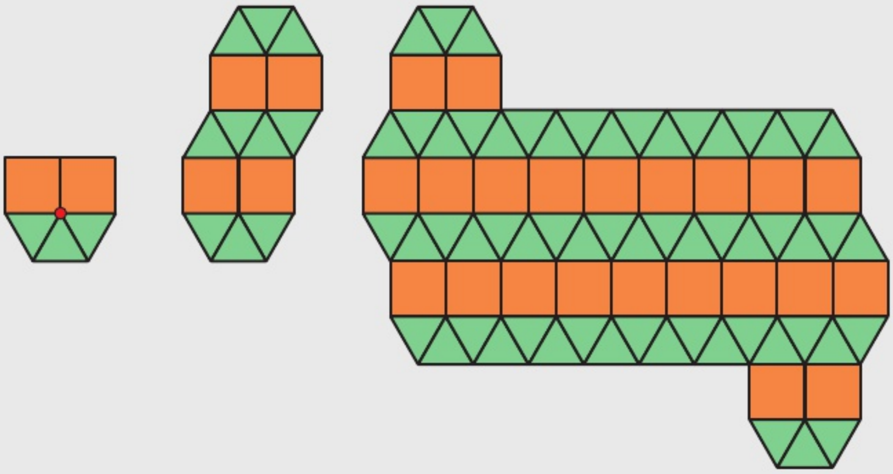
Finally, add the second square to connect at each vertex. Note that at the highlighted vertex, three triangles and two squares meet.



The pattern can then be repeated to fill more space:



A similar technique can be used to determine a different pattern that uses three triangles and two squares meeting at one vertex.



Note that at the highlighted vertex, three triangles and two squares come together.

Investigation 10.2

As you have seen, tessellations can be created using regular polygons and using different shapes. How can you create your own tessellation?

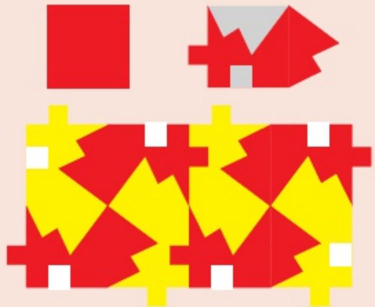
Suppose you are asked to create a tessellation design for one of the following:

- artwork to go in the main lobby of a new social media app company
- bedroom décor for young children (for example, a quilt or pillow case)
- a new logo for your school
- a new design to promote a service project at your school

Communication skills

Research skills

Start by choosing a regular polygon that tessellates (a regular triangle, square or regular hexagon).



An example is shown here, demonstrating how to create your own tessellation using rotational symmetry. You can use a similar technique to create a tessellation using translational symmetry.

You can investigate how to create the other semi-regular tessellations by going to the site:

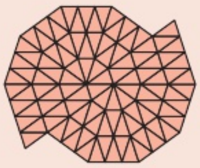
Hint

The regular polygons that can be used to create semi-regular tessellations are triangles, squares, hexagons, octagons and dodecagons.

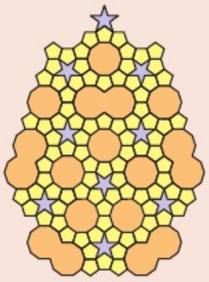
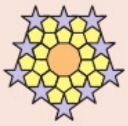
Here you will be able to work with different regular polygons to create unique tessellations.

In addition to regular and semi-regular tessellations, there are other types of tessellations you can investigate.

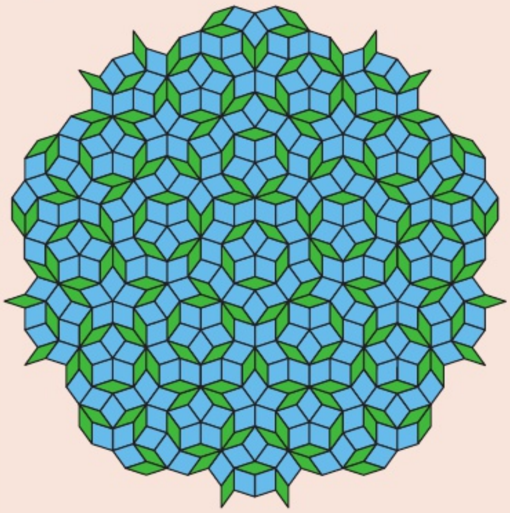
Spiral tessellations:

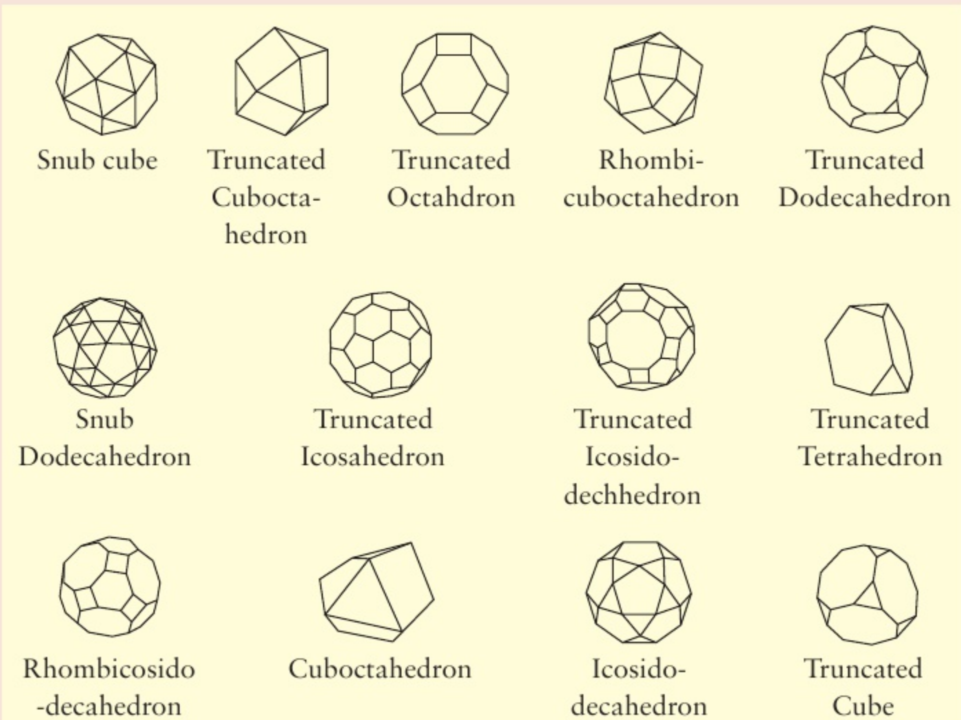

Johannes Kepler's tessellations:

Roger Penrose’s kites and darts tessellations:



3D tessellations, such as the Archimedean solids:



How does your design showcase the area of focus you chose? You can also think about what materials you might use for your design. Could you make your design come to life and be used?

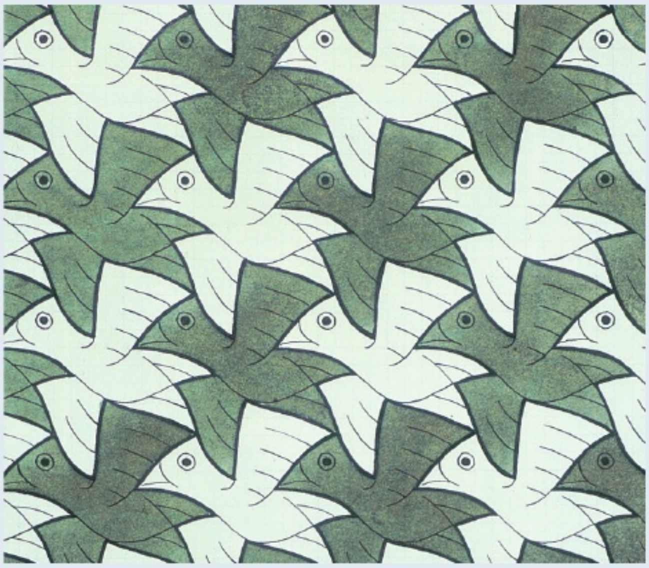
Reflect

Tessellations are visual ways that can be used to express different ideas and show the beauty of mathematics. You have seen that there are different types of tessellations, and that they can be seen in both two dimensions and three dimensions. M.C. Escher shows how tessellations create works of art and optical illusions. How else can tessellations be used? Where else can they be seen? How can you recognise different tessellations in the world around you?

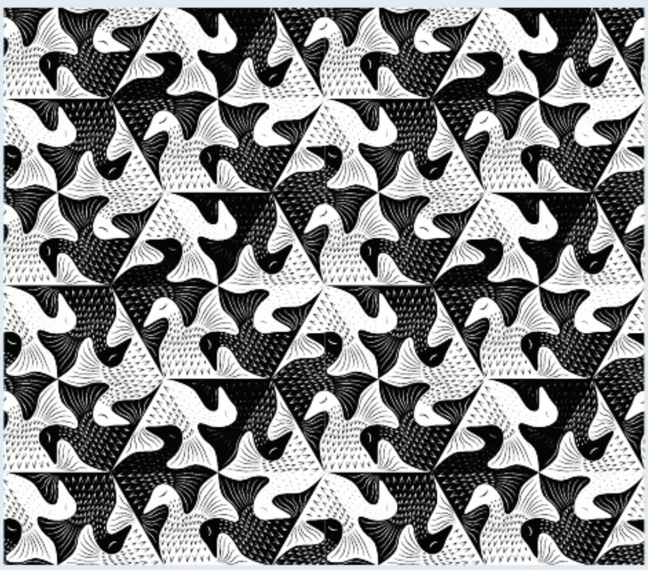
Practice questions 10.3

1 Identify each tessellation as a translation, reflection, or rotation tessellation, or state which combination of tessellations is used.

a

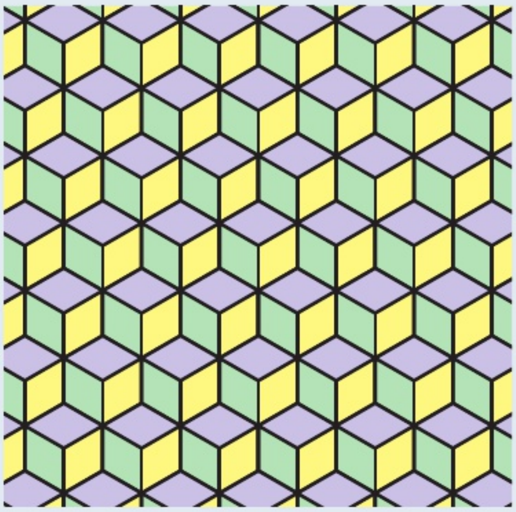


b

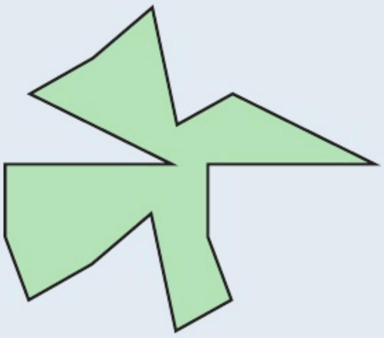


c

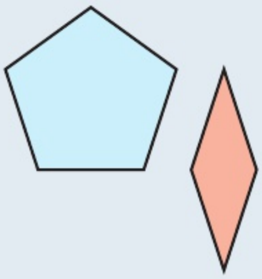
d



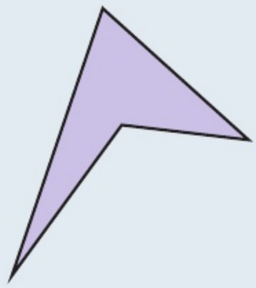
2 Use this shape to create a translation tessellation, repeating the pattern at least six times.



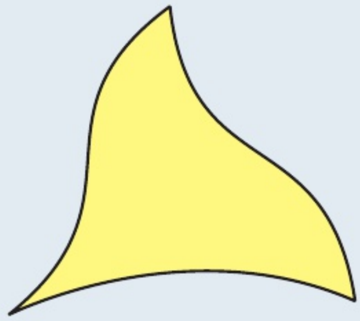
3 Use a regular pentagon and a rhombus to create a translation and rotation tessellation, repeating the pattern at least six times.



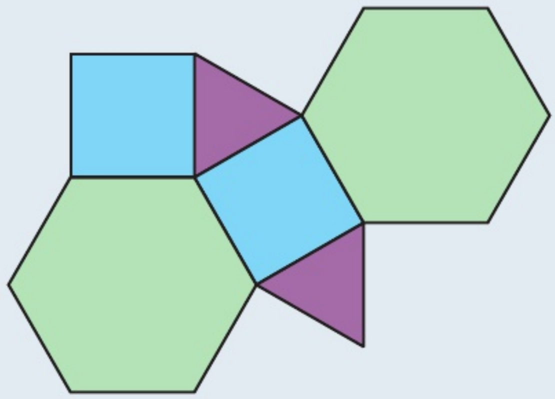
4 Use this concave polygon to create a reflection and translation tessellation, repeating the pattern at least six times.



5 Use this shape to create a rotation tessellation, repeating the pattern at least six times.



6 The diagram shows part of a semi-regular tessellation that uses regular triangles, squares and regular hexagons. Continue the pattern to include at least six hexagons.

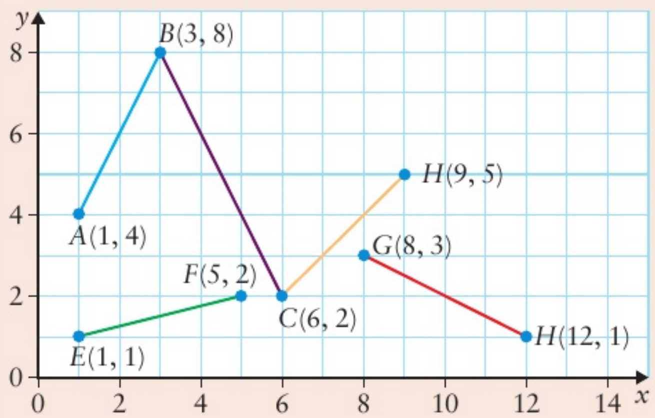


Self assessment

- I can find the distance between two points using the distance formula.
- I can find the midpoint of a line segment using the midpoint formula.
- I can identify if two sides of a polygon are perpendicular to each other.
- I can determine the perimeter of a triangle or quadrilateral in the coordinate plane.
- I can determine the area of a triangle or quadrilateral in the coordinate plane.
- I can identify a transformation in the coordinate plane as a translation, reflection or rotation.
- I can perform transformations in the coordinate plane including translations, reflections, rotations and dilations.
- I can identify tessellations in the plane including translation, reflection and rotational tessellations.

Check your knowledge questions

Use this diagram to answer questions 1 and 2.



1 Determine whether $\overline{AB}$ is the same length as $\overline{GH}$

2 Determine whether $\overline{EF}$ is the same length as $\overline{CD}$

3 If XY has endpoints $X(7, -4)$, $Y(-3, -2)$ and HJ has endpoints $H(5, 7)$ and $J(-1, -13)$, determine whether or not they have the same midpoint.

4 AB has a midpoint of $M(11, 1)$. If A has coordinates $(-9, 6)$, what are the coordinates of B?

5 A scale drawing of a skating rink is in the shape of a quadrilateral with vertices at $A(-4, 0)$, $B(3, 5)$, $C(4, 1)$ and $D(-1, -3)$. Entrances to the skating rink are to be constructed at the midpoint of each side, and straight paths through the rink are formed by connecting the entrances that are across from each other.

a Where do the paths intersect?

b What is the total distance of the paths if each unit represents 100 m?

6 In triangle ABC, two of the vertices are $A(1, -5)$ and $B(7, -7)$. Suggest integer coordinates of C if:

a triangle ABC is an isosceles right-angled triangle, with AB and AC of equal length

b triangle ABC is an isosceles triangle, with AB and AC of equal length.

7 In quadrilateral $ABCD$, three of the vertices are $A(-9, -6)$, $B(5, -4)$ and $C(-4, 8)$. What are the coordinates of D if $ABCD$ is a parallelogram?

8 What shape is formed by the following lines?

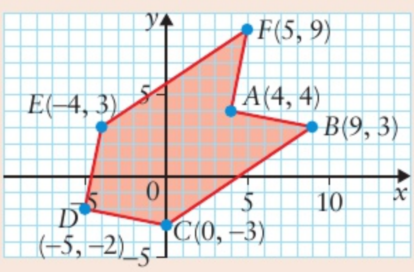
Line a: $y = 2x + 22$

Line b: $y - 2 = \left(-\frac{2}{7}\right)(x + 10)$

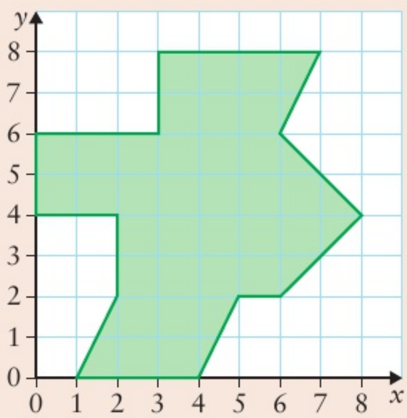
Line c: $2x + 7y = 26$

Line d: $y + 2 = 2(x - 4)$

9 Find the area of $ABCDEF$.



10 The exterior outline of an outdoor vegetable garden is shown. To get the ground ready for planting, it must have a 20 cm deep fresh layer of topsoil, which costs €32.50 per cubic metre. In addition, a new fence, including a gate, must be constructed around the whole perimeter. The fencing materials cost €3.50 per metre. If each unit on the diagram represents 2 metres, what is the total cost of the topsoil and fencing materials?



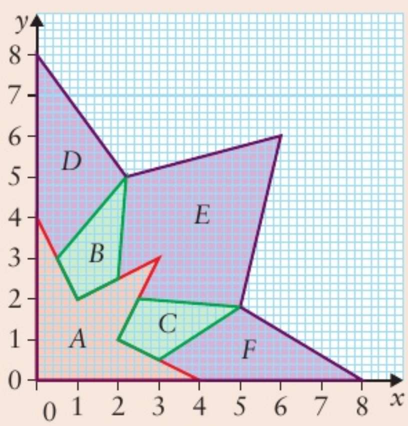
11 $A(3, -4)$, $B(-1, 11)$, and $C(2, 6)$ are the coordinates of the vertices of a triangle. If this triangle is reflected in the y-axis and then rotated 90° about the origin, what are the coordinates of the transformed triangle?

Challenge Q11

12 Jean is designing a pattern for a star quilt for Stephanie. Part of the pattern is shown.

a What is the most precise name for quadrilaterals B and C?

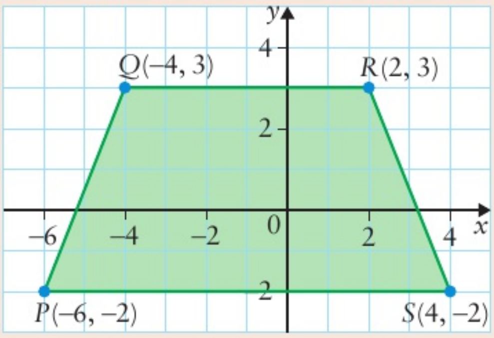
b The pattern is reflected in the y-axis and in the x-axis, and rotated 180° about the origin. Draw the final design of the star quilt.



13 Plot ABC and its image $A'B'C'$ for a dilation with centre (0, 0) and the given scale factor. Then compare the area of the preimage and image.

$A(2, 8)$, $B(-2, -2)$, $C(6, -2)$; scale factor $= \frac{1}{4}$

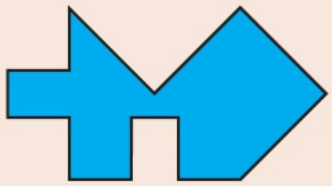
14 Given quadrilateral $PQRS$:



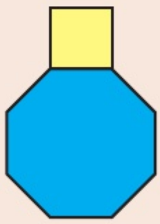
a Work out and sketch the image after a dilation with centre (0, 0) and a scale factor of -2.5

b Compare the sides and areas of the preimage and image.

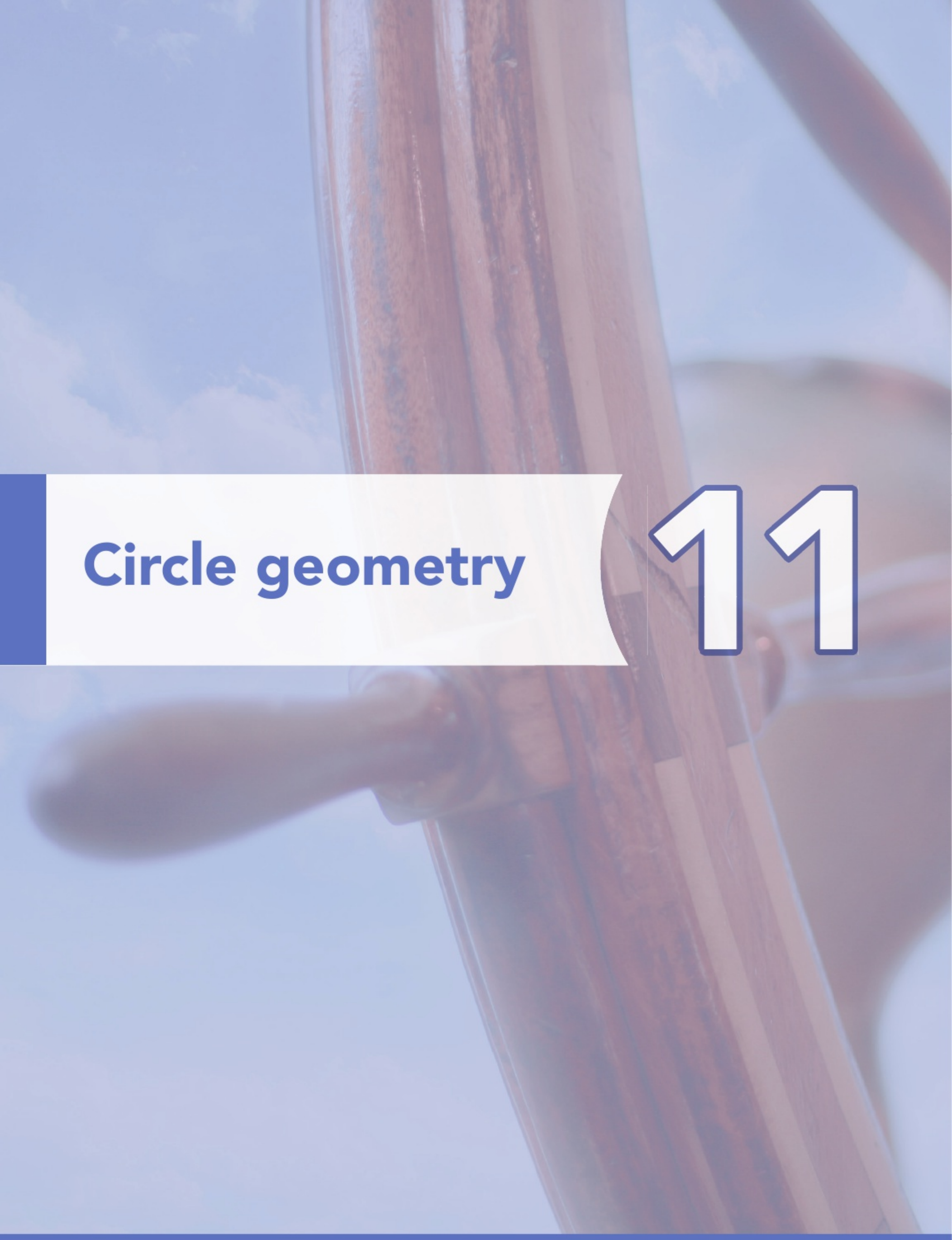
15 Use this shape to create a rotational tessellation with at least eight shapes.



16 Create a semi-regular tessellation using only squares and octagons. Use at least five octagons.



11 Circle geometry



11 Circle geometry

KEY CONCEPT

Logic

RELATED CONCEPTS

Models, Representation, Validity

GLOBAL CONTEXT

Identities and relationships

Statement of inquiry

The validity of the generalisations we make when exploring geometric relationships can be evaluated with logical reasoning.

Factual

- How do we measure different circle parts?

Conceptual

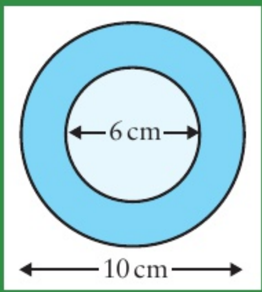
- Why are angles standing on the same arc congruent?

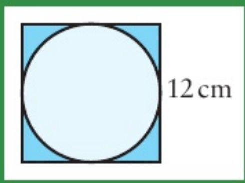
Debatable

- Are circles perfect shapes?
- Can we inscribe/draw any regular polygon in a circle?

Do you recall?

1 Calculate the darker shaded area for each shape.

a 

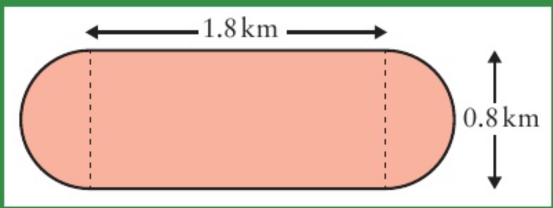
b 

2 Given that the total height of this doorway is 2.8 m and the width is 2.2 m, calculate its area and perimeter. You can assume that the curved part of the doorway makes a semicircle.

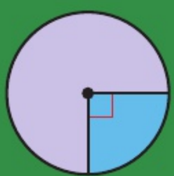
3 The diagram shows the plan of a race track. The curved ends are semicircles.

a Find the length of one lap.

b A motorcycle race is 12 laps of the track. Find the distance of the race, to the nearest kilometre.



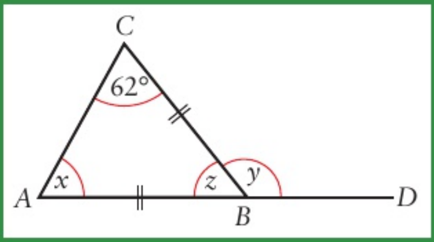
4 This circle has diameter 2.4 m.
Find the area of both shaded regions.



5 A household has the following general budget items. Draw a pie-chart to represent this data.

Expenditure category	Annual average cost in US$
Housing	10,000
Transportation	90,000
Food/entertainment	9500
Tax, insurance, health care, etc.	21,000
Other expenses	10,000

6 $\triangle ABC$ is isosceles with $AB = BC$. Angle C measures 62°.



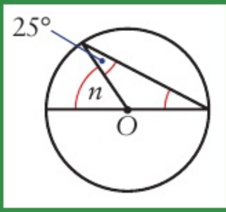
Find the following angles, giving reasons for your answers.

a x

b y

c z

7 Find the measure of angle n.



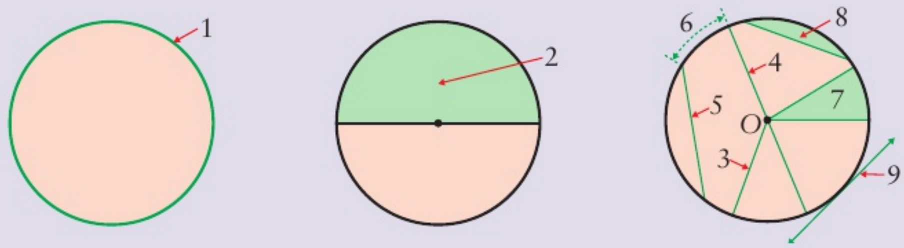
8 Draw a 6 cm line segment on plain paper and label the endpoints A and B. Now construct the perpendicular bisector of AB.

11.1 Circles: review

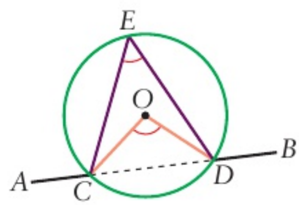
You are already familiar with circles and parts of circles from previous years. In this chapter we will discover the many attributes of circles.

Explore 11.1

Here are some diagrams to refresh your memory about the parts of a circle. O marks the center of the circle. Can you name the green, numbered parts?



In this chapter we will explore interesting properties of circles and then prove some of them. We will examine some important ideas about circles first.



A line that intersects a circle at two points is a **secant**. AB is a secant.

An **angle at the centre** is an angle formed by joining the ends of an arc or chord to the centre of a circle. (We say $\sphericalangle COD$ is an angle at the centre standing on the arc or chord CD.)

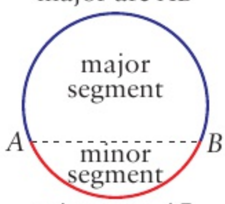
An **angle at the circumference** is an angle formed by joining the ends of an arc or chord to another point on the circumference. (We say $\sphericalangle CED$ is an angle at the circumference standing on the arc or chord CD.)

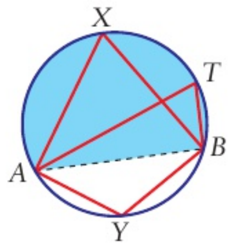
Hint

$\sphericalangle COD$ is also called an angle **subtended** at the centre by the arc (or chord) CD. $\sphericalangle CED$ is also called an angle **subtended** at the circumference by the arc (or chord) CD.

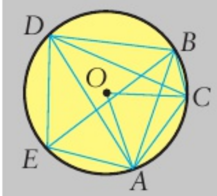
> **Fact**
>
> If instead of segments, we refer to the arcs, the angles are said to be standing on the same arc. Angles AXB and ATB are standing on the major arc AB, while $\sphericalangle AYB$ is standing on the minor arc AB.

Angles in the same segment: A chord divides a circle into two segments. The larger segment is called the **major segment** and the smaller is called the **minor segment**. In the diagram, $\sphericalangle AXB$ is in the major segment while $\sphericalangle AYB$ is in the minor segment. Angles AXB and ATB are in the same segment.





Worked example 11.1



a Which two angles are standing on the arc EA?

b Which angles are standing on the arc AC?

c How many angles does the arc DB subtend at the circumference?

d The chord DA divides the circle into minor and major segments. Name the angle in the minor segment.

e Only one angle is subtended at the centre. On which chord is it standing?

Solution

a The only angles whose sides contain A and E are $\sphericalangle EDA$ and $\sphericalangle EBA$

b On the major arc AC we have only two points, B and D, that are connected to both endpoints A and C. So $\sphericalangle ABC$ and $\sphericalangle ADC$ are two of the angles. However, we also have O connected to A and C, and thus $\sphericalangle AOC$ is a third angle standing on the arc (angle at the centre).

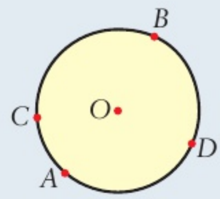
c The endpoints D and B of DB are both connected to A, C, and E. Therefore the arc subtends three angles at the circumference: $\sphericalangle BAD$, $\sphericalangle BCD$ and $\sphericalangle BED$

d There is only one point on the minor arc of AD. Thus, the only angle in the minor segment is $\measuredangle AED$.

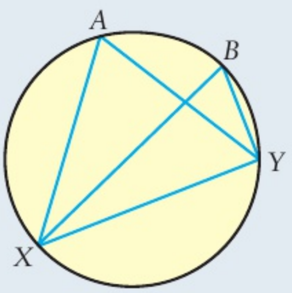
e $\measuredangle AOC$ is the only angle at the centre. Its sides contain the points A and C on the circumference, and thus the chord it is standing on is AC.

Practice questions 11.1

1 Make tracings of the diagram, and on separate drawings show:



- **a** the angle at the centre standing on the arc AC
- **b** the angle at the centre subtended by the arc AD
- **c** the angle subtended at D by the arc BC
- **d** the angle at C standing on the arc AD.

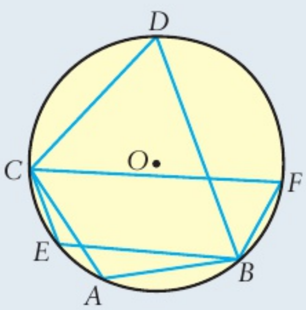


2 **a** Name an angle at the circumference that is standing on the arc:

i AX **ii** BY.

b Name two angles at the circumference that are standing on the arc:

i XY **ii** AB.



3 **a** Name two angles that are:

- **i** standing on the minor arc BC
- **ii** standing on the major arc BC.

b The chord BC divides the circle into major and minor segments. Name two angles in:

- **i** the minor segment
- **ii** the major segment.

11.2 Angle properties of circles

Round house in Moscow.

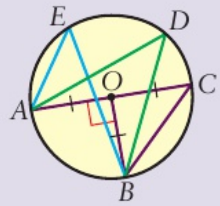
Fact

∢AOB and angles at C, D, and E stand on the same chord (or arc) AB

Reminder

Remember that figures are congruent and numerical values are equal. So, the two line segments AB and CD are congruent, that is, $\overline{AB} \cong \overline{CD}$, but $AB = CD$. Also for angles, ∢$A \cong$ ∢B and the measure of angle A = the measure of angle B

Explore 11.2



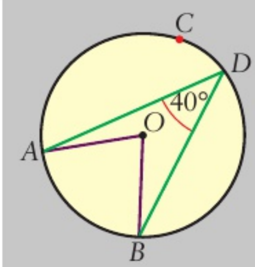
Consider the following diagram, where O is the centre of the circle.

Can you find the measure of ∢ACB? Justify your answer.

What do you think the measures of ∢ADB and ∢AEB are?

Can you make a general statement about the connection between ∢AOB and angles at C, D, and E?

Worked example 11.2



The circle has centre O and points A, B, C and D on the circumference. ∢D has a measure of 40°.

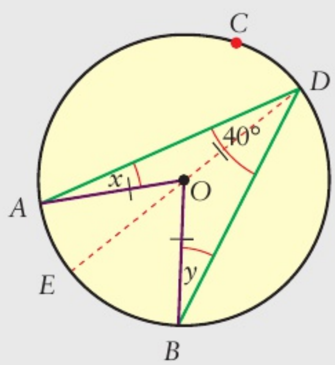
a Work out the size of ∢AOB.

b Make a conjecture about ∢ACB and prove it.

c Generalise what you observe.

Solution

a To be able to find the size of ∢AOB, we need to try to use the available measure, that of ∢D. If we connect O to D, we can establish a relationship through the exterior–interior angles in a triangle.



Draw a diameter through D. Now observe that $\triangle AOD$ is isoceles: $OA = OD$ because they are radii of the same circle. Thus $\sphericalangle A$ and $\sphericalangle ODA$ have the same measure, which we will call x. We can make the same argument for $\triangle BOD$, and will call the measure of $\sphericalangle B$ and $\sphericalangle ODB$ y.

Now, $\sphericalangle AOE$ is the sum of the measures of the opposite interior angles in $\triangle AOD$, and thus $\sphericalangle AOE = 2x$

Similarly in $\triangle BOD$, $\sphericalangle BOE$ is the sum of the measures of the opposite interior angles in $\triangle BOD$, and thus $\sphericalangle BOE = 2y$

Therefore, $\sphericalangle AOB = 2x + 2y = 2(x + y) = 80°$

Looking back, we could have worked this out in a different way.

$\sphericalangle AOD = 180 - 2x$, since angles in $\triangle AOD$ add up to 180°

$\sphericalangle BOD = 180 - 2y$, since angles in $\triangle AOD$ add up to 180°

Finally,

$$\begin{aligned}\sphericalangle AOB &= 360 - (\sphericalangle AOD + \sphericalangle BOD)\\ &= 360 - (180 - 2x + 180 - 2y) = 2(x + y) = 80°\end{aligned}$$

b In part a we showed that $\sphericalangle AOB$ is twice the measure of $\sphericalangle D$. Thus we can say that $\sphericalangle D$ is half the measure of $\sphericalangle AOB$, so we can conjecture that $\sphericalangle C$ is half the measure of $\sphericalangle AOB$, that is 40°.

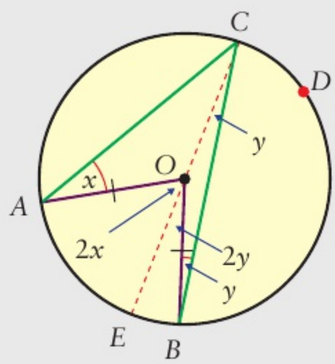
The proof is similar to the previous one.

Draw a diameter through point C to meet the circle at E.

With similar arguments as above:

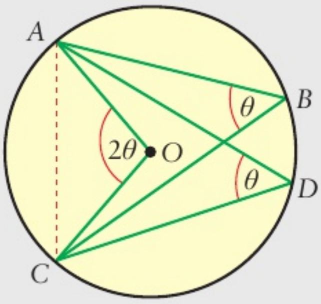
$\sphericalangle AOB = 2x + 2y$ and $\sphericalangle C = x + y$, which means that

$$\sphericalangle C = \frac{1}{2}\sphericalangle AOB = \frac{1}{2} \times 80 = 40°$$



c Examining the outcomes, we can claim:

- The angle subtended by an arc (or chord) at the centre of a circle is double the angle subtended by the same arc (or chord) at the circumference.
- Angles subtended at the circumference by the same arc (or chord) are congruent.



Fact

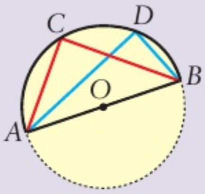
An angle such as $\sphericalangle C$ or $\sphericalangle D$ is called an **angle subtended on the circle by the diameter** or an **angle in a semicircle**. This problem is known as Thales' theorem. Thales of Miletus was a Greek mathematician who lived from mid 600 – late 500 BC. He is the first person known to apply deductive reasoning to geometry.

Reflect

What if angles are subtended at the circumference by equal arcs (or chords)?

Explore 11.3

Consider the following circle with centre O and diameter AB.



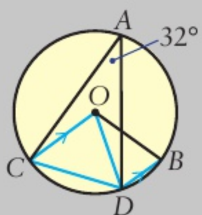
What can you say about $\sphericalangle C$ and $\sphericalangle D$?

Can you generalise and justify what you observe?

Worked example 11.3

Consider the circle with centre O, $\sphericalangle A = 32°$ and $OC \parallel BD$

Work out $\sphericalangle DOB$



Solution

$\sphericalangle DOB$ is the vertex angle of an isosceles triangle. So if we can find one of the base angles, we can find its measure.

Since $\sphericalangle COD$ is an angle at the centre standing on the same arc as $\sphericalangle A$ then $\sphericalangle COD = 2 \times \sphericalangle A$, thus $\sphericalangle COD = 2 \times 32 = 64°$

Now, because $OC \parallel BD$, $\sphericalangle COD$ and $\sphericalangle ODB$ are alternate interior angles, so $\sphericalangle COD \cong \sphericalangle ODB$

$\sphericalangle ODB$ is a base angle in isosceles triangle DOB.

Then $\sphericalangle DOB = 180 - 2 \times 64 = 52°$

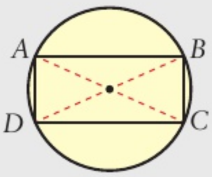
> **Hint**
>
> *DOB* is a triangle with two of its sides radii of the circle.

Worked example 11.4

Show that a quadrilateral whose diagonals are diameters of the same circle must be a rectangle.

Solution

We first draw a picture.

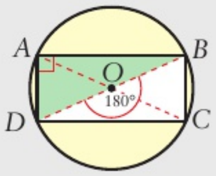


Since the diagonals of this quadrilateral bisect each other, $ABCD$ is a parallelogram.

> **Reminder**
>
> In a parallelogram:
> - opposite sides are parallel and congruent
> - opposite angles are congruent
> - diagonals bisect each other.

Now consider chord DB, which subtends $\sphericalangle A$ at the circumference, and thus

$\sphericalangle A = \frac{1}{2}\sphericalangle DOB = \frac{1}{2} \times 180 = 90°$



> **Reminder**
>
> The measure of a straight angle is 180°.

If one of the angles is a right angle, then the opposite angle, $\sphericalangle C$, is a right angle. Since $\sphericalangle B$ and $\sphericalangle D$ are supplementary to either $\sphericalangle A$ or $\sphericalangle C$, then they are both right angles.

Therefore the parallelogram $ABCD$ is a rectangle.

Looking back at the problem we can see that we could have used a different method.

As you recall from earlier work, a quadrilateral whose diagonals bisect each other is a parallelogram, and a parallelogram whose diagonals are congruent must be a rectangle.

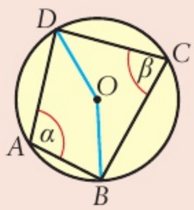
Reflect

Did we need to use angles supplementary to $\sphericalangle A$? Can you think of how to approach this problem differently?

Does this enable you to state Thales' theorem and prove it?

Investigation 11.1

A quadrilateral inscribed in a circle is called a cyclic quadrilateral. *ABCD* is such a quadrilateral.



Fact

Points *A*, *B*, *C* and *D* are called **concyclic**.

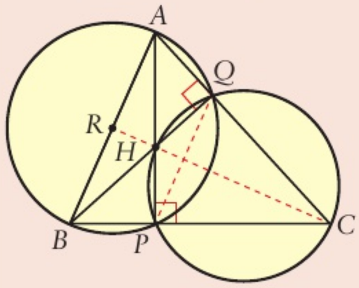
1. Can you find the measure of the obtuse and reflex angles at *O* in terms of α and β?
2. Can you find $\alpha + \beta$? Hence, what can you conclude about angles *A* and *C*?
3. What can you say about angles *B* and *D*?
4. Use your response to question 2 to find β in terms of α.
5. Extend *BC* to any point *E*. What is $\sphericalangle DCE$?
6. Extend *DA* to any point *F*. What is $\sphericalangle BAF$?
7. $\sphericalangle DCE$ and $\sphericalangle BAF$ are called **exterior angles to the cyclic quadrilateral.** Can you state a general conjecture about the exterior angles of a cyclic quadrilateral?

Reflect

What can you say about the sum of the exterior angles of a cyclic quadrilateral? (Consider only one angle at each vertex.)

Investigation 11.2

In the diagram below, the vertical heights AP and BQ meet at H. The segment CH is produced to meet AB at R.



1 Explain why $QHPC$ is cyclic.

2 Explain why $ABPQ$ is cyclic.

3 Join the common chord PQ, and explain why $\sphericalangle QCH \cong \sphericalangle QPH$

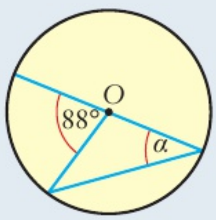
4 Explain why $\sphericalangle APQ \cong \sphericalangle ABQ$

5 Use $\triangle ABQ$ and $\triangle ACR$ to explain why CR is perpendicular to AB

Practice questions 11.2

1 Find the value of the variables for each circle.

a



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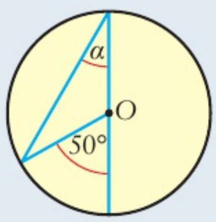

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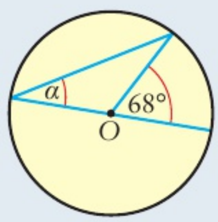
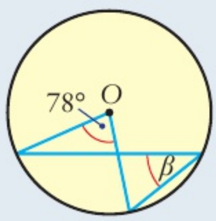


d



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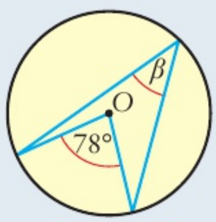

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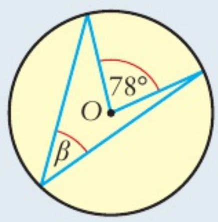

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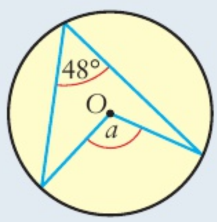
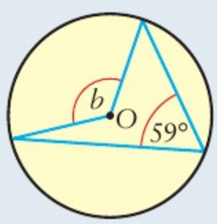


h



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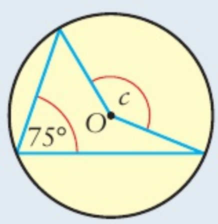
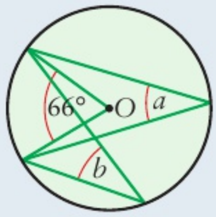


Hint

The 'measure', 'size' or 'value' of an angle all have the same meaning.

2 Find the value of the variables for each circle.

a



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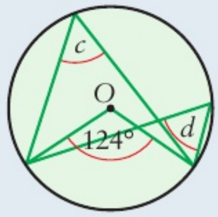

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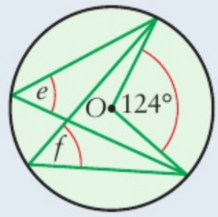

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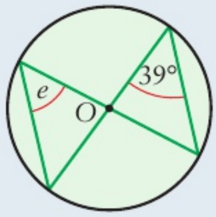

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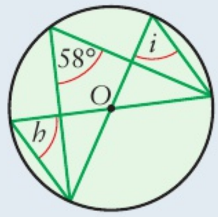

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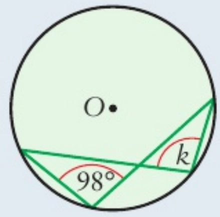

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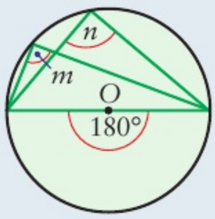

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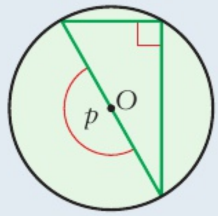

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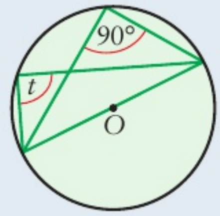
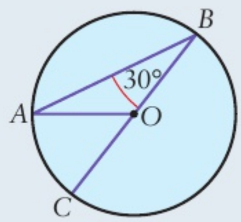
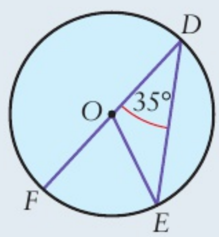


3 **a**



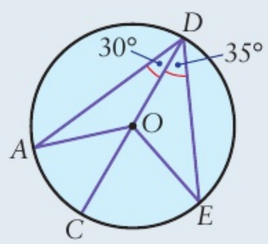


Find the size of:

i ∢*BAO** **ii** ∢*AOC* **iii** ∢*DEO*

iv ∢*EOF* **v** ∢*ABO* + ∢*ODE* **vi** ∢*AOC* + ∢*EOF*

b



Find the size of:

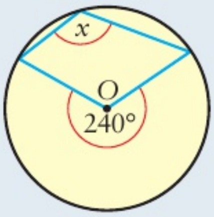
i ∢*DAO* **ii** ∢*AOC* **iii** ∢*DEO*

iv ∢*EOC* **v** ∢*ADE* **vi** obtuse ∢*AOE*

vii Is the angle at the centre (∢*AOE*) twice the angle at the circumference (∢*ADE*)?

4 Find the value of the variables for each circle.

a



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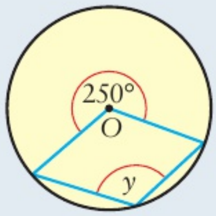
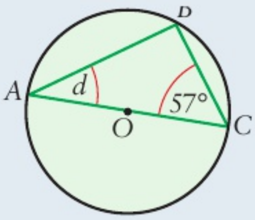


5 Find the value of each variable, giving reasons for your answers.

a



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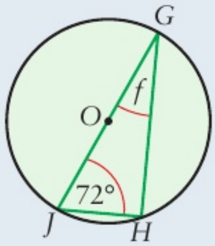

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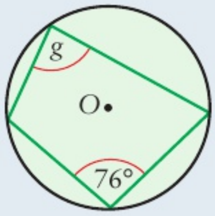

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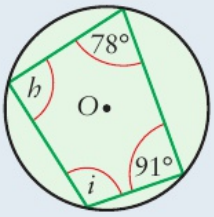

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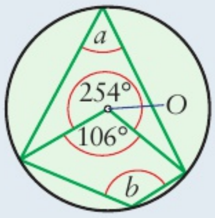

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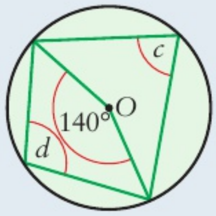

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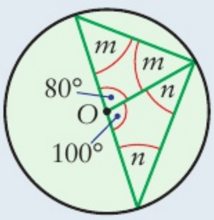

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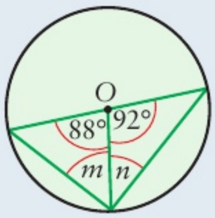

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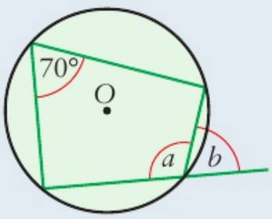

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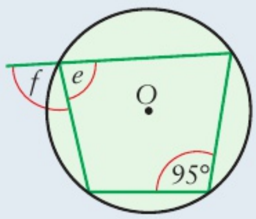
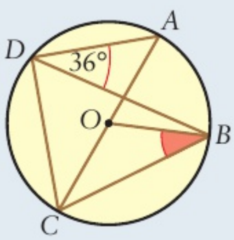
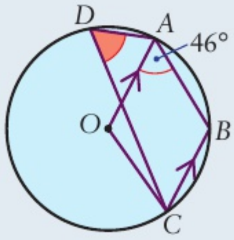


6 Work out the size of the angle at *B* in two different ways. *O* is the centre of the circle.

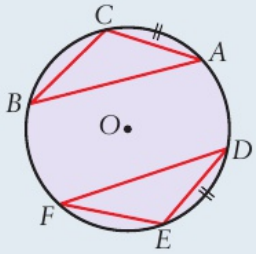


7 O is the centre of the circle. Work out the size of the angle at D.



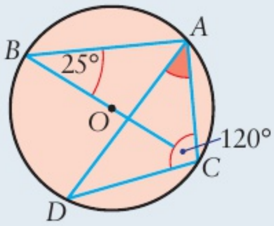
Challenge Q8

8 A circle with centre O is shown. Arcs AC and DE have equal measure. Show that the angles at B and at F have equal measure.



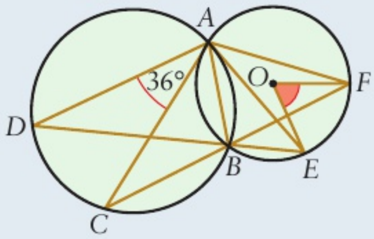
Challenge Q9

9 Find the measure of $\sphericalangle CAD$.



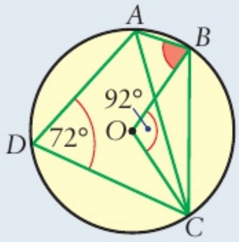
Challenge Q10

10 O is the centre of the small circle. Find the measure of $\sphericalangle EOF$.



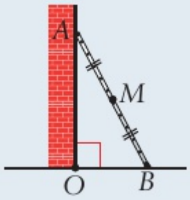
Challenge Q11

11 Find the measure of $\sphericalangle ABO$.

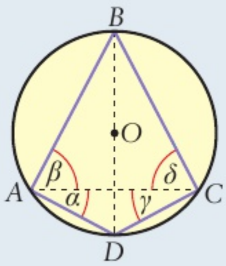


Challenge Q12

12 A ladder of length ℓ metres is initially resting against a wall, but it slips outwards, with its top sliding down the wall and its foot moving on the ground at right angles to the wall. What path does the midpoint of the ladder trace out?



13 Join the diagonals AC and BD of the cyclic quadrilateral $ABCD$. Let α, β, γ and δ be the measures of angles shown.



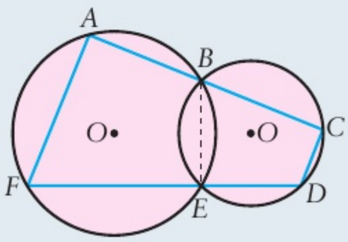
a Give a reason why $\sphericalangle DBC = \alpha$

b What other angles have sizes β, γ and δ?

c Prove that $\alpha + \beta + \gamma + \delta = 180°$

d Hence prove that the opposite angles of $ABCD$ are supplementary.

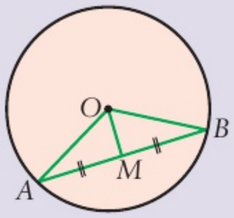
14 Show that $AF \parallel CD$



11.3 Chord properties of circles

Explore 11.4

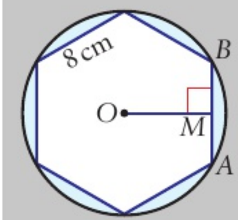
Consider the circle with centre O on the right and the chord AB. M is the midpoint of the chord.



Can you write down all you know about OM and justify your statements?

Worked example 11.5

Consider the regular hexagon with side 8 cm inscribed in the circle, with centre O. OM is the distance from O to side AB



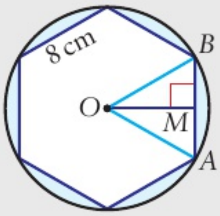
a Work out the length of OM.

b Work out the area of the shaded region.

Give your answers as exact values.

Solution

a Drawing the radii to A and B, will help us to relate OM to the given length.



Since the hexagon is regular, it divides the circle into six sectors of equal measure. Thus the angle at the centre is 60°

Thus, $\triangle OAB$ is an equilateral triangle.

Consider triangles OMA and OMB. These two triangles are congruent because of the hypotenuse–leg theorem for congruent triangles.

Hint

$\triangle OAB$ is isosceles with vertex angle of 60°

Thus $MA = MB = 4\,\text{cm}$

Now, consider either of the two triangles, OMA for example, and use Pythagoras' theorem:

$OA^2 = OM^2 + MA^2 \Rightarrow OM = \sqrt{8^2 - 4^2} = 4\sqrt{3}\,\text{cm}$

b The shaded area is the difference between the area of the circle and that of the hexagon.

area of circle $= \pi r^2 = 64\pi\,\text{cm}^2$

The area of the hexagon is six times the area of triangle OAB.

area of hexagon $= 6 \times \frac{1}{2} \times 8 \times 4\sqrt{3} = 48\sqrt{3}\,\text{cm}^2$

shaded region $= 64\pi - 48\sqrt{3}\,\text{cm}^2$

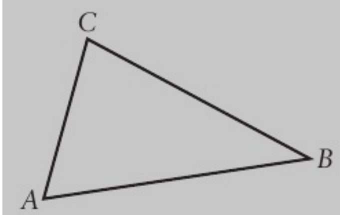
Fact

In general:

- If a line is drawn from the centre of a circle perpendicular to a chord, then it bisects the chord.
- If a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.
- The perpendicular bisector of a chord passes through the centre of the circle.

Worked example 11.6

Construct a circle that passes through the vertices of triangle ABC.



Solution

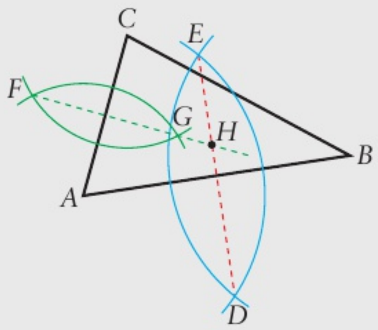
If the circle passes through the vertices, then the sides of the triangle are chords.

Thus the circle centre is where the perpendicular bisectors meet.

Hint

If you need to refresh your memory about constructing perpendicular bisectors, check Year 3, chapter 8.

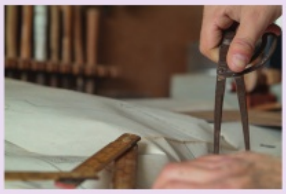
With A as centre and radius larger than half of AB, draw an arc of a circle. With the same radius and B as centre, draw another arc. The two arcs meet at points D and E. Draw DE. This is the perpendicular bisector of side AB. Similarly with A and C. FG is the perpendicular bisector of side AC. The perpendicular bisectors intersect at H. This is the centre of the required circle.



Note that we do not need to construct the perpendicular bisector to segment BC. The reason is that H is on the perpendicular bisector of AB and so is equidistant from A and B. For the same reason it is equidistant from A and C, and thus it is equidistant from A, B and C. So with H as centre and radius HA or HB or HC, we draw a circle that passes through the three vertices.

Connections

This area of mathematics is used by carpenters.

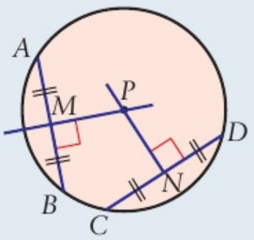


Hint

We did not draw the circle for aesthetic purposes.

Practice questions 11.3

1 For the diagram below:



a **i** give reasons why MP passes through the centre of the circle

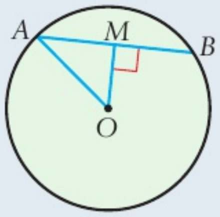
ii give reasons why NP passes through the centre of the circle.

iii Which point is the centre here? Why?

b Draw any circle. Use the method in part a to find the centre of your circle.

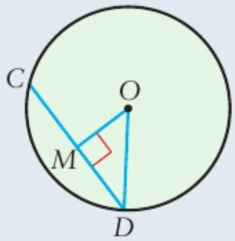
c The same method can be used to draw a circle that passes through any three non-collinear points. Choose any three non-collinear points and, by constructing two perpendicular bisectors, locate the centre and then draw the circle that passes through these points.

2 **a**



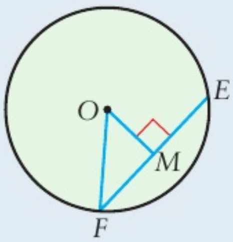
Given that $OM \perp AB$, $AO = 5\,\text{cm}$ and $OM = 4\,\text{cm}$, find the length of AB.

b



Given that $OM \perp CD$, $MO = 6\,\text{m}$ and $OD = 8\,\text{m}$, find the length of CD.

c

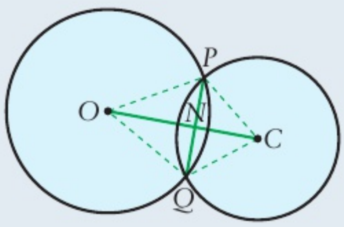


Given that $OM \perp EF$, $OM = 12\,\text{mm}$ and $OF = 13\,\text{mm}$, find the length of FE.

3 **a** A chord of length 12 cm is drawn on a circle of radius 8 cm. How far is this chord from the centre of the circle?

b A chord of length 10 cm has a perpendicular distance of 4 cm from the centre of the circle. What is the radius of the circle?

4 These two circles have as their centres points O and C. PQ is the common chord joining the points of intersection of the two circles. N is the point where PQ intersects the line OC, which joins the centres.



a Show that the $\triangle POC \cong \triangle QOC$

b Hence, show that $\sphericalangle POC \cong \sphericalangle QOC$

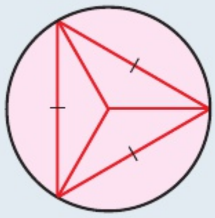
c Now, show that $\triangle PON \cong \triangle QON$

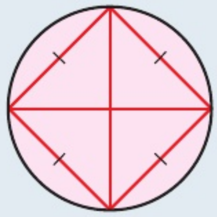
d Hence, show that N bisects PQ and that $PQ \perp OC$

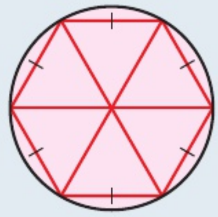
Fact

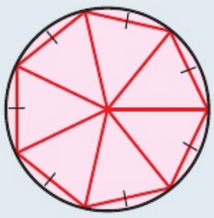
Question 4 proves the following theorem: When two circles intersect, the line joining their centres bisects their common chord at right angles.

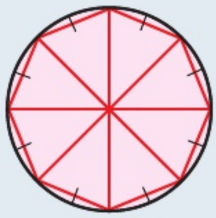
5 Find the size of the angle subtended at the centre by one side of each of these regular shapes.

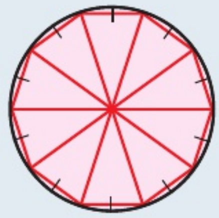
a 

b 

c 

d 

e 

f 

6 In circles of radius 3 cm, construct:

a an equilateral triangle

b a square

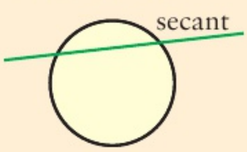
c a regular hexagon

d a regular octagon.

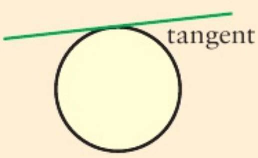
Reminder

Given a line and a circle, there are three possibilities:

- The line may be a secant, cutting the circle at two points.



- The line may be a tangent, touching the circle at just one point.



- The line may not intersect the circle at all.

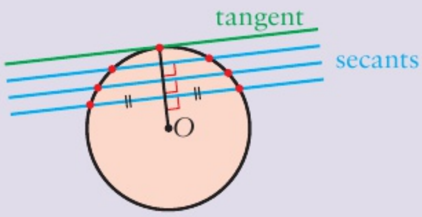
The words 'secant' and 'tangent' are from Latin: 'secant' means 'cutting' and 'tangent' means 'touching'.

11.4 Tangent properties of circles

A **tangent** to a circle is a line that meets the circle at just one point. This point is called the **point of tangency** or **point of contact**. All other points on the tangent are outside the circle.

Explore 11.5

You draw parallel secants that cut a circle at two points, creating chords within the circle. By joining the centre to the midpoints, the line you draw is the perpendicular bisector of these chords.



If you continue until the secants become a tangent, what do you observe?

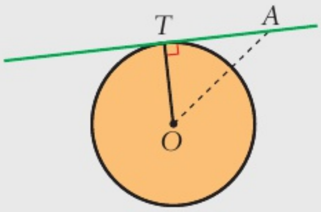
Fact

The shortest distance from a point to a straight line is the perpendicular distance.

Worked example 11.7

Let T be a point on a circle with centre O. Show that the line through T, perpendicular to the radius OT, is a tangent to the circle.

Solution



To show that the line is tangent, we need to show that it has only one point of intersection with the circle.

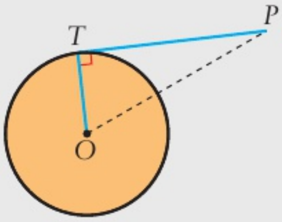
Consider any point A on the line, other than T.
Draw a line segment joining O to A. Now, in $\triangle OTA$, $\sphericalangle T$ is the largest angle, and thus OA is the largest side. Specifically, $OA > OT = r$
Therefore, A, any point on the line other than T, lies outside the circle.
This means that T is the only point on the line that is on the circle. Hence the line must be a tangent to the circle because it meets the circle at a single point but does not cross it.

Worked example 11.8

Construct tangents to a circle with centre O from a point P outside the circle.

Challenge

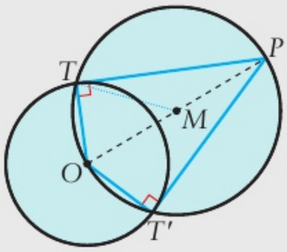
Solution



As with most constructions, we first draw a sketch in an attempt to understand the problem. Then we proceed with construction.

Since we know only P and O, we can use the fact that the tangent is perpendicular to the radius at the point of contact. The task is reduced to finding T.

T is a point on the circle with centre *O*. We know that the tangent meets the radius of the circle at right angles. We also know that an angle subtended by a diameter is a right angle. Therefore, if we draw a circle with diameter *OP*, then *T* will be the intersection of that circle and the circle with centre *O*.



We can construct *M*, the midpoint of the line segment, *OP*. Then, with *M* as centre and radius *MO*, we draw a circle. We have two positions for *T* because the two circles intersect at two points.

Now, since *PT* is perpendicular to a radius at the point of contact, then it is a tangent. The same is true about *PT*′.

Connections

You learned how to construct the midpoint of a segment in Year 3.

Reflect

What do you observe about *PT* and *PT*′?

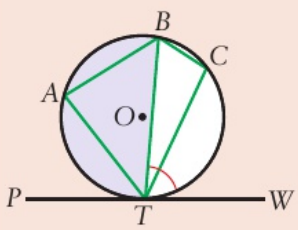
Can you make a general statement about tangents drawn to a circle from a point outside the circle?

Reminder

Opposite angles in a cyclic quadrilateral are supplementary.

Investigation 11.3

Before you carry out this investigation, you will need to know some new definitions.



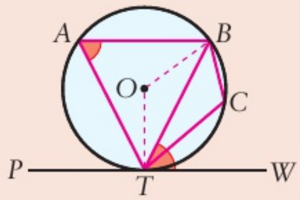
$\sphericalangle BTW$ is the acute angle between the tangent *PW* and the chord *BT*.

The shaded segment of the circle is called the **alternate segment** to $\sphericalangle BTW$, while $\sphericalangle BAT$ is an **angle in the alternate segment**.

$\sphericalangle BCT$ is an angle in the alternate segment to $\sphericalangle BTP$.

Consider this diagram.
The chord BT meets the tangent PW at the point of contact, T.

O is the centre of the circle.



$\measuredangle BAT$ is any angle in the segment alternate to $\measuredangle BTW$.

$\measuredangle BCT$ is any angle in the segment alternate to $\measuredangle BTP$.

Let $\measuredangle BTW = x$

1 In terms of x, what is the measure of $\measuredangle OTB$?

2 In terms of x, what is the measure of $\measuredangle TOB$?

3 Can you deduce the measure of $\measuredangle TAB$?

4 Let A' be any other point in the same segment as A.
What can you say about $\measuredangle TA'B$?

5 Fill the spaces in a copy of the following statement to make it true:

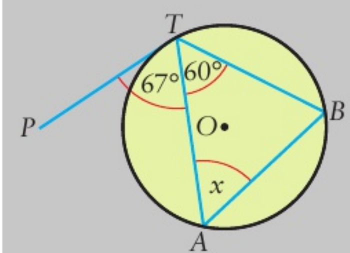
An ______ formed by a ______ to a circle with a ______ drawn to the point of ______ is equal to any ______ in the ______ segment.

6 In terms of x, what is the measure of $\measuredangle PTB$?

7 Given what you found in part 3, in terms of x, what is the measure of $\measuredangle TCB$?

Worked example 11.9

PT is tangent to the circle at T. Find the value of x.

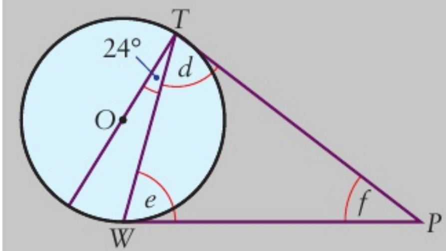


Solution

$\measuredangle B = 67°$	Angle in the alternate segment for $\measuredangle PTA$
$x = 180 - (60 + 67) = 53°$	The sum of angle measures in a triangle is 180°

Worked example 11.10

Find the value of each variable, giving reasons for your answers.



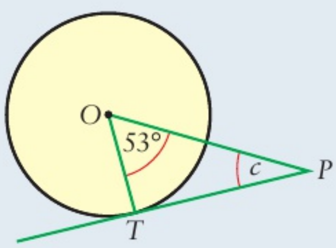
Solution

$d + 24 = 90$	Radius $OT \perp$ tangent TP.
So, $d = 66°$	
$PT = PW$	Equal tangents from a point outside the circle.
Therefore, $e = 66°$	$\triangle PTW$ is isosceles.
Finally, $f = 48°$	Angle sum of a triangle.

Practice questions 11.4

1 Find the value of each variable, giving reasons for your answers. (PT and PW are tangents.)

a



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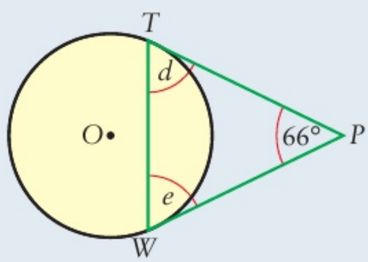

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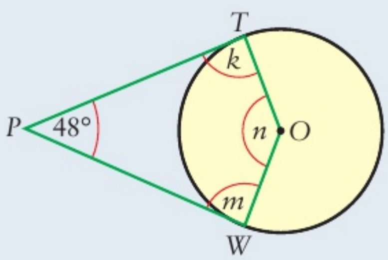
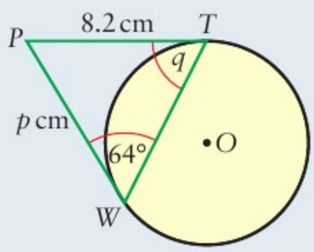


d

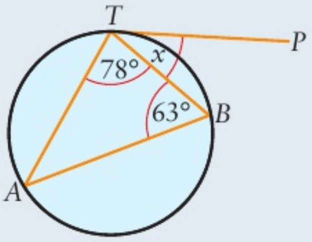


Fact

- An angle formed by a tangent to a circle with a chord drawn to the point of contact is equal to any angle in the alternate segment.
- The angle between a tangent and the radius drawn to the point of contact is a right angle.
- Two tangents drawn to a circle from a point outside the circle have equal lengths.

2 Find the value of each variable, giving reasons for your answers. (*PT*, *PW* and *QR* are tangents.)

a



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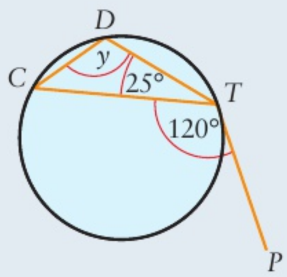

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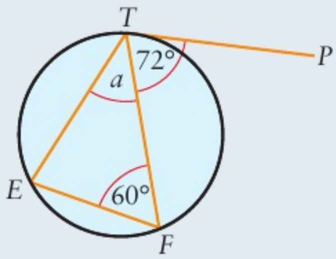

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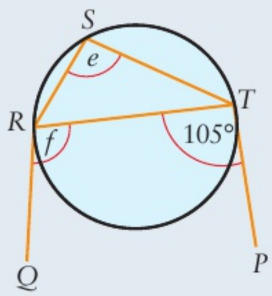
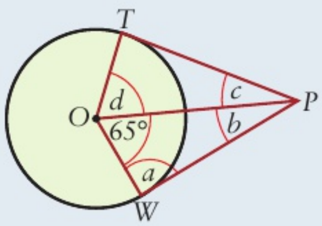


3 Find the value of each variable, giving reasons for your answers. (*PT* and *PW* and *QW* are tangents.)

a



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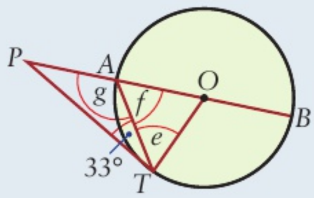

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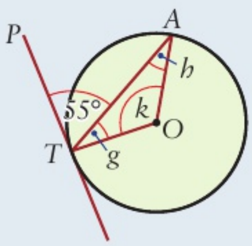
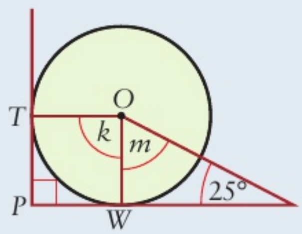


d



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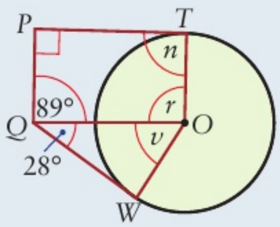

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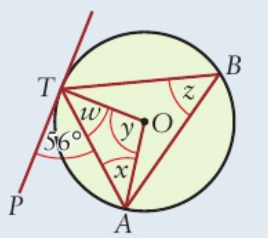

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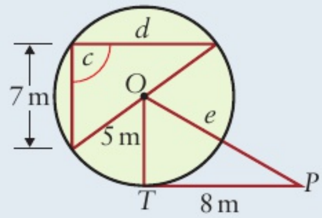

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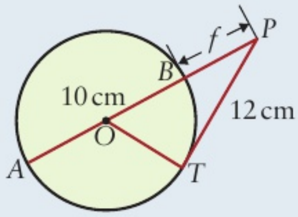

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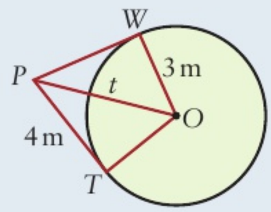
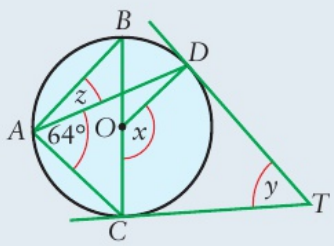
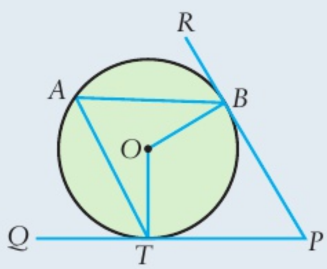


4 *ABCD* is a quadrilateral inscribed in a circle with centre *O*. *BC* is a diameter and $\sphericalangle CAD = 64°$. *TC* and *TD* are tangents. Work out the values of x, y, and z.



5 *O* is the centre of the circle. *BT* is a chord that subtends $\sphericalangle BAT$ at the circumference and $\sphericalangle TOB$ at the centre. *PT* and *PB* are tangents to the circle.



a Prove that $\sphericalangle BOT = 2\sphericalangle BTP$

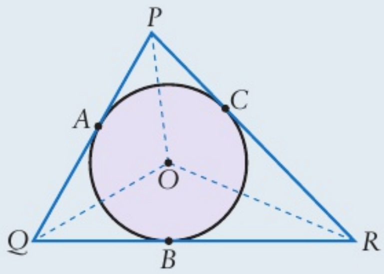
b Prove that $\sphericalangle ATQ + \sphericalangle RBA + \sphericalangle PBT = 180°$

c Prove that $\sphericalangle BPT = 180° - 2\sphericalangle BAT$

6 From a point *T* on a circle, chords of equal length are drawn to meet the circle at *A* and *B*. Show that the tangent at *T* is parallel to the chord *AB*.

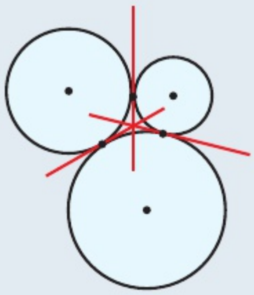
7 Show that the three angle bisectors of triangle PQR are concurrent and that their point of intersection is the centre of a circle which touches each side of the triangle. (This is called the **incircle** of the triangle.)

Challenge Q7



8 Three different sized circular discs touch each other. Prove that the three common tangents are concurrent.

Challenge Q8

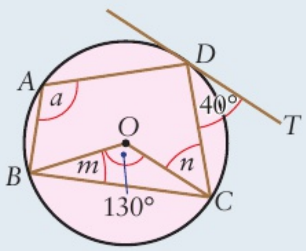


9 Consider a circle with centre O. A, B, C and D are points on this circle. Tangent DT is parallel to OC

a Work out the values of a, m and n

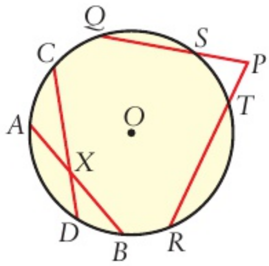
b Work out the measure of $\sphericalangle ODC$

c Work out the measure of reflex angle BOD



11.5 Further circle properties

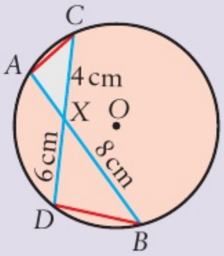
When two chords or two secants intersect, they divide each other into **intercepts**.



- *AB* and *CD* divide each other **internally** at *X*.
  - *AX* and *XB* are the intercepts of chord *AB*.
  - *CX* and *XD* are the intercepts of chord *CD*.
- *PQ* and *PR* divide each other **externally** at *P*.
  - *PQ* and *PS* are the intercepts of secant *PQ*.
  - *PR* and *PT* are the intercepts of secant *PR*.

Explore 11.6

Consider the circle with centre *O* and two chords *AB* and *CD*. The measures of three intercepts are given.



Can you work out the measure of *AX*?

How did you arrive at your calculation?

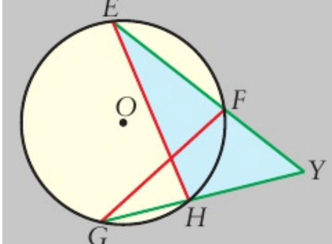
Can you generalise your method about relationships between intercepts?

Hint

Consider joining *AC* and *DB*, as has been done in the diagram.

Worked example 11.11

Consider the diagram below with two secants *YE* and *YG* that cut the circle at *F* and *H*, respectively.



$EY = 8\,\text{cm}$, $YG = 6\,\text{cm}$, and $HY = 4\,\text{cm}$

a Work out FY.

b If $EY = y$, $YG = n$, and $HY = m$ and $FY = x$, use your answer from part a to find a relationship between m, n, x and y.

c State a relationship between intercepts of secants.

Solution

a We can use the 'hidden' hint in the shaded triangle. Since FY is a side of $\triangle GFY$, we look for a triangle similar to it. We look at the shaded triangle EHY and check if the triangles are similar. Indeed they are:

$\sphericalangle Y$ is common to both triangles and $\sphericalangle E \cong \sphericalangle G$ because they are subtended by the same arc, FH

Therefore $\triangle GFY \sim \triangle EHY$ by the AAA postulate.

If two triangles are similar, then the corresponding sides are proportional. Thus,

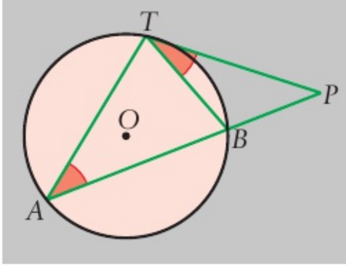
$\frac{FY}{HY} = \frac{GY}{EY} \Rightarrow FY \times EY = HY \times GY \Rightarrow 8FY = 4 \times 6$, and so $FY = 3$

b From part a, $\frac{FY}{HY} = \frac{GY}{EY} \Rightarrow FY \times EY = HY \times GY \Rightarrow x \times y = m \times n$

c From part b we observe that the segments involved are nothing but the intercepts of the two secants. We can claim that if two secants are drawn from a point outside a circle, then the product of the intercepts are equal.

Worked example 11.12

In the diagram, PT is a tangent to the circle. PA is a secant that cuts the circle at A and B such that $AP = 9$ cm and $BP = 4$ cm. Work out PT.



Solution

We consider similar triangles PTB and PAT

$\sphericalangle PTB \cong \sphericalangle PAT$ Angle in alternate segment

$\sphericalangle P$ is common to both triangles.

$\therefore \triangle PTB \sim \triangle PAT$ by the AAA postulate.

Using the fact that corresponding sides are proportional, we have:

$$\frac{PT}{AP} = \frac{BP}{PT} \Rightarrow PT^2 = AP \times BP \Rightarrow PT^2 = 9 \times 4 = 36$$

So $PT = 6\,\text{cm}$

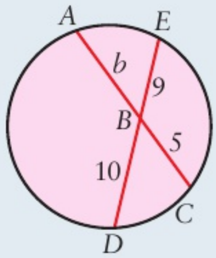
Fact

- The products of intercepts of intersecting chords are equal.
- The products of intercepts of intersecting secants are equal.
- The square of the length of a tangent is equal to the product of the intercepts of a secant drawn from the same external point. i.e. $(PT)^2 = AP \times PB$

Practice questions 11.5

1 Find the value of each variable. All lengths are in centimetres and PT is a tangent wherever it is used.

a



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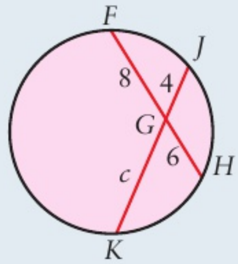

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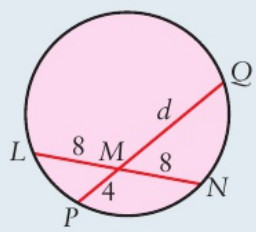

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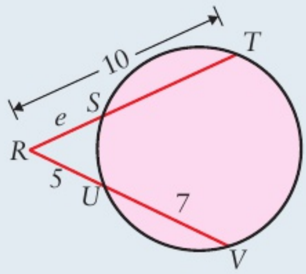

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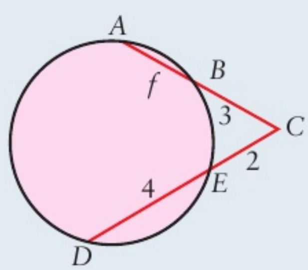

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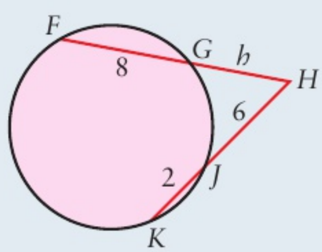

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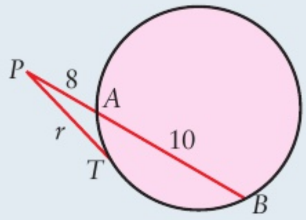

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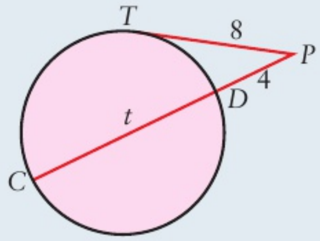

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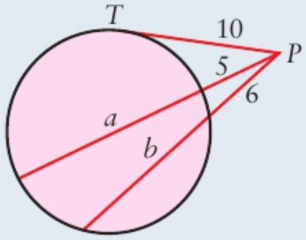

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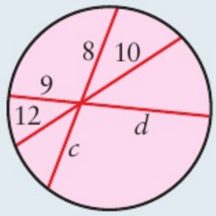

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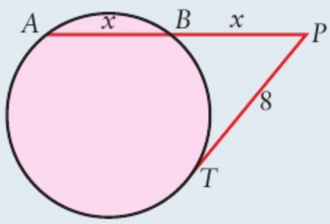
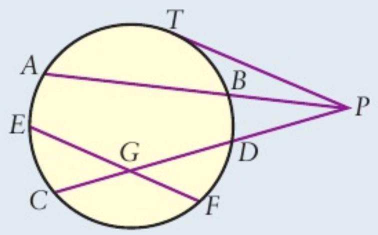


2 Consider the figure below. PT is a tangent.



a Find PT if $AB = 9\,\text{m}$ and $BP = 3\,\text{m}$.

b Find AB if $BP = 10\,\text{cm}$ and $PT = 13\,\text{cm}$.

c Find CD if $DP = 5\,\text{m}$, $AB = 8\,\text{m}$ and $BP = 6\,\text{m}$.

d Find EG if $GF = 20\,\text{m}$, $CG = 30\,\text{m}$ and $GD = 25\,\text{m}$.

e Find CD if $CG = 15\,\text{m}$, $EF = 35\,\text{m}$ and $EG = 22\,\text{m}$.

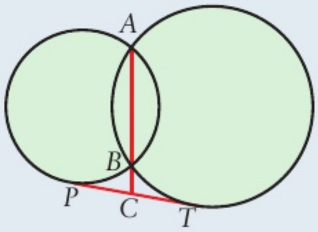
f Find CD if $TP = 9\,\text{cm}$ and $DP = 5\,\text{cm}$.

g Find PT correct to one decimal place if $CD = 8\,\text{cm}$ and $DP = 10\,\text{cm}$.

3 AB is the diameter of a circle. AB bisects a chord CD at the point E. Find the length of CE if $AE = 3$ m and $BE = 9$ m.

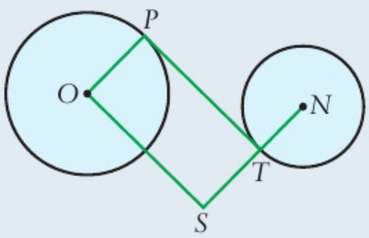
Challenge Q4

4 PT is a direct common tangent of the circles drawn. AB is a common chord that has been produced to meet the common tangent at C. $BC = 2$ cm and $AB = 6$ cm. Find the length of PT.



5 PT is a tangent to a circle, centre O, and the tangent touches the circle at T. A is a point on the circle and AP cuts the circle at B such that $AB = BP$. Find the length of AB if PT is 8 cm.

6 PT is an indirect common tangent of the two circles that have centres O and N as shown.



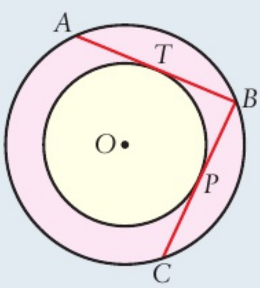
$OP = 6$ cm, $NT = 5$ cm and $ON = 15$ cm. $OS \parallel PT$

a Show that $\sphericalangle OSN = 90°$

b Show that $OPTS$ is a rectangle.

c Find the length of PT.

7 The two circles are concentric. AB and CB are chords of the larger circle and tangents to the smaller circle. $CP = 4$ cm. Work out AB.



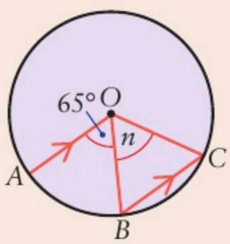
Self assessment

- I can work with different parts of a circle:
  - segment and sector
  - chord and secant
  - minor and major arcs and segments.
- I can use the fact that the perpendicular bisector of a chord passes through the centre.
- I can use the fact that two chords equidistant from the centre are congruent.
- I can find an angle subtended by an arc if I know the angle at the centre.
- I can find the angle between a radius and a tangent at the point of tangency.
- I can use the fact that tangents from the same point outside the circle are congruent.
- I can work with cyclic quadrilaterals and find connections between:
  - opposite interior angles
  - exterior angles and the opposite interior angles.
- I can work with intercepts of intersecting chords and secants.
- I can connect tangents and secants drawn from the same point outside the circle.
- I can compare the angle between a tangent and a chord with the angle in the alternate segment.

Check your knowledge questions

1 Find the value of each variable, giving reasons for your answers.

a



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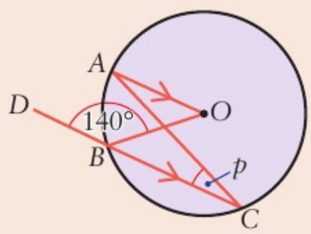

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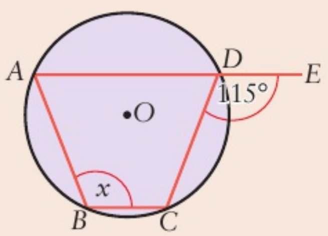

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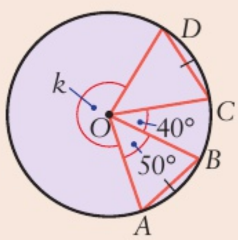

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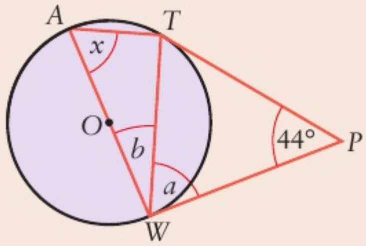

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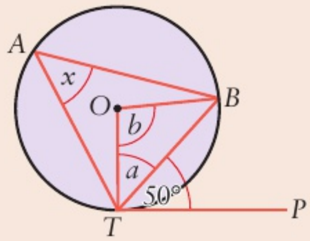
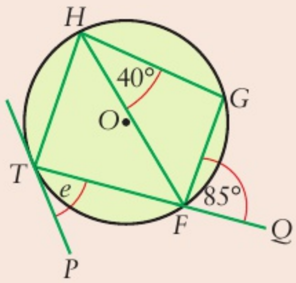


2 Find the value of each variable, giving reasons for your answers.

a



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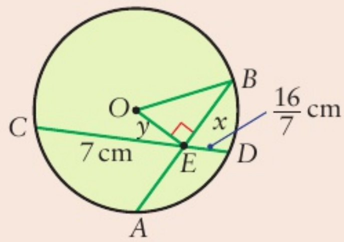

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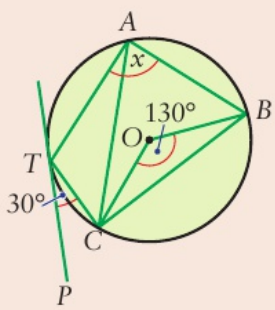
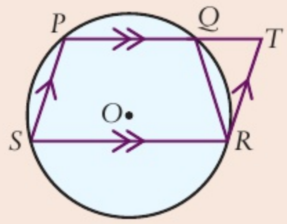
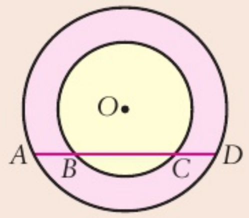


3 $PQRS$ is a cyclic quadrilateral. Side PQ has been extended to T so that $PTRS$ is a parallelogram and $\sphericalangle PSR = 70°$. Find the measures of all angles of $\triangle QRT$.



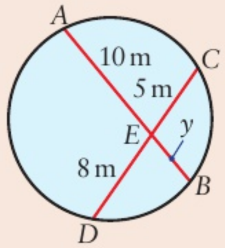
4 In the diagram, O is the centre of two concentric circles. $ABCD$ is a straight line. Prove that $AB = CD$



5 EB is the common chord of two intersecting circles. AB is a diameter of the smaller circle which is produced to meet the larger circle at C. DA passes through E, and D is on the larger circle. Find $\sphericalangle ACD$

6 Find the value of the variable in each circle, giving reasons for your answers.

a



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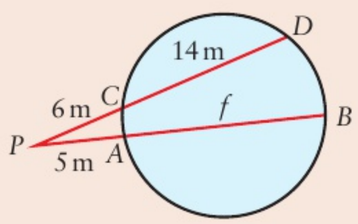

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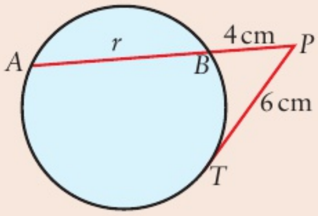
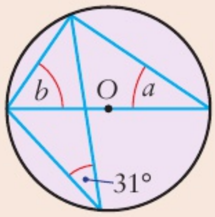


7 O is the centre of each circle. PT and PW are tangents. Find the size of the variables in each diagram.

a



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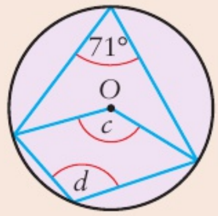

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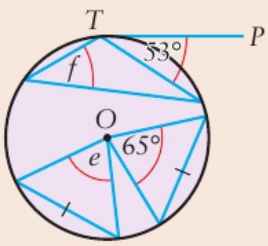

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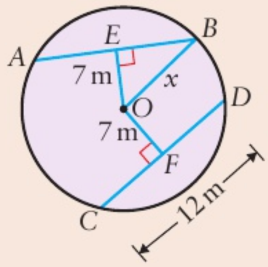

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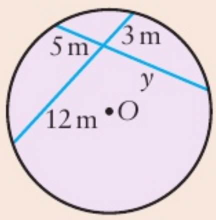

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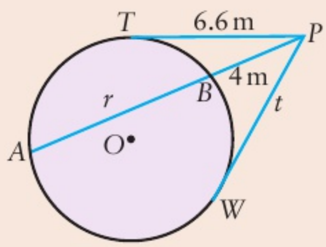

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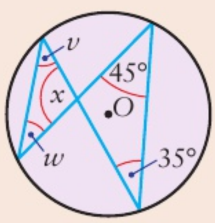

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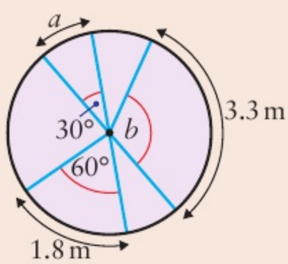
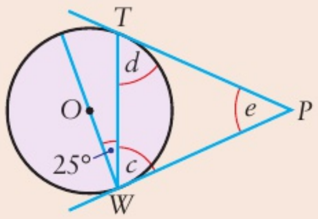


i



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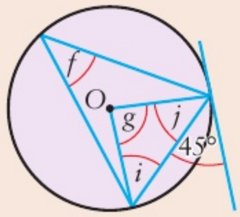

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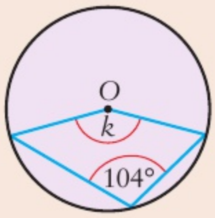
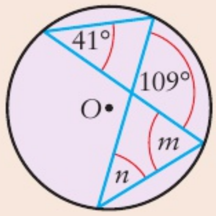


l



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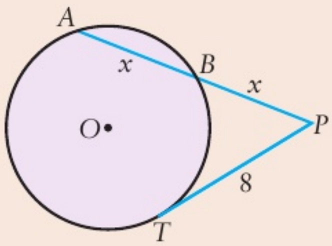

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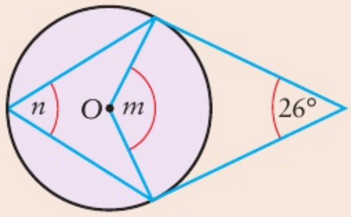
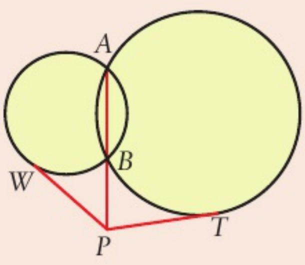


8 **a** AB is the common chord of the circles and has been extended to P. From P, tangents PT and PW have been drawn to the circles.

Challenge Q8a

i Show that $WP = PT$

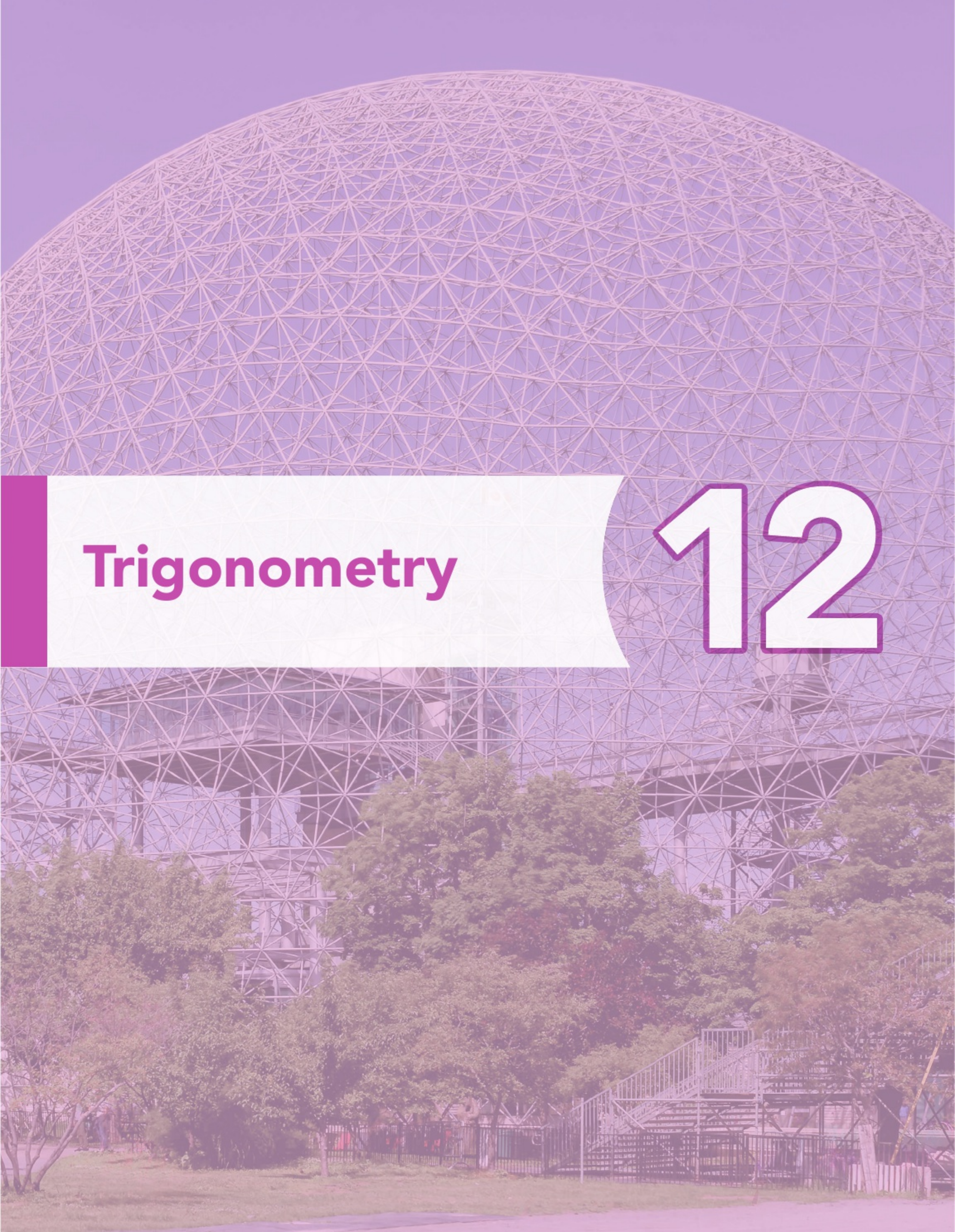
ii If $BP = 14.6$ cm and $PT = 19.4$ cm, find the length of AB correct to three significant figures.



b Prove that the bisector of the angle between the tangents drawn to a circle from an external point passes through the centre.

c If AB and AC are two tangents to a circle and $\angle BAC = 84°$, what are the sizes of the angles in the two segments into which BC divides the circle?

d P is a point within a circle of radius 13 cm and XY is any chord drawn through P so that $XP \times PY = 25$. Find the length of OP if O is the centre of the circle.



12 Trigonometry

12 Trigonometry

KEY CONCEPT

Relationships

RELATED CONCEPTS

Generalisation, Quantity, Space

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Understanding the relationships between quantities, and being able to generalise them, allows us to calculate our position in space which can help us to meet our everyday needs.

Factual

- How are the three basic trigonometric ratios defined?
- What are the values of the three basic trigonometric ratios for special triangles?

Conceptual

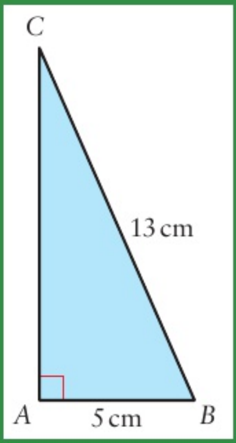
- What different methods exist to calculate the angles of a triangle when we know the length of its sides?
- Are trigonometric ratios always defined for any triangle?

Debatable

- Can we always generalise an observable pattern?
- Can trigonometry help us to measure objects that are inaccessible or not directly measurable?

Do you recall?

1 Find the length of the unknown side in the right-angled triangle.



2 Simplify the following fractions.

a $\dfrac{\frac{2}{3}}{4}$ b $\dfrac{\frac{3}{10}}{\frac{7}{20}}$

3 Rationalise the denominator in each of the following fractions.

a $\dfrac{3}{\sqrt{7}}$ b $\dfrac{1}{\sqrt{2}}$ c $\dfrac{6}{\sqrt{3}}$

4 Find the value of x in these algebraic fractions.

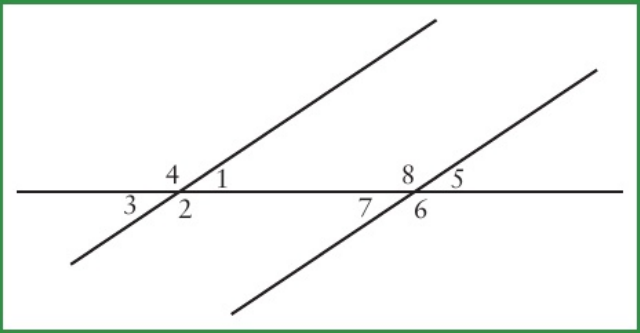
a $\dfrac{x}{10} = \dfrac{2}{5}$ b $\dfrac{2}{x} = \dfrac{1}{4}$

5 How can you tell whether two triangles are similar?

6 What is the sum of the interior angles in a triangle?

7 Complete the following definition: Two angles α and β are complementary if….

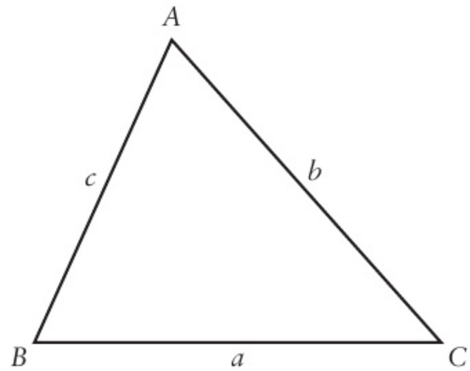
8 In the diagram below, the two lines are parallel. State which angles are congruent.



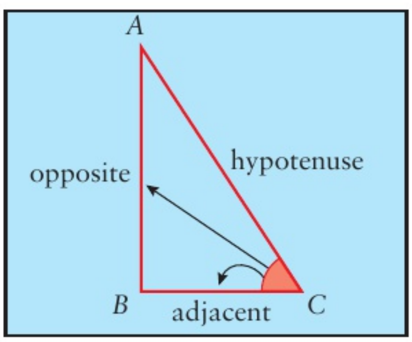
12.1 Labelling triangles

As you have seen before, Pythagoras' theorem establishes a relationship between the sides of a right-angled triangle. There are other important relationships in right-angled triangles: those between the sides and the angles. **Trigonometry** is the branch of mathematics that studies these relationships. The word trigonometry comes from two Greek words: *trigonon*, which means triangle, and *metron*, which means measure.

There is a standard way of naming vertices and sides in triangles. The vertices are labelled with capital letters. Each side is labelled with the lower-case letter corresponding to the capital letter of the vertex opposite to that side, as shown in the diagram.



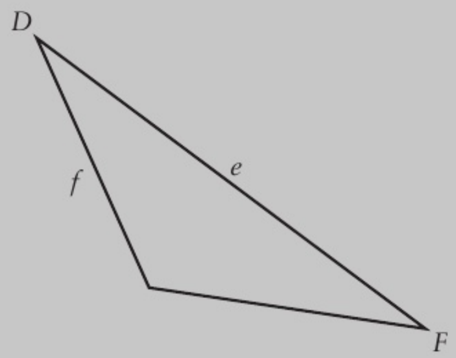
In right-angled triangles, we know from Pythagoras' theorem that the side opposite to the right angle is called the **hypotenuse**. The hypotenuse is the longest side. Now, choose one of the two acute angles. The other two sides are named according to their position with respect to the chosen angle. The side opposite to the chosen angle is called the **opposite** side, and the side next to the chosen angle is called the **adjacent** side.



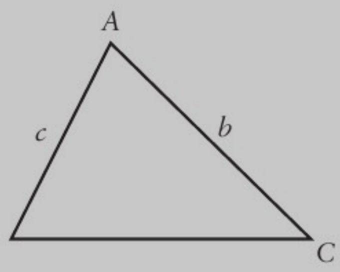
Worked example 12.1

Label the missing sides and vertices in these triangles.

a

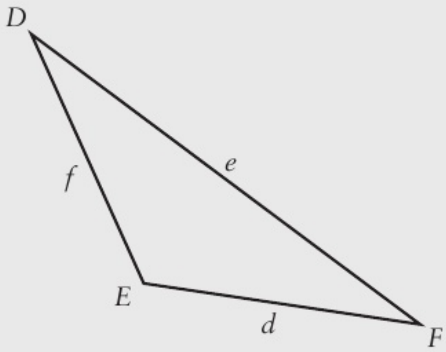


b

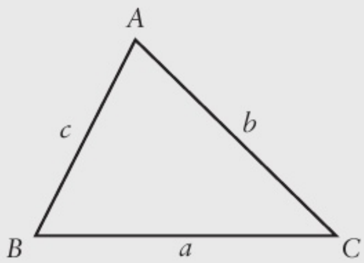


Solution

a The missing label for the vertex opposite to side e must be E.
The missing label for the side opposite vertex D must be d.



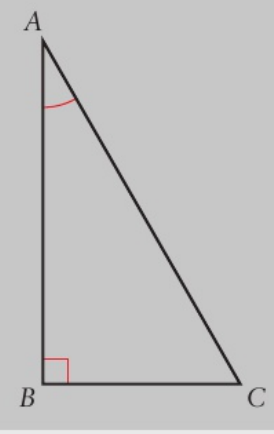
b The missing label for the vertex opposite to side b must be B.
The missing label for the side opposite vertex A must be a.



Worked example 12.2

Name the sides as opposite or adjacent with reference to the angle marked.

a



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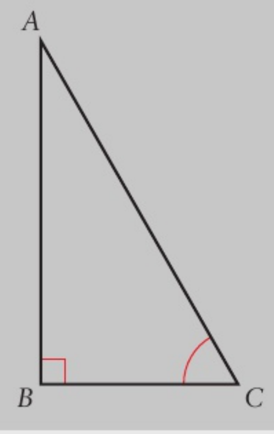
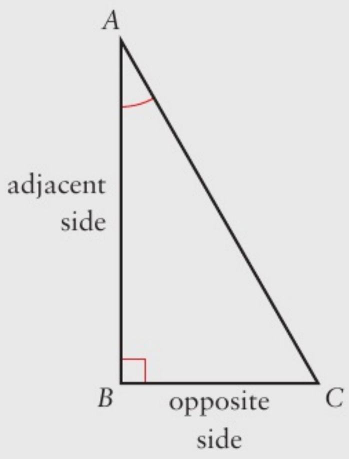
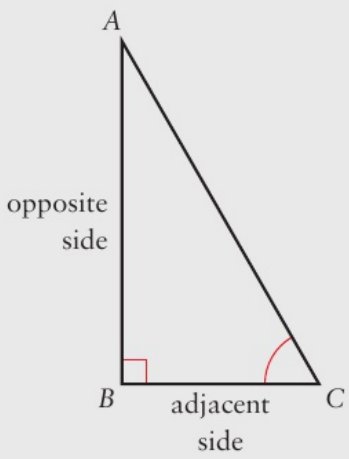


Solution

a The angle A is marked. The hypotenuse is AC. The side, BC, opposite A is called the opposite side and the remaining side, AB, is called the adjacent side.



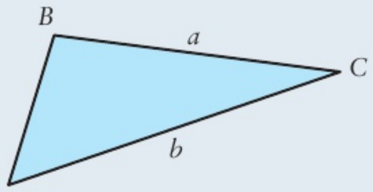
b This time the angle C is marked. The hypotenuse is still AC but we switch the opposite and adjacent sides. The side, AB, opposite C is the opposite side and the remaining side, BC, is the adjacent side.



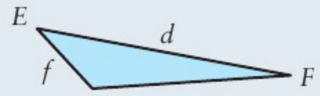
Practice questions 12.1

1 Label the missing sides and vertices in these triangles.

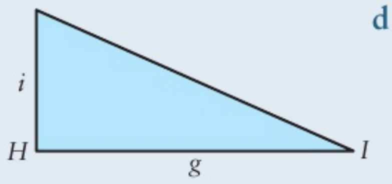
a



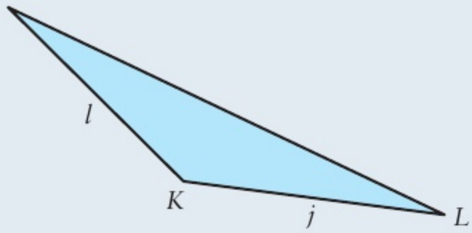
b



c

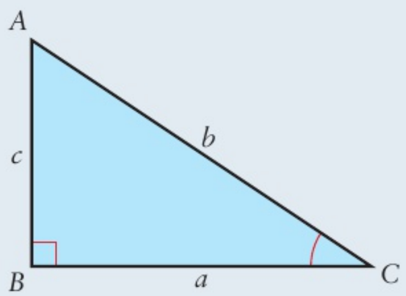


d



Use these diagrams for questions 2–4.

a



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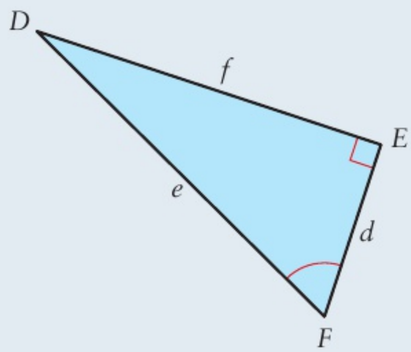

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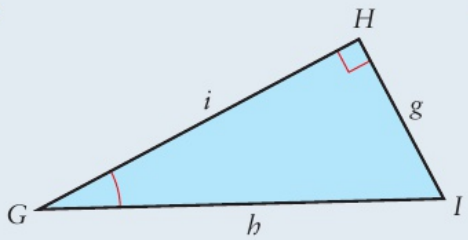

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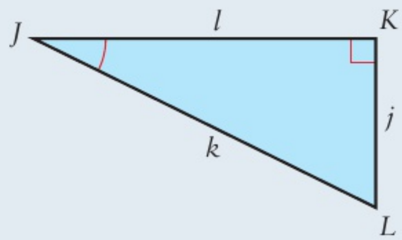


2 Name the opposite side to the marked angle.

3 Name the adjacent side to the marked angle.

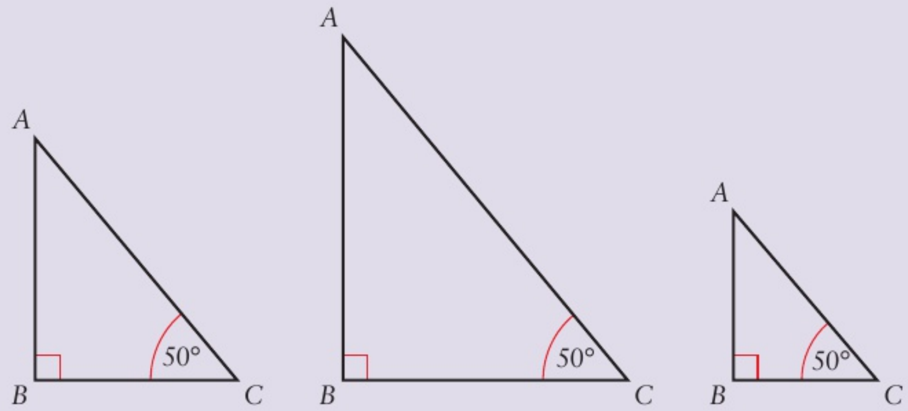
4 Name the hypotenuse.

12.2 The tangent ratio

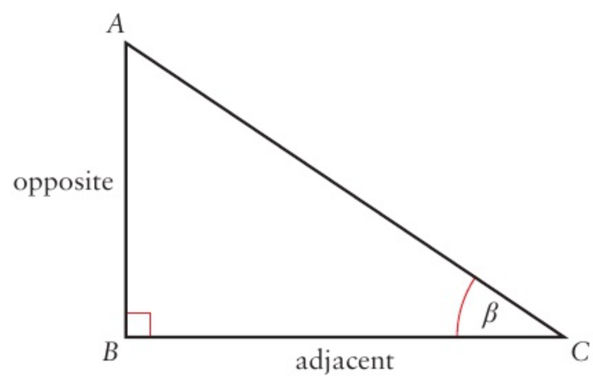
Explore 12.1

What can you say about the ratio $AB:BC$ in each of these right-angled triangles?

Can you prove your observation?



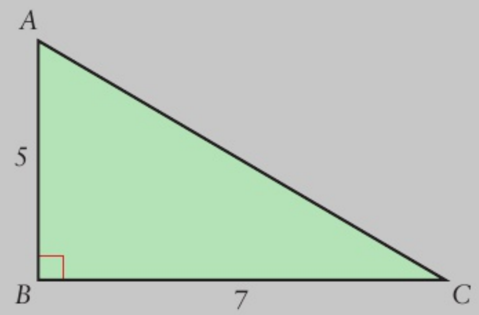
In a right-angled triangle, we define the **tangent** of an angle as the ratio of the length of the side opposite to the angle to the length of the side adjacent to the angle. This ratio is well defined and does not change when the size of the triangle changes, as long as it retains a similar shape. It depends only on the angle. The mathematical reason for this is that the ratio of corresponding sides in similar triangles remains constant.



$$\tan\beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{BC}$$

Worked example 12.3

Use the diagram to find $\tan C$ and $\tan A$.



Solution

The question asks us to find the tangents of two angles. To find the tangent of an angle we identify the sides opposite and adjacent to the angle, then find the ratio between them.

First we identify the opposite and adjacent sides for each angle.
Then we use the definition of tangent.

$$\tan C = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{BC} = \frac{5}{7}$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AB} = \frac{7}{5}$$

Reflect

Thinking skills

How are angle C and angle A related in the right-angled triangle in Worked example 12.3? How are $\tan C$ and $\tan A$ related?

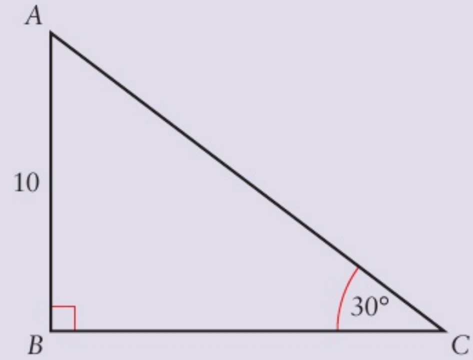
For what triangle is the value of the tangent ratio 1?

Does the tangent ratio have a unit of measure?

In Worked example 12.3, we were given a right-angled triangle and used the measures of its sides to find the value of the tangent ratio. Most scientific calculators have a tangent key that gives precise or rounded values for the tangent of a given angle.

Explore 12.2

Can you find the length of side BC?

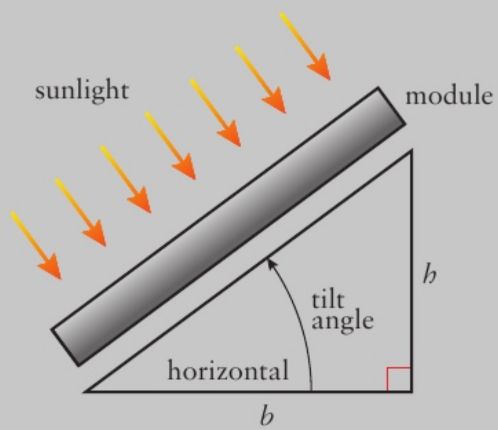


In a right-angled triangle, if we know the length of one of the shorter sides and one acute angle, we can use the tangent ratio to find the length of the other short side.

Fact

Solar energy, together with wind power, hydropower and geothermal energy, is one of the renewable energies.

Worked example 12.4



The angle formed by a solar panel with the horizontal is called the **tilt angle**.

Suppose that a solar panel has a tilt angle of 37° and a vertical height h of 1 m. Find the horizontal distance b that the solar panel covers. Give your answer to three significant figures.

Solution

The length of the solar panel, the horizontal distance b and the vertical height h form a right-angled triangle. The opposite side of the tilt angle is the vertical height and the adjacent side is the horizontal distance.

We will use the definition of tangent to write an equation with b as variable. Then we will solve for b. We can use a calculator to find the value of the tangent of a 37° angle:

$$\tan 37° = \frac{1}{b}$$

This is an algebraic fraction with variable b. We can now solve for b:

$$b = \frac{1}{\tan 37°}$$

We can use a calculator to find the value of $\frac{1}{\tan 37°}$

$b = 1.327\ldots$

The horizontal distance covered by the solar panel is 1.33 m (3 s.f.).

In order not to lose accuracy, we either evaluate $\tan 37°$ in the last step or store the value of $\tan 37°$ in our calculator and use the stored value for further calculations.

Connections

Globalisation and sustainability

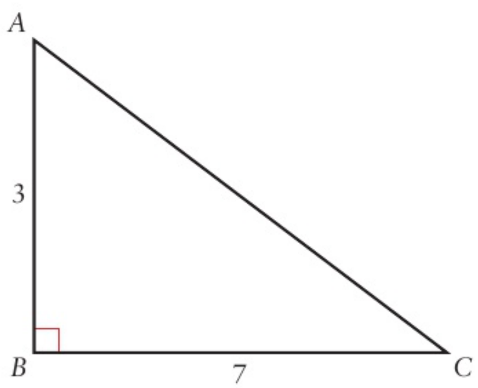
Reminder

Remember to set up your calculator in degree mode when evaluating $\tan 37°$.

As we have seen, we can use the tangent ratio to find the measure of one side of a right-angled triangle when we know the measure of an angle and the measure of the other side.

If we know the lengths of the sides opposite and adjacent to an angle in a right-angled triangle, can we find the measure of the angle?

Yes. We can use a scientific calculator to find the measure of an acute angle, given its tangent.



Hint

The notation $\tan^{-1}$ in place of arctan is very common, but can be confusing. There is a gradual move towards using 'arctan' (or 'Arctan' depending on context). You will notice this particularly on a calculator where the keypad may use arctan, but the display will probably show $\tan^{-1}$, inherited from an older operating system.

In the right-angled triangle shown, $\tan C = \frac{3}{7}$

The measure of C is the **inverse tangent** of $\frac{3}{7}$

We write this as $C = \tan^{-1}\left(\frac{3}{7}\right)$

Entering this information into a calculator gives us $C \approx 23.20°$

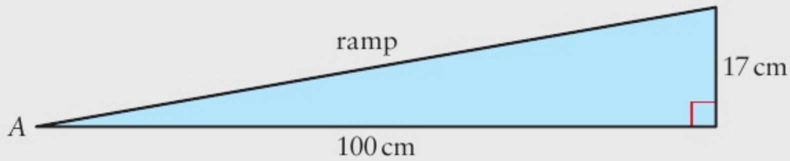
Worked example 12.5

Ascending or descending the access ramp on a bus can be complicated for wheelchair users. For safety reasons, the angle of a ramp in public transport should be below 9.5°.

A wheelchair ramp has a horizontal run of 1 m and a vertical rise of 17 cm. Giving your answer to one decimal place, work out the angle of the wheelchair ramp. Does this ramp meet the safety requirements?

Solution

We can sketch a diagram to make it easier to understand the information given in the question. We need to make sure that both measurements are in the same units. We will convert both measurements to centimetres.



Angle A is the angle of the ramp. The opposite side is 17 cm and the adjacent side is 100 cm.

Using the definition of tangent and the data in the diagram we can write an equation and solve for the angle A.

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{17}{100}$$

$$A = \tan^{-1}\left(\frac{17}{100}\right)$$

Using a calculator in degree mode:

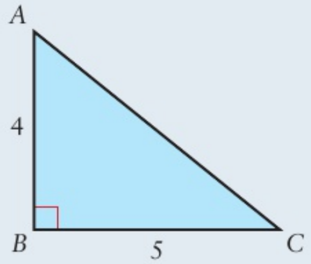
$A = 9.648\ldots° = 9.7°$ (1 d.p.)

This is greater than 9.5° so the ramp does not meet the safety requirements.

Practice questions 12.2

1 Find the tangent of the acute angles for each triangle. Use Pythagoras' theorem to find the length of the third side of the triangle when needed. Give your answers in exact form.

a



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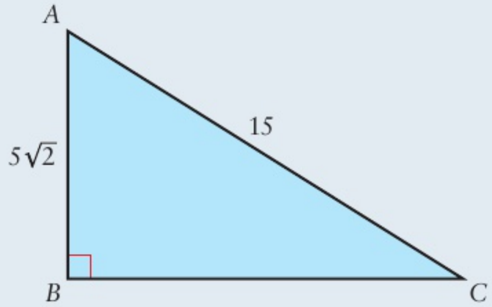
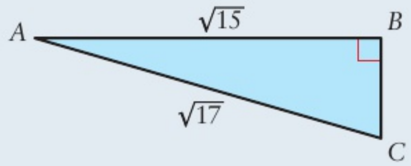
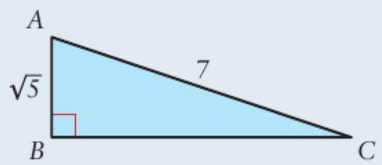


c



d

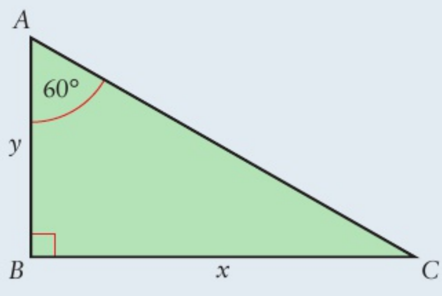


e How are angle C and angle A related in the right-angled triangles in parts a to d? What are they called?

f How are $\tan C$ and $\tan A$ related? Write an equation that expresses the relationship between $\tan C$ and $\tan A$.

2 Write an equation using the definition of the tangent ratio and the indicated angle and variables.

a



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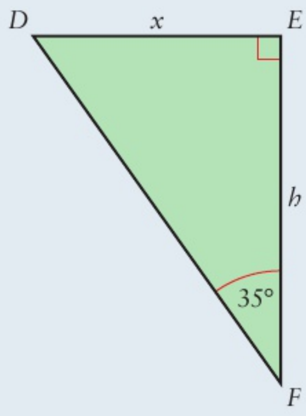
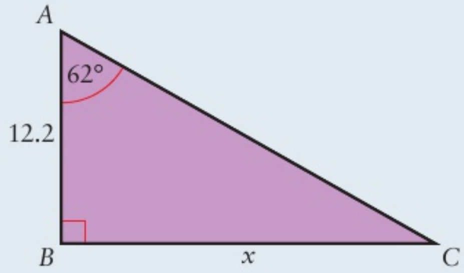
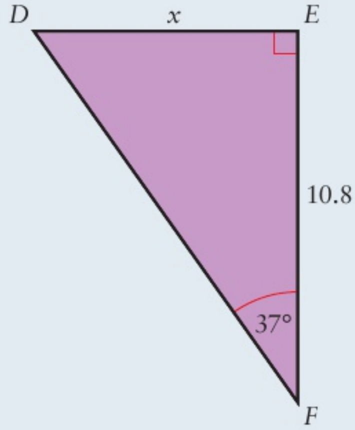


3 Find the value of x in each triangle. Give your answer correct to three decimal places.

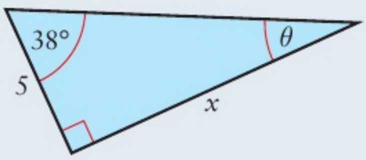
a


b


4 For each triangle, find:

i the value of x ii the value of θ iii the length of the hypotenuse.

Give your answers for x correct to two decimal places.

a


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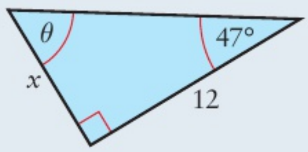

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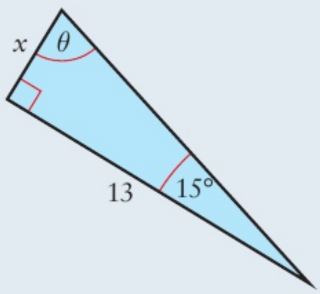

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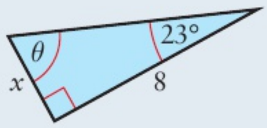


5 Find the value of each tangent ratio. Give your answers correct to three decimal places.

a $\tan 15°$ b $\tan 47°$ c $\tan 66°$

d $\tan 89°$ e $\tan 89.88°$ f $\tan 89.99999°$

g What happens to the tangent as the angle approaches 90°? What do you observe?

h Is there a limit to how big the value of the tangent of an angle can be? Explain your answer.

6 a Draw three different triangles ABC with a right angle at B such that $\tan A = \frac{2}{7}$

b What is the relationship between the three triangles you drew in part a? Explain your answer.

7 Calculate the angle A for the following tangent ratios. Give your answers correct to two decimal places.

a $\tan A = \frac{3}{8}$

b $\tan A = \frac{15}{7}$

c $\tan A = 0.045$

d $\tan A = 3.1357$

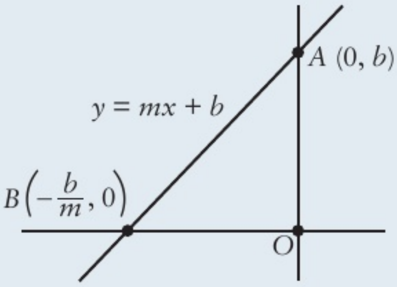
8 a Graph the line $y = -0.75x + 6$ on a coordinate plane.

b Find the coordinates of the x- and y-intercepts.

c The line $y = -0.75x + 6$ forms a right-angled triangle with the x- and y-axes. The vertices are the origin $(0, 0)$ and the x- and y-intercepts. Use the tangent ratio to find the angle that the line forms with the x-axis.

9 A line l with equation $y = mx + b$, $m \neq 0$, forms a right-angled triangle OAB with the x-axis and the y-axis. The vertices are $O(0, 0)$, the y-intercept $A(0, b)$ and the x-intercept $B\left(-\frac{b}{m}, 0\right)$.

Challenge Q9



a Explain why the lengths of the legs of the right-angled triangle OAB are $|b|$ and $\left|-\frac{b}{m}\right|$

b Use triangle OAB to write an expression for the tangent of the acute angle that the line l forms with the horizontal axis.

c How are the tangent of the angle that the line l makes with the x-axis and the slope of line l related? Explain your answer. Write down a statement that summarises your findings.

Challenge Q10

10 Consider the lines $y = 0.5x + 4$ and $y = 0.25x + 2$

a Use the tangent ratio to find the angle that each line makes with the positive direction of the x-axis.

b Find the acute angle that the lines form.

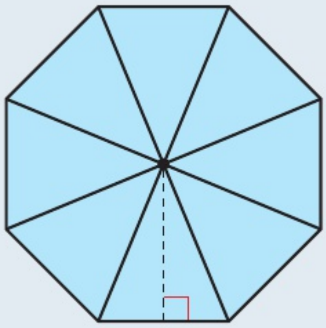
c Is the tangent of the acute angle that the two lines form equal to the tangent of the difference of the two angles that each line makes with the x-axis?

Thinking skills

d Determine whether or not the statement $\tan(A + B) = \tan A + \tan B$ is true or false. Justify your answer.

Challenge Q11

11 A regular polygon with n sides, each of length x, can be divided into n isosceles triangles as shown.



a Find the measure of the vertex angle of each isosceles triangle, in terms of n.

b Find the measure of the base angles of each isosceles triangle.

c The **apothem** of a regular polygon is, as shown, the line segment joining the centre of the polygon to the mid-point of any side. It is therefore the altitude of each of the isosceles triangles in this polygon. Find a trigonometric expression for the apothem in terms of x and n.

d Find an expression for the area of the regular polygon in terms of x and n.

e Find the area of a square with side length 6 cm, using your expression from part d. Is your result consistent with what you expected?

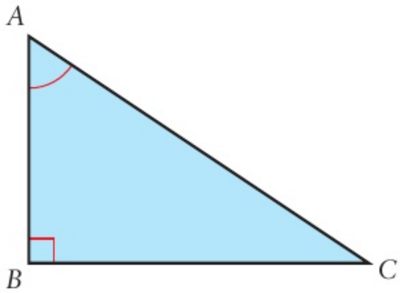
f Find the area of a regular pentagon with side length 5 cm.

12.3 The sine and cosine ratios

Explore 12.3

Can you draw several different sized right-angled triangles all with an angle of 65°? Can you measure all three sides accurately? Can you find other ratios like the tangent ratio which remain the same between the different sized triangles? Name them in terms of opposite, adjacent and hypotenuse.

The tangent ratio relates the opposite and adjacent sides to an angle. We can also relate the angle to the hypotenuse instead of either the adjacent or the opposite side. This leads to two other important ratios: **sine** and **cosine**.



Given a right-angled triangle ABC, we define the **sine of angle A** as:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AC}$$

Investigation 12.1

Can you find a relationship between the sine, cosine and tangent of an angle in a right-angled triangle? Can you see a link to Pythagoras' Theorem?

Fact

We recommend that these definitions become part of your problem-solving tools, but the following acronym helps those of us who like to memorise facts. The SOH-CAH-TOA acronym helps for sine, cosine and tangent of an angle α in a right-angled triangle is:

$$\text{Sin}\,\alpha = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \text{(SOH)}$$

$$\text{Cos}\,\alpha = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{(CAH)}$$

$$\text{Tan}\,\alpha = \frac{\text{Opposite}}{\text{Adjacent}} \quad \text{(TOA)}$$

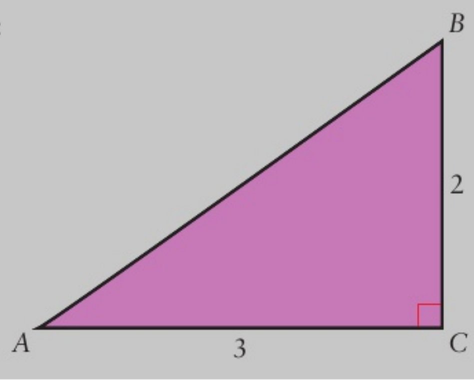
Worked example 12.6

For the triangle ABC, calculate:

a $\sin A$

b $\cos A$

c $\tan A$.



Solution

To apply the definition of sine, cosine and tangent, we need to identify the opposite and adjacent sides to the angle, and to find the value of the hypotenuse, which is not provided by the diagram.

We can find the hypotenuse using Pythagoras' theorem:

$AB = \sqrt{9 + 4} = \sqrt{13}$

Then,

$$\sin A = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$\cos A = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\tan A = \frac{2}{3}$$

Investigation 12.2

Two angles are said to be complementary if they add up to a right-angle (90°).

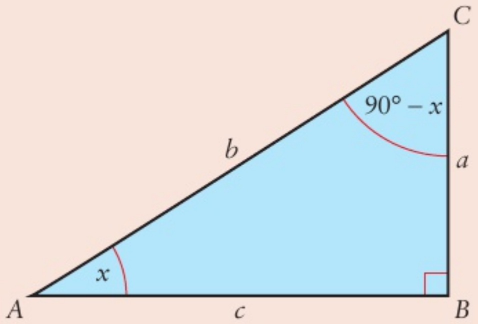
1 Find an expression for the complementary angle of x.

2 Use a calculator to fill in a copy of the following table.

x	$\sin x$	$\cos x$	$\tan x$	$\sin(90° - x)$	$\cos(90° - x)$	$\tan(90° - x)$
21°						
43°						
76°						

3 What is the relationship between the sines, cosines and tangents of an angle and its complementary angle? Write down your observations.

4 Use the diagram to complete a copy of the next table. Write sines, cosines and tangents in terms of a, b, c.



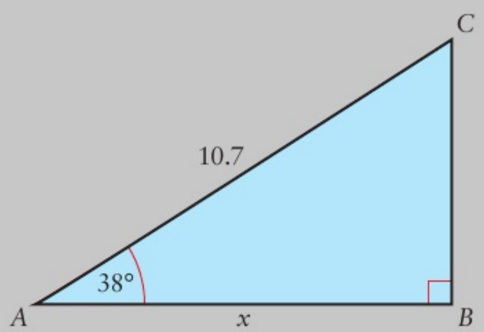
$\sin x$	$\cos x$	$\sin(90° - x)$	$\cos(90° - x)$

5 Use the table above to produce an argument that proves your observations.

In the same way that we did with the tangent ratio, we can use the sine and cosine ratios to find the lengths of sides.

Worked example 12.7

Find the value of x. Give your answer correct to three significant figures.



Solution

The labelled sides are the adjacent side to the angle 38° and the hypotenuse. Therefore, we use the cosine ratio.

We can apply the definition of cosine to get an equation and then solve for x.

$$\cos 38° = \frac{x}{10.7}$$

$$10.7 \times \cos 38° = x$$

$$x = 8.4317\ldots$$

$$= 8.43 \text{ (3 s.f.)}$$

Reminder

When you check your answer, use the unrounded result for x.

We can check our answer by substituting our value for x into the original equation and calculating both sides.

$\cos 38° = 0.7880...$

$\frac{8.43171506359}{10.7} = 0.7880...$

Reflect

In Worked example 12.7, x is the value of the opposite side to angle C. We could solve the question using $\sin C$. What is the measure of C?

Worked example 12.8

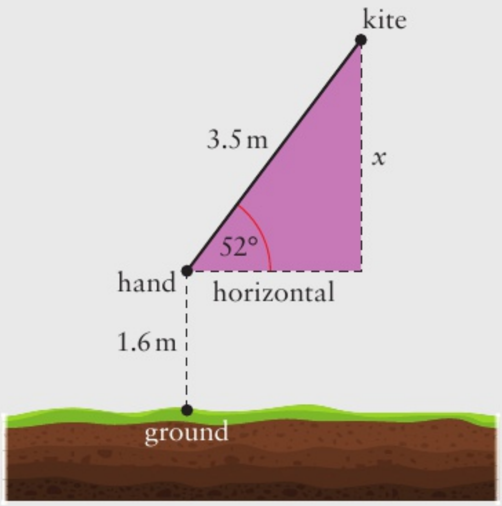
Joe is flying a kite. The hand that holds the string of the kite is 1.6 m above the ground. The string is taut and 3.5 m long, and the angle that the string creates with the horizontal is 52°. How high is the kite is flying above the ground? Give your answer correct to three decimal places.

Solution

Hint

Remember to include heights that are not part of the right-angled triangle, but contribute to the overall height, in this case the height of Joe's hand above ground level.

We will start by sketching a diagram showing all the given information. Let x be the height of the kite above the hand holding the string.



We can find the height of the kite from the ground by adding x and 1.6 m. To solve the problem, we have to find x.

We know the length of the hypotenuse, and the side we want to find is opposite the angle of 52°. Therefore, we will use the sine ratio to get an equation and solve for x.

$$\sin 52° = \frac{x}{3.5}$$

$$3.5 \times \sin 52° = x$$

$$x = 2.7580\ldots$$

$$= 2.758 \text{ (3 d.p.)}$$

The height of the kite from the ground is $x + 1.6 = 4.358$ m (3 d.p.)

In the same way as we did for the tangent, we can find the measure of an angle given its sine or cosine.

Suppose that $\sin A = p$. Then $\sin^{-1} p = A$

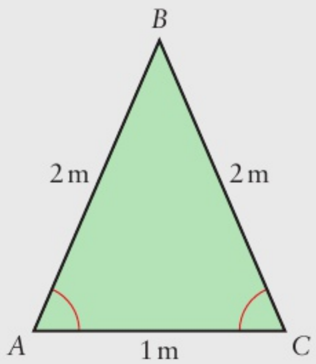
We say that A is the **inverse sine** of p. We can use a calculator to evaluate inverse sines.

Inverse cosine is defined and evaluated in a similar way.

Worked example 12.9

Two lettuce stands in a vertical farm, each 2 m long, are placed against each other to form an isosceles triangle with a base of 1 m. Calculate the base angles of the isosceles triangle that the stands form. Give your answer correct to three significant figures.

Solution

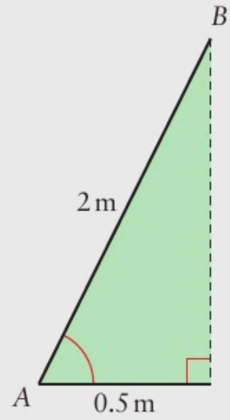


The base angles A and C of the isosceles triangle ABC are congruent, so we only need to calculate one of them. To use trigonometry we need to identify a suitable right-angled triangle.

We can use the right-angled triangle formed by drawing a line from B perpendicular to and bisecting the base. We know the hypotenuse and the adjacent side to angle A. We can use the cosine ratio and solve for angle A.

Fact

Vertical farming is a method of producing food on vertically inclined surfaces. Thanks to its limited land usage, it is less disruptive for the local fauna and flora. It uses a lot of energy, which is usually provided by renewable energy sources.



$$\cos A = \frac{\frac{1}{2}AC}{AB} = \frac{0.5}{2} = 0.25$$

$A = \cos^{-1}(0.25) = 75.522...° = 75.5°$ (3 s.f.)

Therefore, the base angles of the lettuce stand are both 75.5°

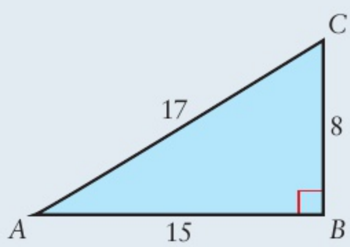
Reminder

Remember to have your calculator in degree mode when finding A.

Practice questions 12.3

1 Find $\sin A$, $\cos A$, $\sin C$ and $\cos C$ in each triangle. Give your answer in simplified exact form.

a



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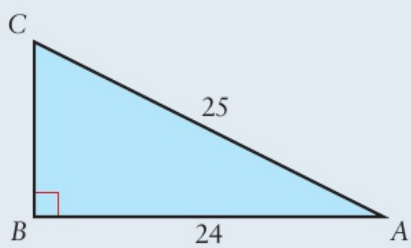
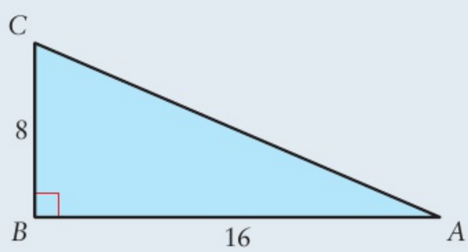


c

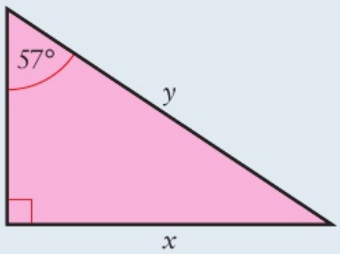


2 Find the value of each ratio. Give your answers correct to five decimal places.

a $\sin 37.7°$ **b** $\sin 89.02°$

c $\cos 0.4°$ **d** $\cos 45°$

3 Write a trigonometric equation using either sine, cosine or tangent that connects the given angle and sides.

a


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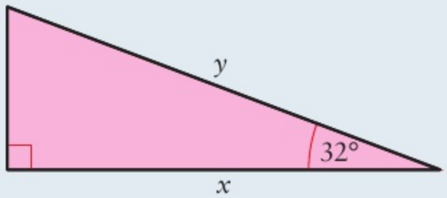
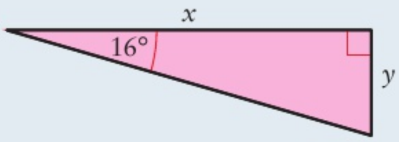
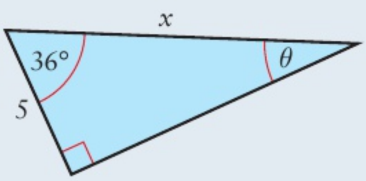


c


4 Find the values of x and θ. Give your answers for x correct to three decimal places.

a


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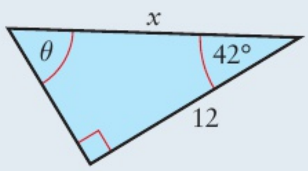

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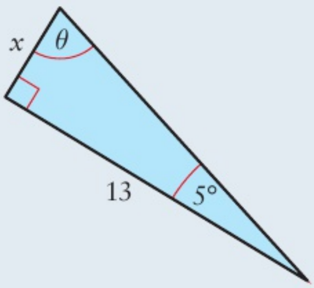

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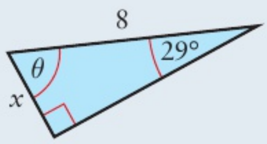
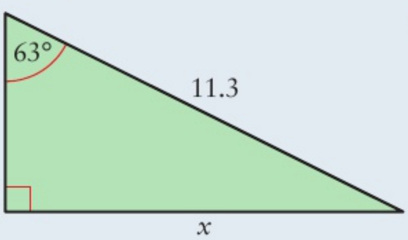
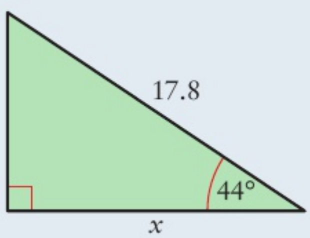


5 Find the value of x in each triangle. Give your answers correct to one decimal place.

a


b


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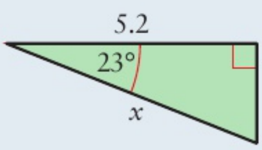
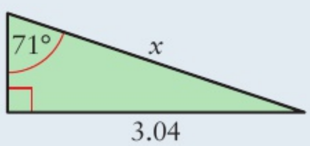
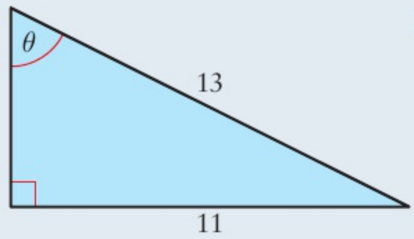


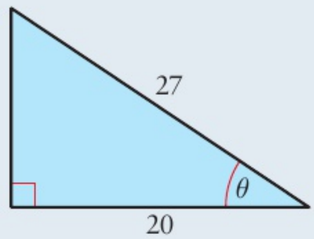
d


6 Find the value of θ in each triangle. Give your answers correct to three decimal places.

a



b



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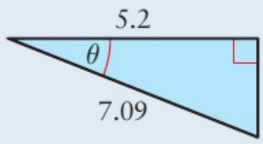
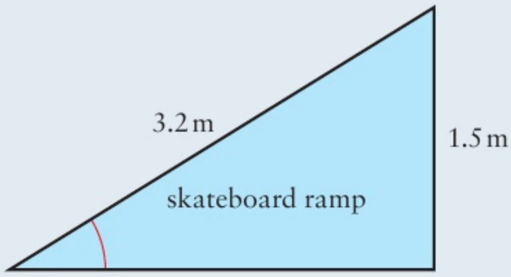
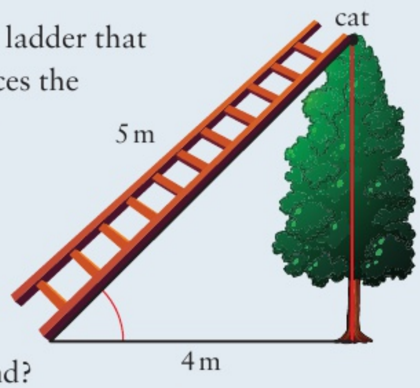


d

7 Carlos wants to build a skateboard ramp. The height of the ramp is 1.5 m and the length is 3.2 m. What angle does the ramp makes with the horizontal?



8 Lucy's cat is stuck in a tree. Lucy has a ladder that can be extended up to 5 m. If Lucy places the base of the ladder at 4 m from the base of the tree and uses the maximum extension of the ladder:



a what is the angle between the ladder and the horizontal?

b how high is the cat above the ground?

Research skills

9 The Pilatus cogwheel railway in Switzerland is the steepest railway in the world. It travels a vertical distance of 1635 m with an average inclination angle of 20.8°.

a Estimate the length of the railway based on its average inclination.

b Research the actual length of the railway and compare it with your answer to part a. Was your estimate close to the actual length of the railway?

10 A rhombus has sides of length 12 cm. If one of the angles between two adjacent sides is 57°, find the length of the minor diagonal.

11 In an isosceles triangle, the equal sides are each $\frac{7}{5}$ of the length of the base.

Determine the measure of the base angles.

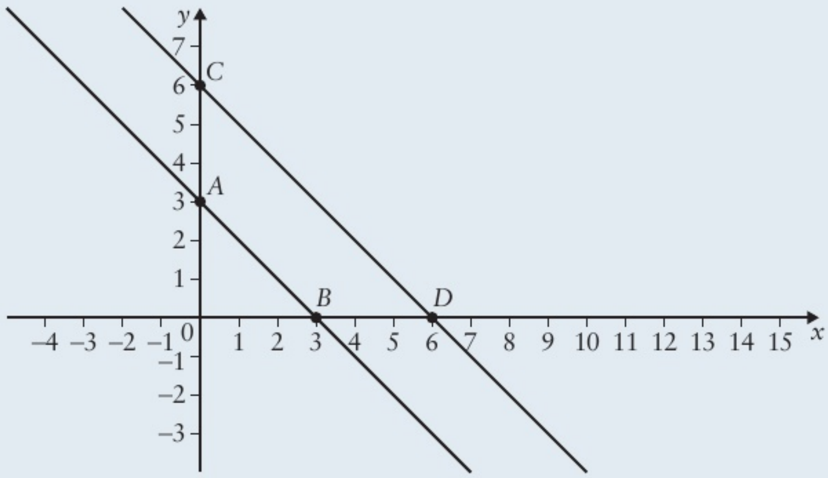
12 The vertex angle of an isosceles triangle with height 8 cm measures 35°. Calculate its area.

13 The diagonal of a rectangle *ABCD* is 12 cm long and forms an angle of 47° with one of the sides. Find the side length of a square whose perimeter is double the perimeter of *ABCD*.

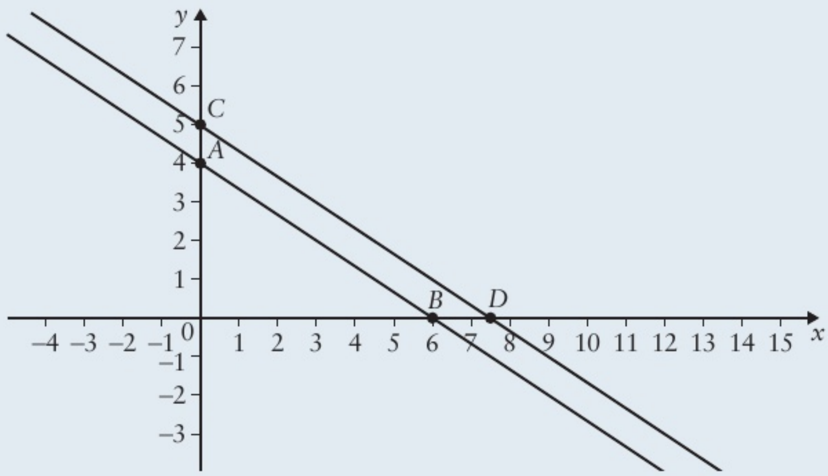
14 A point *P* is 5.4 cm from the centre *C* of a circle of radius 3 cm. Find the angle that the line segment *PC* forms with the tangent to the circle from *P*.

15 Find the shortest distance between the two parallel lines.

a



b



c Describe how you found your answers to parts a and b.

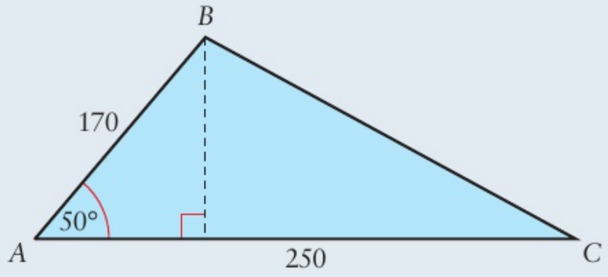
Challenge Q16

16 A triangular field ABC is surveyed by measuring the sides AC and AB and the angle between these sides as shown.

a What is the distance between B and AC?

b What is the length of BC?

c What is the area of the field?



17 Through what angle must a door to be opened if from the other end of a long corridor it appears to cover one third of the doorway?

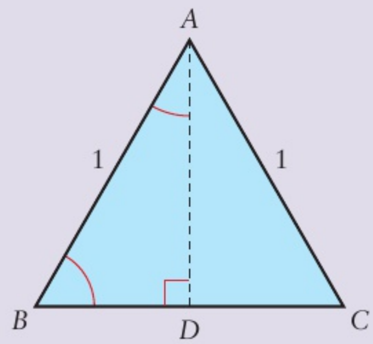
12.4 Exact values of the trigonometric ratios for 30°, 45° and 60° angles

You need to be able to remember the exact values of some trigonometric ratios.

Explore 12.4

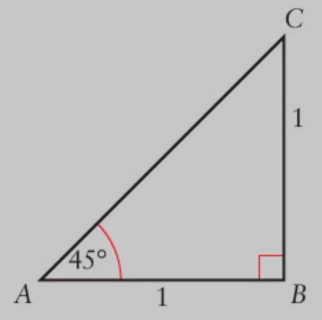
ABC is an equilateral triangle of side 1 unit.

Can you use the triangle to find the values of $\sin 30°$, $\sin 60°$, $\cos 30°$, $\cos 60°$, $\tan 30°$ and $\tan 60°$?°



Worked example 12.10

ABC is an isosceles right-angled triangle.



Work out the exact values of $\sin 45°$, $\cos 45°$ and $\tan 45°$.

Solution

We need to use the triangle to find exact values for $\sin 45°$, $\cos 45°$ and $\tan 45°$.

We can find the length of the hypotenuse then apply the trigonometric ratios.

By Pythagoras' theorem:

$AC = \sqrt{1 + 1} = \sqrt{2}$

Hence:

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$

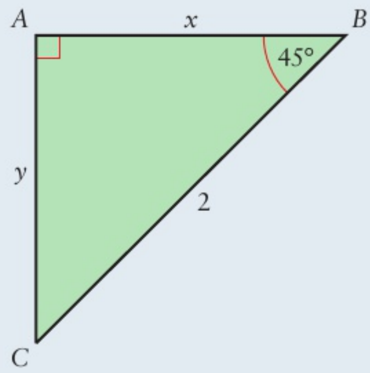
We can check our answers with a scientific calculator.

The exact values for the angles from Explore 12.4 and Worked example 12.10 are given in the table.

	$x = 30°$	$x = 45°$	$x = 60°$
$\sin x$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos x$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan x$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Practice questions 12.4

1 Find the value of the unknowns in each of the triangles. Give your answer in simplified exact form.

a


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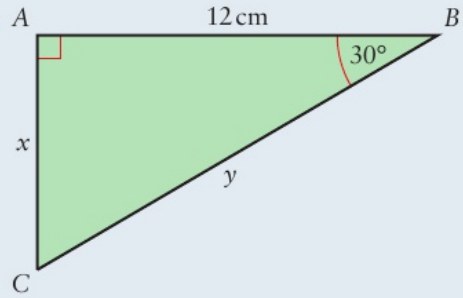
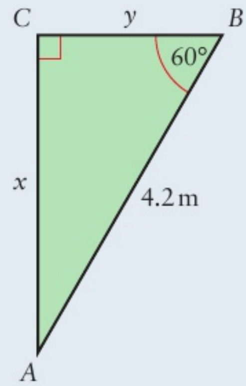
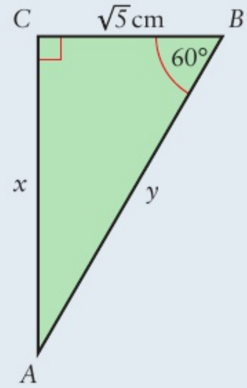
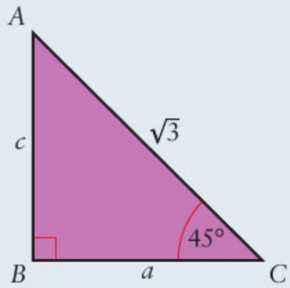


c


d


2 Find the lengths of the unknown sides. Give your answers in simplified exact form.

a


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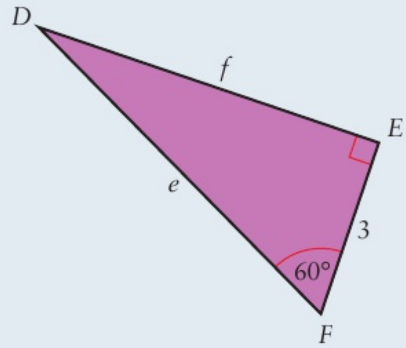

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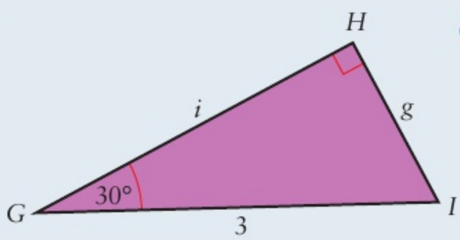

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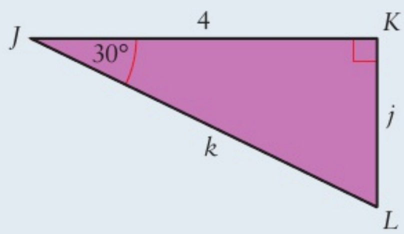


3 A square has a diagonal of 5 cm. Use trigonometry to find the length of the sides. Give your answer in simplified exact form.

4 An equilateral triangle ABC has sides of 12 cm. Find the height of the triangle. Give your answer in simplified exact form.

5 A hexagon has sides of 5 cm. Use the diagram to answer the following.

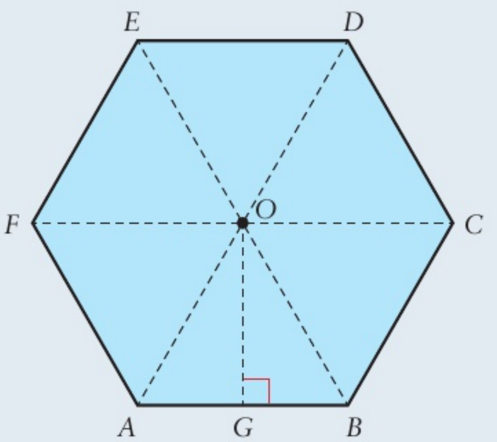
Challenge Q5

a Write down the measure of the angle AOB.

b Classify the triangle AOB.

c Use trigonometry to find the height OG. Give the answer in simplified exact form.

d Find the area of the hexagon. Give the answer in simplified exact form.



12.5 Angles of elevation and depression

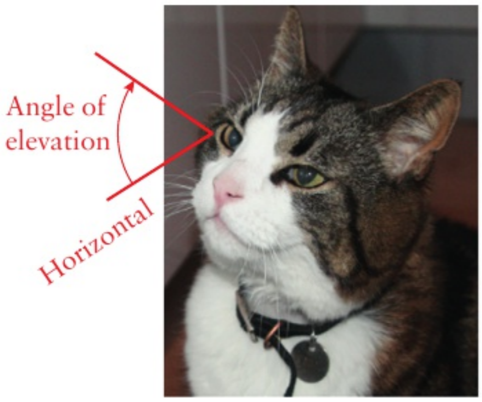
Explore 12.5

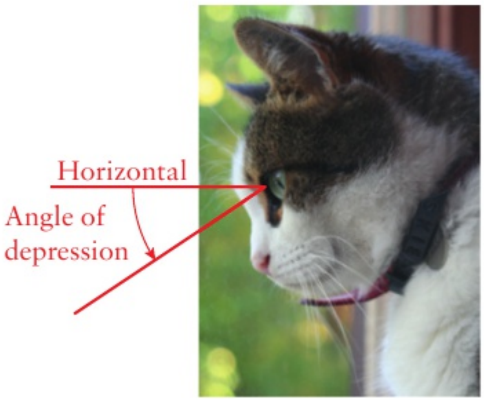
For each cat in the photo on the right, can you describe the angle at which it is looking at the other? What do you estimate each angle to be?

We (and cats!) tend to look horizontally, so it is from the horizontal that we measure the angle when eyes are turned up or down. The angle that the line of sight when raised makes with the horizontal is called the **angle of elevation**. The angle that the line of sight when lowered makes with the horizontal is called the **angle of depression**.

Hint

Note that in the photo on the left, the horizontal is drawn taking the camera angle into account.





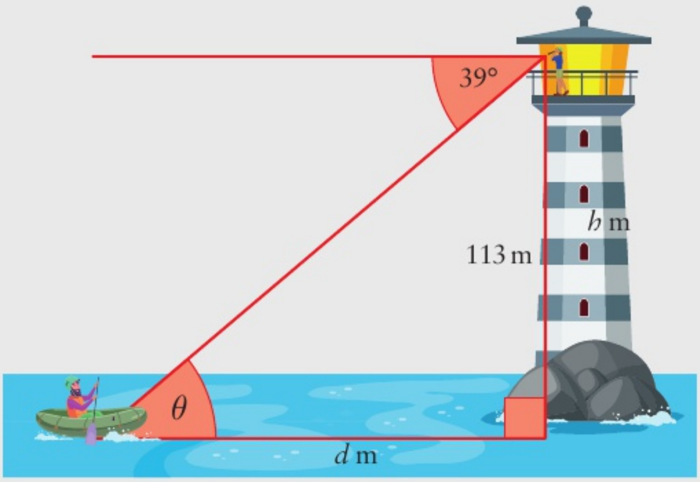
Worked example 12.11

A lighthouse keeper is standing on the observation deck and spies a person floating on a life raft in the waters below. The angle of depression made is 39°.

The lighthouse keeper's eye level is 113 metres above the level of the sea. Find the horizontal distance, d, from the lighthouse keeper to the person on the life raft. Give your answer correct to the nearest metre.

Solution

We can sketch a diagram to help answer the question.



From the diagram, we know one side of the right-angled triangle (113 m) and we are asked to determine the length of the other side. The angle θ is also 39° because it is an alternate angle to the angle of depression.

We can use this information together with the tangent ratio to find the value of d.

$$\tan\theta = \frac{113}{d}$$

$$\tan 39° = \frac{113}{d}$$

$$\Rightarrow d = \frac{113}{\tan 39°} = 139.543 \text{ (3 d.p.)}$$

The horizontal distance from the lighthouse keeper to the person on the life raft is 140 metres, to the nearest metre.

Reflect

Why is the measure of the angle of depression from one point of view equal to the measure of the angle of elevation from the opposite point of view?

Practice questions 12.5

1 The angle of elevation from a boat to the top of a lighthouse 10 m above sea level is 23°. Calculate the horizontal distance from the boat to the lighthouse.

2 From a vertical cliff 60 m above sea level, a coast guard observes a shark at an angle of depression of 37°. Find the diagonal distance from the coast guard to the shark.

3 A cable car goes up the slope of a mountain with an angle of inclination of 63°. It starts at an altitude of 600 m and travels 900 m along the slope. What altitude does it reach?

4 A tree casts a shadow 3 m long. The angle of elevation from the tip of the shadow to the top of the tree is 52°. Find the height of the tree.

5 A kite is on the end of a taut cord 13 m long. The hand of the person holding the kite is at a height of 2 m. If the vertical height of the kite is 12 m, find the angle of elevation of the kite from the hand.

Challenge Q6

6 A hiker standing on flat ground can see a mountain in the distance. The angle of elevation to the top of the mountain is 36°. If the hiker moves 630 m closer, the angle of elevation is now 43°. Find the height of the mountain above the flat ground.

Challenge Q7

7 Two skyscrapers are 32 m apart. Ben, who is standing on the top of the shorter skyscraper, measures the angle of elevation from his eyes to the top of the taller skyscraper to be 13°. He measures the angle of depression from his eyes to the bottom of the taller skyscraper to be 64°. Find the height of the two skyscrapers, given that Ben's eyes are 1.8 m above the roof of the shorter skyscraper.

Challenge Q8

8 An aeroplane is approaching a runway. The angle of depression from the aeroplane to the far end of the runway is 10°, while the angle of depression from the aeroplane to the close end of the runway is 17°. The aeroplane is flying at an altitude of 600 m. Find the length of the runway.

12.6 Bearings

Explore 12.6

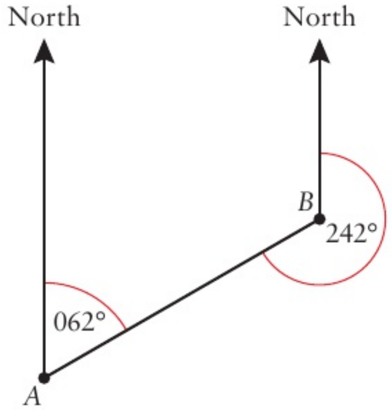
A boat sails in a straight line from Dover to Le Havre.

Can you describe the direction in which the boat sails?

The direction that a boat travels relative to north is called its bearing. The bearing of Le Havre from Dover is the angle measured clockwise between the North line and the straight line joining Dover and Le Havre.

In the diagram, the **bearing of *B* from *A*** is 062°. The angle is measured in a clockwise direction from the North line to $\overline{AB}$.

The **bearing of *A* from *B*** is 242°.



> **Fact**
>
> Bearings are always measured in a clockwise direction from North and are always given as three-digit numbers. For numbers less than 100, put zeroes at the front of the number to make three digits.

Worked example 12.12

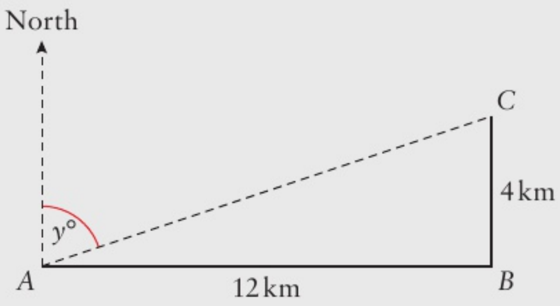
Starting from point *A*, a ship travels 12 km East to *B* and then 4 km north to *C*. Find:

a the bearing of *C* from *A*, correct to the nearest degree

b the distance from *A* to *C*, correct to two decimal places.

Solution

We can draw a diagram to show all the information.



The bearing, y, is the complementary angle to BAC. The side AC is the hypotenuse of a right-angled triangle ABC.

We can use the tangent ratio to find angle BAC. Then subtract the answer from 90° to find the bearing.

To find the length of AC we can use Pythagoras' theorem.

a $\tan A = \frac{4}{12} = \frac{1}{3}$

$A = \tan^{-1}\left(\frac{1}{3}\right) \approx 18.43\ldots°$

$y \approx 90 - 18.43\ldots = 71.57\ldots \approx 72$ (2 s.f.)

The bearing of C from A is 072°, correct to the nearest degree.

b $AC = \sqrt{4^2 + 12^2} = \sqrt{160} = 12.65$ (2 d.p.)

The distance fom A to C is 12.65 km, correct to two decimal places.

Investigation 12.3

1 Use map software such as Google Maps to select two cities. Take a screenshot or use the snipping tool to crop a rectangular image of the map displaying the two cities.

2 Open GeoGebra and use the Image tool to import your screenshot on GeoGebra. Adjust the image with a reasonable scale.

3 Place two points C and D on the two cities and connect them through a line segment, CD. Draw the north lines from C and from D.

4 Use the Angle Measure tool in GeoGebra to measure the bearing of C to D and of D to C.

5 Calculate the difference of the two bearings you found.

6 Work in pairs and compare your results. What do you notice? Use angle properties and parallel lines to explain the reason why the two bearing always differ by 180°.

Reminder Q4

The angles are measured in clockwise direction from the north line to the line segments

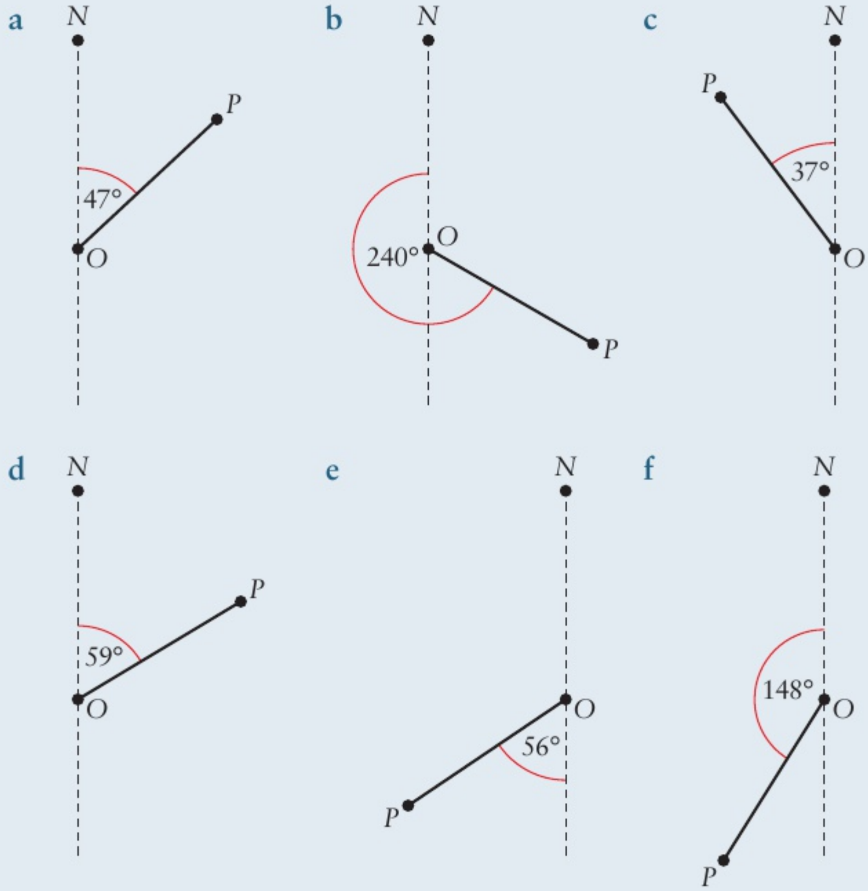
Practice questions 12.6

1 Draw a diagram showing each of the following bearings.

a 062° **b** 132°

c 250° **d** 320°

2 Find the bearing of P from O in each diagram.



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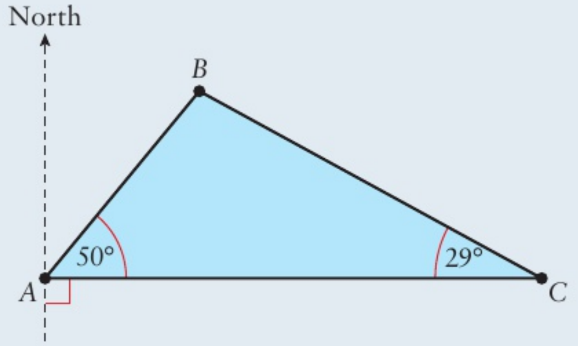


Find the bearing of:

a A from B **b** B from A **c** A from C

d C from A **e** B from C **f** C from B.

4 A ship sails 50 km on a bearing of 071°. How far east of its starting point is the ship?

5 A glider travels on a bearing of 246° until it is 100 km west of its starting point. How far has it travelled on this bearing?

6 In a drone race, Peter's drone flew 1 km north and then 800 m east. What is the bearing of the drones finishing point from its starting point?

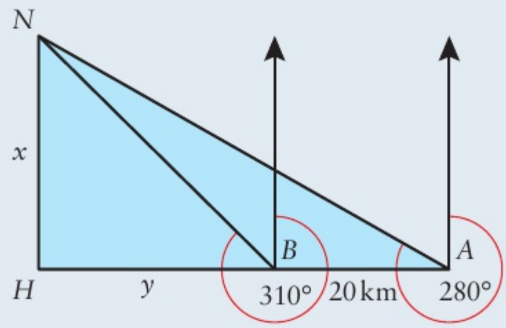
7 A car is travelling from City A to City N on a bearing of 280°. Another car is travelling from City B to City N on a bearing of 310°. City A and City B are 20 km apart.

a Use the diagram to show that:

i $x = y\tan 40°$ ii $x = (y + 20)\tan 10°$

b Use the equations in part a to find the value of x.

c Find the distance from City B to City N.



Challenge Q8

8 A cyclist departs from point E and rides to F on a bearing of 144° for 11 km. The cyclist then changes direction and rides to G on a bearing of 234° for 26 km. Calculate:

a the distance from G to E b the bearing of G from E.

Self assessment

- I can name sides and angles of a given triangle.
- I can identify the adjacent side and the opposite side to a given acute angle in a right-angled triangle.
- I can define the tangent, the sine and the cosine of an acute angle as the ratio of sides of a given right-angled triangle.
- I can explain the relationship between the sine, cosine and tangent of complementary angles.
- I can use a calculator in degree mode to evaluate the trigonometric ratios of an acute angle.
- I can calculate the measure of an angle given its sine, cosine or tangent.
- I can find unknown angles in a right-angled triangle using the appropriate trigonometric ratio.
- I can find unknown sides in a right-angled triangle using the appropriate trigonometric ratio.
- I know the exact values of the sine, cosine and tangent of 30°, 45° and 60°.
- I can explain how to find the exact values of the sine, cosine and tangent of 30°, 45° and 60° using appropriate variables and diagrams.
- I can explain what angles of elevation and depression are.
- I can explain the relationship between angle of elevation and angle of depression.

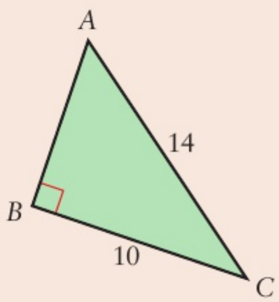
- I can explain what a bearing is.
- I can draw a diagram to represent a problem involving trigonometric ratios.
- I can identify the unknown angles or sides that need to be calculated in a problem involving trigonometric ratios.
- I can locate a right-angled triangle in my diagram for a problem involving trigonometric ratios.
- I can write an equation connecting an angle and two sides using the appropriate trigonometric ratio for a problem involving trigonometric ratios.
- I can solve a trigonometric equation connecting an angle and two sides for a problem involving trigonometric ratios.

Check your knowledge questions

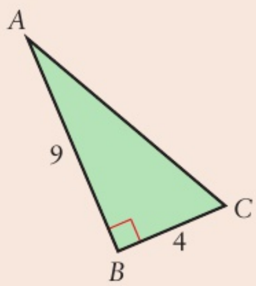
1 For each triangle, find the missing length and find the following trigonometric ratio, giving your answers correct to three decimal places:

i $\sin A$	**ii** $\sin C$	**iii** $\cos A$
iv $\cos C$	**v** $\tan A$	**vi** $\tan C$

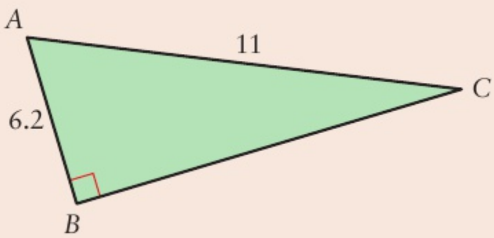
a



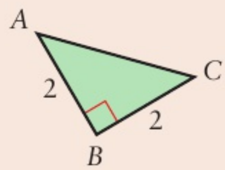
b



c

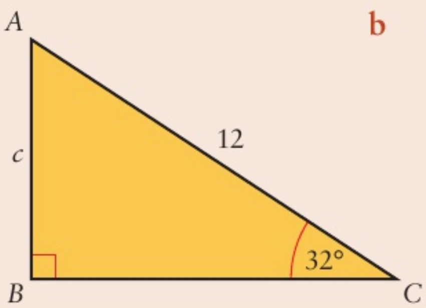


d



2 Calculate the length denoted by each lower-case letter. Give your answers correct to two decimal places.

a



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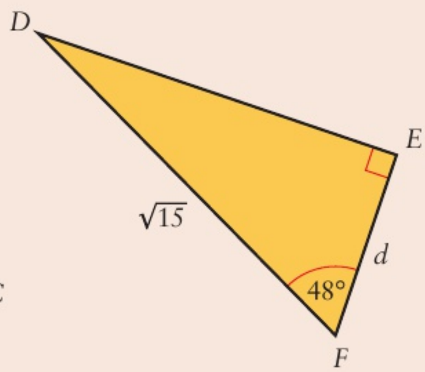

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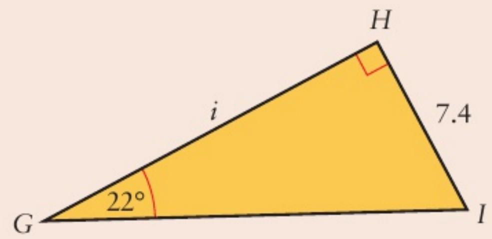
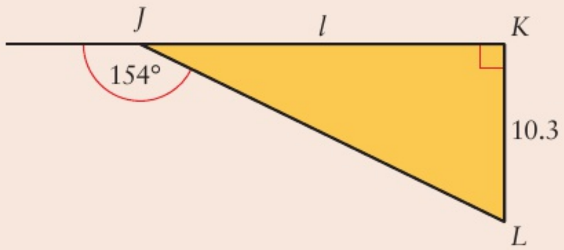
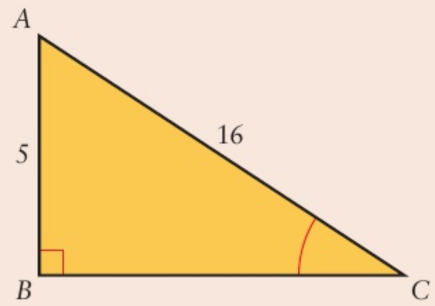


d

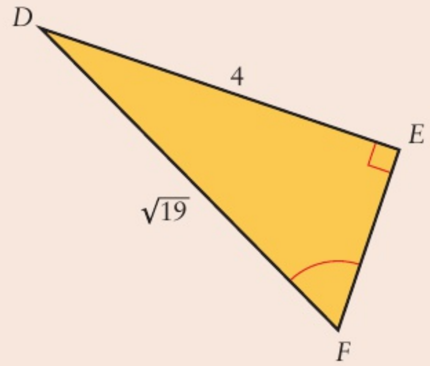


3 Calculate the value of each marked angle using the appropriate trigonometric ratio. Give your answer correct to one decimal place.

a



b



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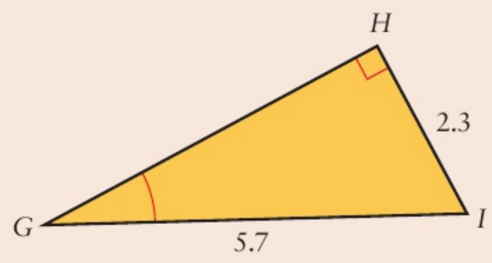
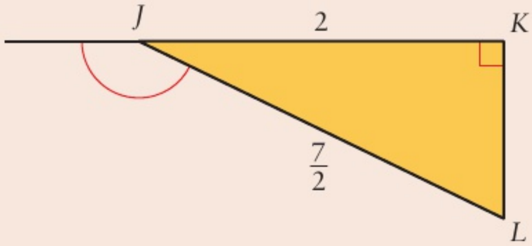
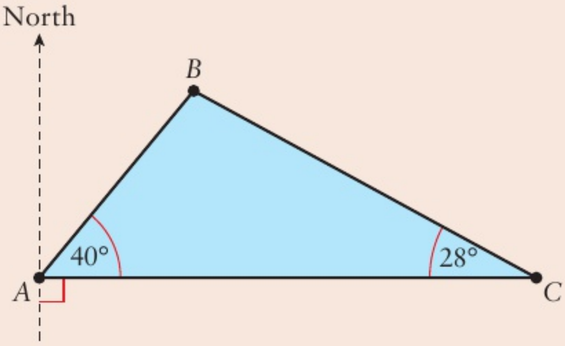


d 

4 

Find the bearing of:

a *A* from *B* **b** *B* from *A*

c *A* from *C* **d** *C* from *A*

e *B* from *C* **f** *C* from *B*.

5 An escalator in a large mall is 15 m long. It makes an angle of 48° with the horizontal. What is the distance between the two floor levels?

6 The ropes of a swing are 3 m long and the seat of the swing is 50 cm from the ground when the ropes are vertical. How high is the seat above the ground when the ropes have swung forward through an angle of 70°?

7 After travelling 120 km on a straight course, a ship made a turn and travelled for 72 km to finally stop at 96 km northeast of the port from which it started. On what bearing was the ship sailing when it stopped?

8 The diagonal of a rectangle is 18 cm long. The angle between the diagonal and the long side is 30°. Find the lengths of the sides of the rectangle. Give your answers in simplified exact form.

9 A plane at an altitude of 4000 m dives at an angle of depression of 20° for 6 km before levelling out. What is its new altitude?

Hint Q7

What type of triangle is a triangle with sides 120, 72 and 96?

10 A road slopes upwards with an angle of elevation of 10°. If a person gains 300 m in altitude walking along the road, what distance have they covered?

11 A ship leaves the port of Niigata, Japan, heading north for 30 km, then changes direction and travels on a bearing of 015° for 60 km and then on a bearing of 040° for 120 km.

a Calculate how far north the ship has travelled in a northerly direction.

b Calculate how far east the ship has travelled.

c How far from the port of Niigata will it then be?

13

Inverse functions, exponentials and logarithms

13 Inverse functions, exponentials and logarithms

KEY CONCEPT

Relationships

RELATED CONCEPTS

Equivalence, Models, Representation

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Complex scientific relationships can be modelled using different forms of representation, and the models used to make predictions.

Factual

- What are exponential growth models?
- What are logarithmic models?

Conceptual

- How do we connect exponential and logarithmic functions?
- How do we transform exponential and logarithmic functions?
- How do we measure the intensity of earthquakes?

Debatable

- Can the population of the Earth grow indefinitely?
- Will we run out of fossil fuels?
- Is matter really three dimensional?

Do you recall?

1 What test can be performed on the graph of a relationship to check whether it is a function?

2 Given that $f(x) = x^2 - 1$:

a find $f(3)$

b solve the equation $f(x) = 3$

c compare the graph of $f(x)$ to:

i $y = f(x) + 3$

ii $y = f(x - 3)$

iii $y = f(-x)$

iv $y = 3f(x)$

v $y = f(3x)$

13.1 Inverse functions

13.1.1 Invertible functions

In Chapter 3, we saw how functions are used to establish a relationship between two sets. This relationship has a **direction**, since it describes how the function f acts on x and sends it into $y = f(x)$. We also saw that **one-to-one** functions can be reversed to give the inverse function.

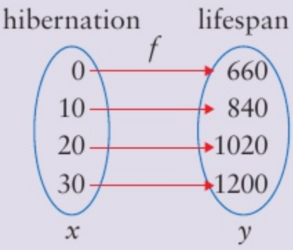
The goal of this section is to investigate further when it is possible to **reverse** this direction.

All function representations, including mapping diagrams, tables, graphs and equations, can be used to discuss this issue.

Explore 13.1

According to an experimental study, the lifespan of a hamster is a function of the time the hamster spends hibernating.

Here is a mapping diagram of the function f that maps the percent of the hamster's lifetime spent hibernating, x, to its lifespan in days, y.

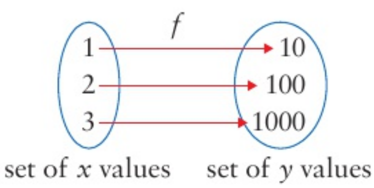


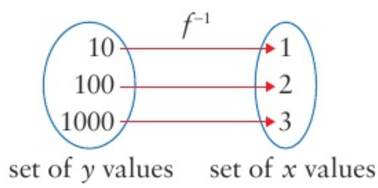
Can you explain what $f(10)$ is?

Can you use a mapping diagram to describe a function, h, that maps lifespan to hibernation?

What is $h(840)$?

If the function f sends each x to a **unique** y, then the backwards mapping from y to x is called the **inverse** of f and it is a function itself, denoted by f^{-1}.



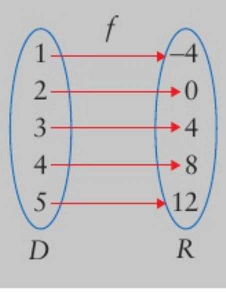


For example, these two statements are equivalent: $f(1) = 10$ and $f^{-1}(10) = 1$

Worked example 13.1

For the mapping below, find:

a $f(3)$ b $f^{-1}(12)$



Solution

a $f(3)$ is the member of R reached by the arrow starting from 3 in D, so $f(3) = 4$

b $f^{-1}(12)$ is the member of D from which the arrow reaching 12 in R starts, so $f^{-1}(12) = 5$

Fact

The sets D and R are called the domain and the range of the function f, respectively.

Reminder

Functions which are **many-to-one** do not have an inverse function.

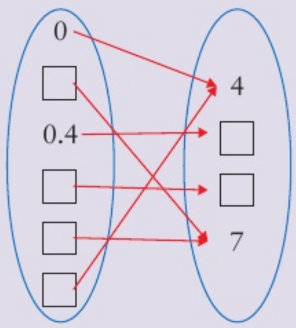
Explore 13.2

A stone is thrown vertically upwards from 4 metres above the ground at time $t = 0$

The height h m of the stone changes as a function of time according to $h(t) = 4 + 8t - 5t^2$, where t is in seconds.

Can you calculate the height of the stone every fifth of a second until it returns to the ground?

Can you use this information to complete the mapping diagram below?

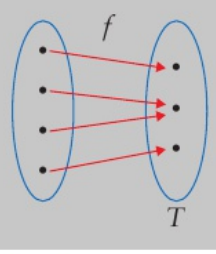


In Explore 13.2, some members of the set of heights are reached by more than one arrow. Therefore the function $h(t)$ does not have an inverse function. In other words, the inverse relationship h^{-1} is one-to-many, and therefore is not a function.

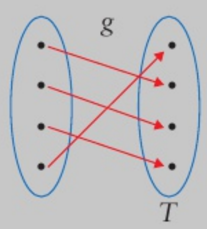
Worked example 13.2

Identify which of these mappings have an inverse function.

a



b



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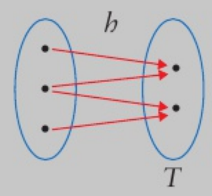
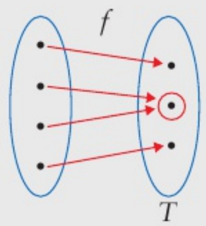


Solution

a For mapping f, there is one member of the set T, circled below, that is reached by more than one arrow. This is enough to conclude that f does not have an inverse function.



b For mapping g, all members of the set T are reached by only one arrow, so g has an inverse function. It does not matter that the arrows cross.

c For mapping h, both members of set T are reached by two arrows, so h does not have an inverse function.

Practice questions 13.1.1

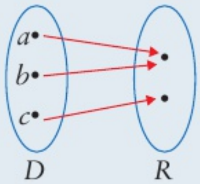
1 Draw a mapping diagram for:

a a function that has an inverse

b a function that does not have an inverse.

2 Which element • must be removed from set D in each mapping diagram such that the relation becomes a function (if it isn't a function already), and that function has an inverse?

a



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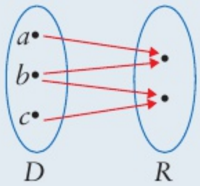
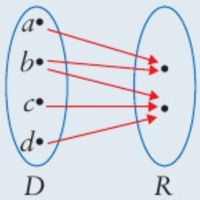
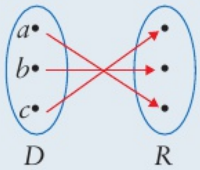


c



d



13.1.2 How to find the inverse function: tables, mapping diagrams and graphs

Now that we know which functions have an inverse, we need to know how to find their inverse.

We have already seen how to find the inverse function using a mapping diagram, but what can be done when other representations are used?

Explore 13.3

How many different ways of representing the same function, for example, the function with equation $f(x) = 2x^2 + 1$, can you think of?

How do the different representations of the function relate to each other? Can you start from any one of them and find all the others?

What are the advantages and disadvantages of each representation in showing the behaviour of the function? Which of them could be the best to work with if your goal is to find f^{-1}?

In this section we will discuss how to find the inverse of a function from its table of values and from its graph.

Worked example 13.3

Find the table of values for the inverse of the function below.

x	1	2	3	5
$y = g(x)$	0.3	0.5	2	4

Solution

Understand the problem

We are given the table of values for a function g. We need to find the table of values for its inverse, g^{-1}.

Make a plan

Can we relate this problem to something we are more familiar with? For example, we know how to invert a mapping diagram. We could:

1 convert the table of values for g into a mapping diagram

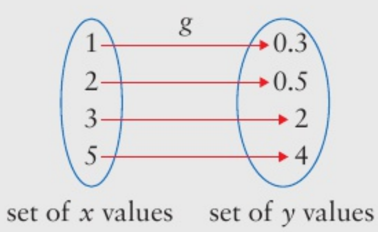
2 invert the mapping diagram for g and find the mapping diagram for g^{-1}

3 convert the mapping diagram for g^{-1} into a table of values for g^{-1}.

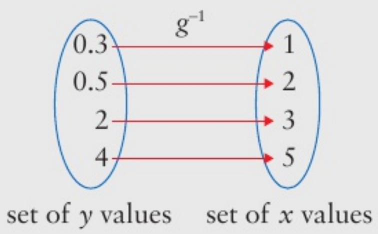
This would combine two pieces of understanding we already have (switching between tables and mapping diagrams, and inverting a mapping diagram) into a new skill (inverting a table).

Carry out the plan

We convert the table of values for g into a mapping diagram.



Then we invert the mapping diagram for g and find the mapping diagram for g^{-1}.



Finally, we convert the mapping diagram for g^{-1} into a table of values for g^{-1}.

y	0.3	0.5	2	4
$x = g^{-1}(y)$	1	2	3	5

Look back

It turns out that finding the inverse of the function produced a table of values with the two rows swapped: y is now in the first row, and x is in the second row. This is something we can remember next time we have to find the inverse of a function using its table of values. We will not need to go through the mapping diagram representation. Instead, we will simply swap the rows.

This is equivalent to reversing the order of ordered pairs.

For example, the function described by

$\{(1, 3), (2, 6), (3, 9), (4, 12)\}$ has an inverse described by

$\{(3, 1), (6, 2), (9, 3), (12, 4)\}$

Now that we have realised that finding the inverse of a function means swapping the x and y values, in other words, going from the range back to the domain using the same pairing in reverse, we can consider what this means in terms of the graph of the function.

Explore 13.4

The function g is given by the table of values.

x	1	2	3	5
$y = g(x)$	0.3	0.5	2	4

How would you represent the inverse function $g^{-1}(y)$? Can you tell the graphs of $y = g(x)$ and $x = g^{-1}(y)$ apart?

Can you think of a better way of graphing the inverse function g^{-1} alongside g?

When working with pairs of inverse functions, it is convenient to call x the independent variable (and have it on the horizontal axis) and to call y the dependent variable (and have it on the vertical axis) for both f and f^{-1}. The very last step when inverting a function will therefore be renaming x as y and vice versa. For the table in Explore 13.4:

x	0.3	0.5	2	4
$y = g^{-1}(x)$	1	2	3	5

One advantage of this approach is that we can graph both $g(x)$ and $g^{-1}(x)$ on the same axes.

Investigation 13.1

What happens if you graph the functions $y = g(x)$ and $y = g^{-1}(x)$ from Explore 13.4 on the same set of axes?

What is the effect of swapping the x- and y-coordinates of a point?

What is the effect of swapping the x- and y-coordinates of all points on a curve, that is reversing the ordered pair for each point?

What can you conclude about the relationship between the graph of a function and the graph of its inverse?

Communication skills

Worked example 13.4

Consider the function $h(x)$ defined by the table:

x	2	3	4	6
$h(x)$	0	$\frac{3}{2}$	2	$\frac{5}{2}$

a Find the table of values for h^{-1}

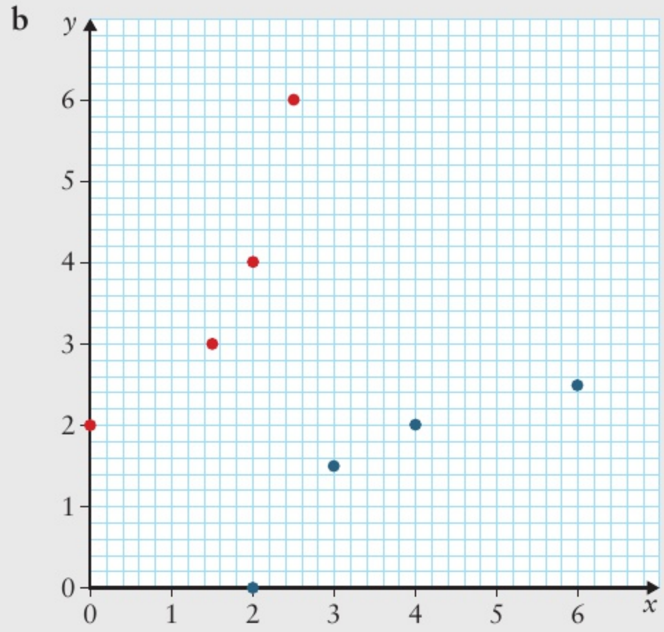
b Draw the graphs of $y = h(x)$ and $y = h^{-1}(x)$ on the same set of axes.

Solution

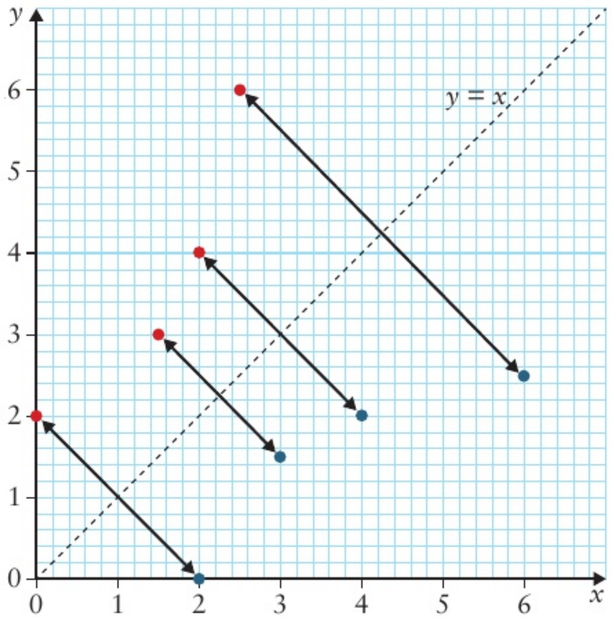
a First we note that the table for h describes a one-to-one function and therefore has an inverse function. We obtain the table of values for the inverse function h^{-1} by swapping the rows of the table of values for $h(x)$.

x	0	$\frac{3}{2}$	2	$\frac{5}{2}$
$h^{-1}(x)$	2	3	4	6

b



Plotting the graphs of the functions $y = h(x)$ and $y = h^{-1}(x)$ from Worked example 13.4 on the same set of axes shows that the x- and y-coordinates are swapped.



The graph of $y = h(x)$ and the graph of $y = h^{-1}(x)$ are **symmetrical** with respect to the line $y = x$. In other words, the graph of $y = h^{-1}(x)$ is obtained from the graph of $y = h(x)$ through a **reflection** in the line $y = x$

This observation allows us to draw the graph of the inverse f^{-1} of any function f directly from the graph of f.

Connections

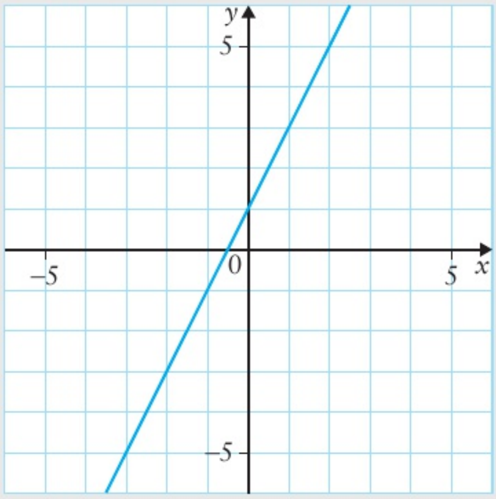
A whole branch of mathematics called analytic geometry is devoted to the exploration of similar links between functions and their graphs.

Worked example 13.5

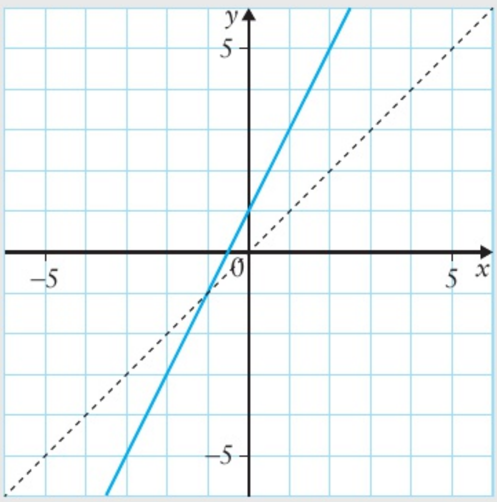
For the function $f(x) = 2x + 1$, draw the graphs of $y = f(x)$ and of $y = f^{-1}(x)$ on the same set of axes.

Solution

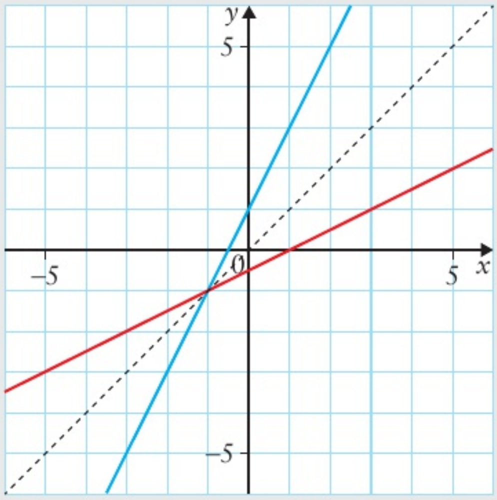
We start by drawing the graph of $y = f(x)$.



We then add the symmetry line, $y = x$



Finally, we reflect the graph of f in the symmetry line to obtain the graph of $y = f^{-1}(x)$.



Hint

To reflect a graph over $y = x$, it is often convenient to reflect a few points first by swapping their coordinates.

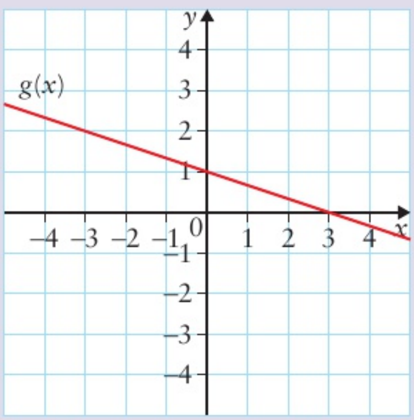
We can use this symmetry to establish a test that the graph of a function f must pass in order for f to have an inverse.

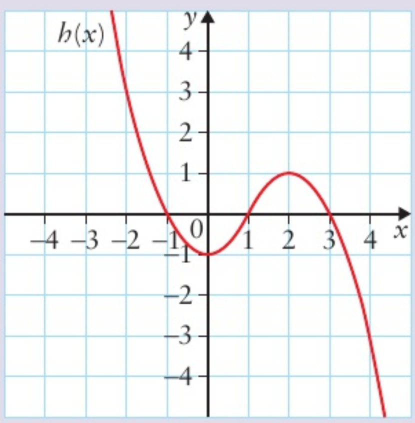
We saw in Section 13.1.1 that a function $y = f(x)$ has an inverse when all y values are reached by only one arrow, that is, when f^{-1} is also a function. We established in Chapter 3 that f^{-1} is a function if its graph passes the **vertical line test**. We have now seen the effect that inverting a function has on its graph.

What type of test should the graph of f pass, so that the graph of f^{-1} passes the vertical line test?

Explore 13.5

Can you sketch the graphs of the inverses of these two functions?





Can you tell which of the two inverses is a function? How?

Do both $g(x)$ and $h(x)$ have inverse functions? Why?

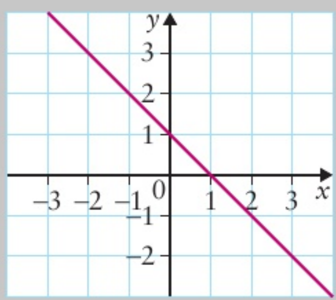
Can you relate this fact to a property of the original graphs for $g(x)$ and $h(x)$?

From Explore 13.5, you might have realised that carrying out the vertical line test on the graph of the inverse is the same as carrying out a horizontal line test on the graph of the original function. Therefore we can state that a function is one-to-one, and therefore invertible, if its graph passes the horizontal line test, that is, if any horizontal line crosses the graph at most once.

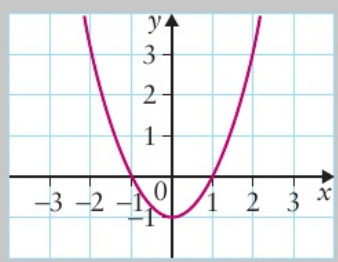
Worked example 13.6

Use the horizontal line test to check if each of these functions has an inverse function.

a

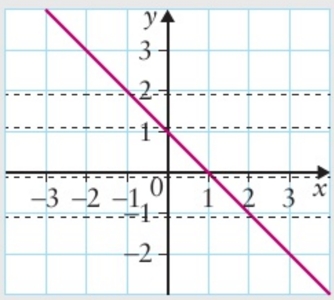


b



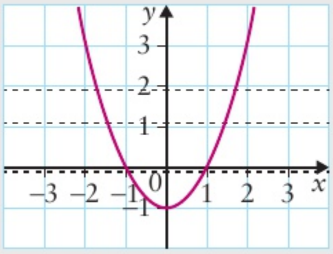
Solution

a To perform the horizontal line test, we cut the graph with all possible horizontal lines (only a few are shown).



None of them intersects the graph more than once, therefore this function has an inverse.

b We repeat the same procedure for the second graph.



There are horizontal lines that cross the graph twice, therefore the function does not have an inverse.

Practice questions 13.1.2

1 For each table of values determine whether the function f has an inverse.

If it has, find the table of values for the inverse f^{-1}. If it has not, explain why.

a

x	0	1	2	3
$f(x)$	−1	−2	−3	−4

b

x	1	2	3	4
$f(x)$	1	2	1	2

c

x	0	2	4	6
$f(x)$	0	2	4	6

d

x	−2	0	2	4
$f(x)$	3	1	3	5

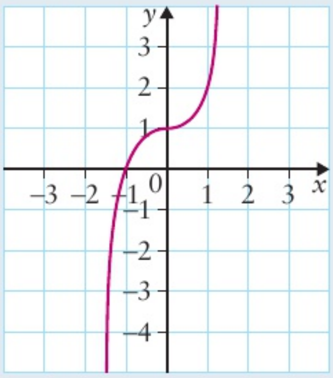
2 For each table of values in question 1:

a draw the mapping diagram

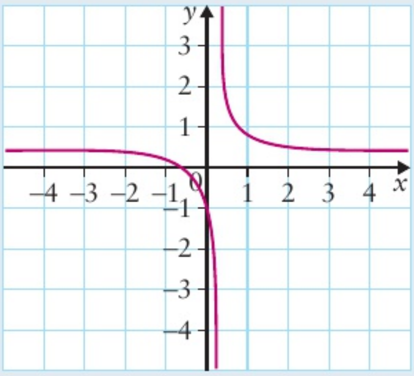
b state and justify whether the function is one-to-one or many-to-one.

3 Which of the following functions have an inverse?

a



b



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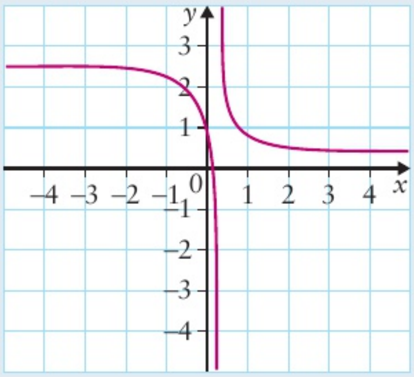

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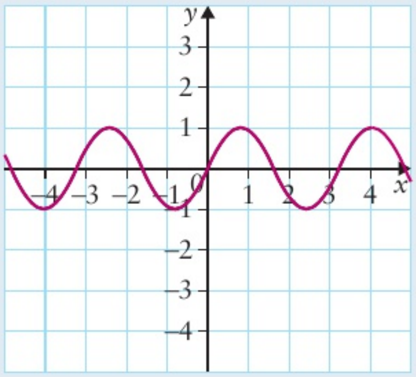

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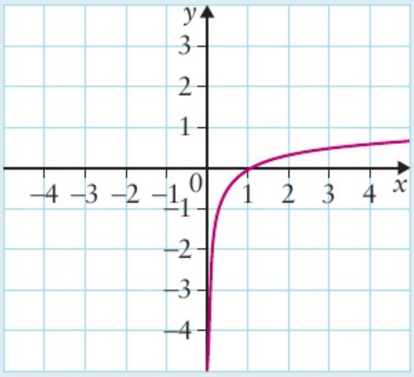

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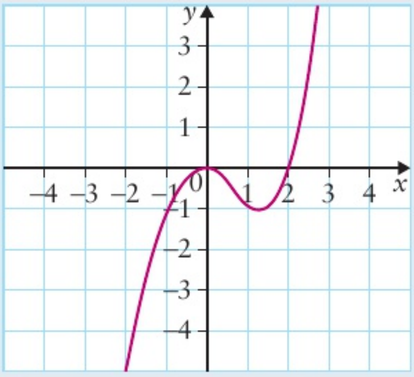

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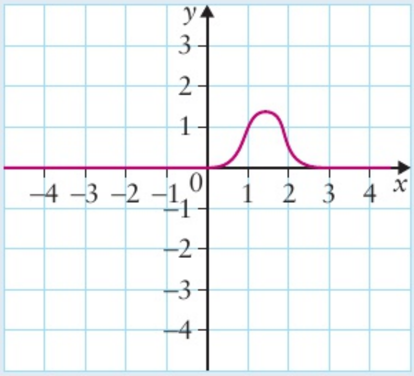

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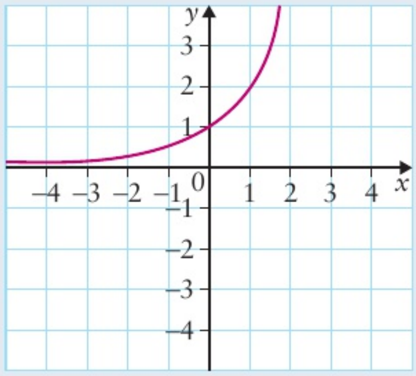

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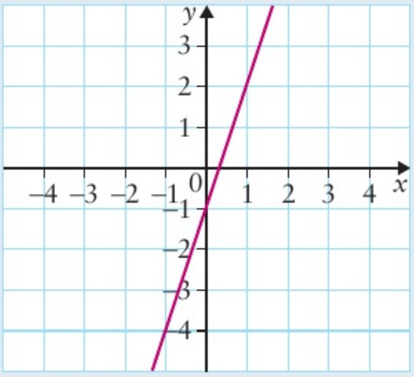

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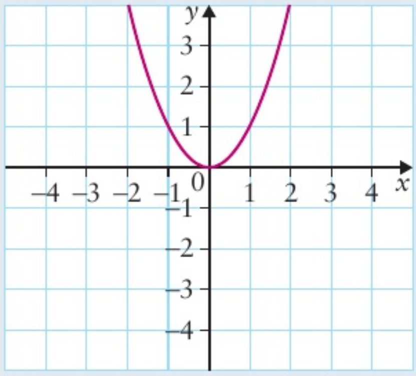

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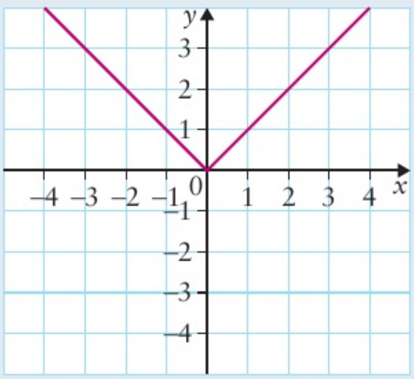
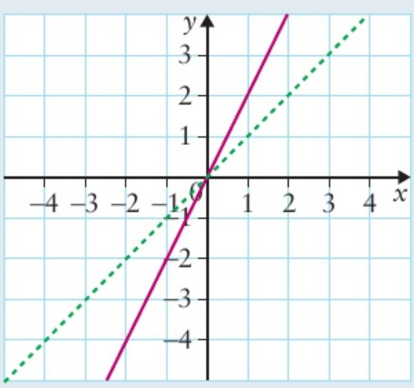


4 Copy the following diagrams, and sketch the graph of the inverse function. The dashed line is the line $y = x$

a



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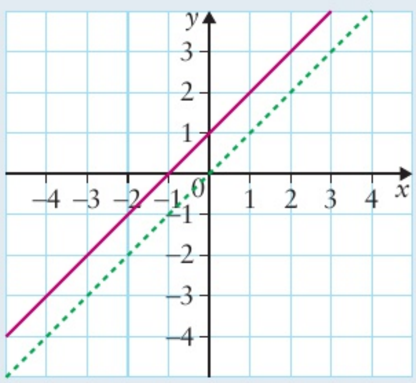

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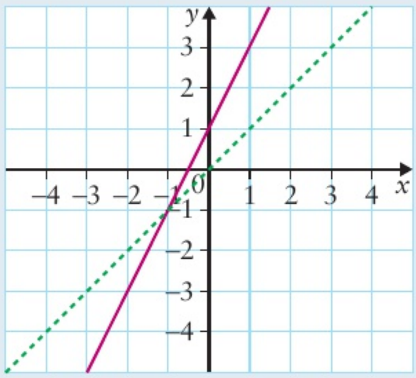
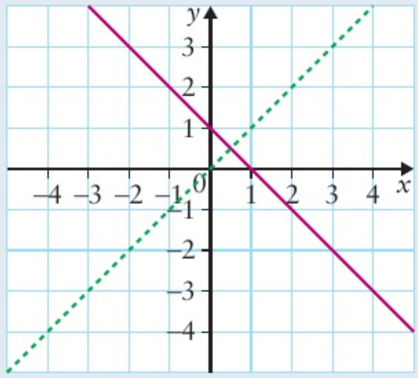


d



5 For each graph in question 4:

a find an equation of the given straight line

b find an equation of the inverse function.

13.1.3 How to find the inverse function: equations

The most commonly used applications of inverse functions require the ability to find the inverse of a function starting from the function's definition.

Explore 13.6

Can you find an expression for $f^{-1}(x)$ when $f(x) = 2x + 1$?

We learned in the previous sections that finding the inverse of a function requires interchanging the roles of the domain and range, for example by reversing arrows in a mapping diagram, by swapping rows or columns in a table of values, or by mirroring a graph in the line $y = x$, to have the effect of swapping the x and y coordinates of all points on the graph.

We will do the same when we are given the equation of a function $y = f(x)$:

- we will first solve the given equation for x, so that we have $x = f^{-1}(y)$
- then we will rename the variables (x becomes y and y becomes x) so that we end up with $y = f^{-1}(x)$

Worked example 13.7

a Given the function defined by $f(x) = 3x - 1$, find, if possible, an expression for $f^{-1}(x)$.

b Solve the equation $f(x) = 5$, where $f(x) = 3x - 1$

Solution

In part a, we first need to check whether the given function has an inverse. If it does, then we need to find it. We are given the function's equation, so we will look for the equation of $f^{-1}(x)$. In part b, we need to find the value of x that the function f sends to 5.

In part a, the horizontal line test will enable us to say whether $f(x)$ has an inverse or not. Since $y = 3x - 1$ has a familiar graph, we might be able to apply the test without actually graphing the function. We will then follow the steps suggested above: solving for x, then renaming variables. In part b, we now know that the value of x that the function f sends to 5 is precisely the meaning of $f^{-1}(5)$, so the answer is $x = f^{-1}(5)$. We will use our expression from part a to solve the equation.

a The graph of the function $f(x)$ is a straight line with gradient 3, so it passes the horizontal line test and therefore $f(x)$ has an inverse, so we can find it as follows:

$y = 3x - 1$ Rewrite the equation, introducing y.

$y + 1 = 3x$ Solve for x.

$x = \dfrac{y+1}{3}$

$y = \dfrac{x+1}{3}$ Rename the variables.

We have not explicitly mentioned the inverse function $f^{-1}(x)$. This is obtained by replacing y with $f^{-1}(x)$:

$$f^{-1}(x) = \frac{x+1}{3}$$

b $x = f^{-1}(5) = \dfrac{5+1}{3} = 2$

Looking back, we can check that this is the correct solution by replacing $x = 2$ in the given equation: $f(2) = 3(2) - 1 = 5$

In practical applications, when the variables have a specific contextual meaning, a slightly different notation is used for functions and their inverses, which helps us interpret the meaning of the inverse function more clearly.

Fact

The fact that y is a function of x, or in other words that y depends on x, can be expressed by $y = f(x)$ or simply by the notation $y(x)$. The inverse function, which describes how x depends on y, can be written as $x = f^{-1}(y)$ or $x(y)$.

Connections

Can you apply this idea to formulas from chemistry or physics?

For example, in science we could have an expression $p(T)$ for how the pressure p of a gas depends on its temperature T. The inverse function, which describes how temperature depends on pressure, is denoted $T(p)$.

Explore 13.7

In a factory, the cost, c dollars, for the production of n boxes of soap is given by the function $c(n) = 34 + 10n$

a Can you find how many boxes were produced if the cost was:

i \$104 **ii** \$1264 **iii** \$5924?

b Can you find $n(c)$, then evaluate $n(104)$, $n(1264)$ and $n(5924)$?

Reflect

Which of the questions in Explore 13.7 took longer to answer, part a or b? Why?

What are some advantages and uses of finding the equation of the inverse function?

Investigation 13.2

1 Write down an expression for $c(r)$, where r is the radius of a circle and c is the circumference of the circle.

a Find the circumference of a circle with radius 4.

b A circle has a circumference of 32. Find the radius of the circle to three significant figures.

c Find an expression for $r(c)$.

2 Write down an expression for $A(r)$, where r is the radius of a circle and A is the area of the circle.

a Find the area of a circle with radius 3.

b A circle has an area of 16. Find the radius of the circle to three significant figures.

c Find an expression for $r(A)$.

3 **a** A circle has an area of 16. Find the circumference of the circle.

b Find an expression for $c(A)$.

4 Find an expression for $A(c)$.

One of the most common applications of inverse functions is to solve equations. In general, the solution to the equation $f(x) = M$, where M is a number, is given by $x = f^{-1}(M)$

Practice questions 13.1.3

1 Find the inverse $f^{-1}(x)$ for each function.

a $f(x) = 3x + 1$

b $f(x) = \dfrac{x+1}{2}$

c $f(x) = x^3 + 1$

d $f(x) = -x - \dfrac{2}{3}$

e $f(x) = x$

f $f(x) = \dfrac{1}{x+1}$

g $f(x) = \dfrac{1}{x}$

h $f(x) = \dfrac{1+x}{1-x}$

Challenge Q1h

2 a Consider the function defined by $f(x) = x^2$ for values of x between -5 and 5.

i Sketch the graph of $y = f(x)$.

ii Determine whether f has an inverse.

b Now consider the function defined by $g(x) = x^2$, for values of x between 0 and 5.

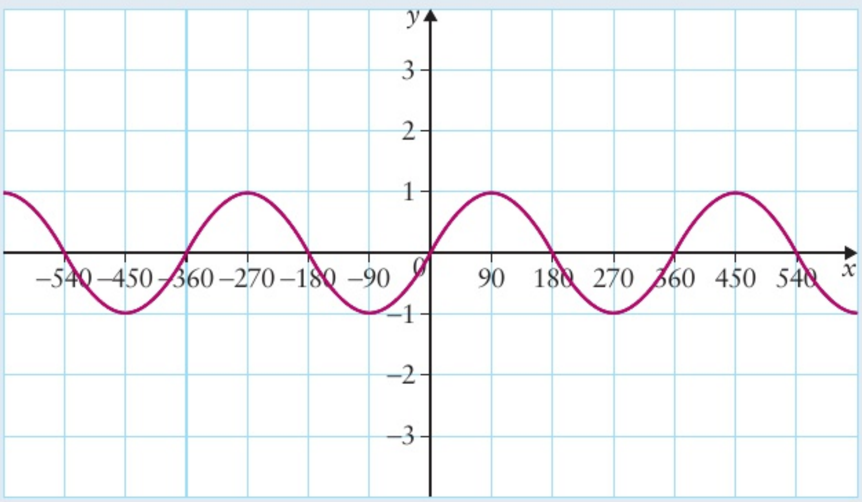
i Sketch the graph of $g(x)$.

ii Determine whether g has an inverse.

iii Find an expression for $g^{-1}(x)$.

c Consider the graph of $h(x)$ shown below for values of x between -540 and 540.

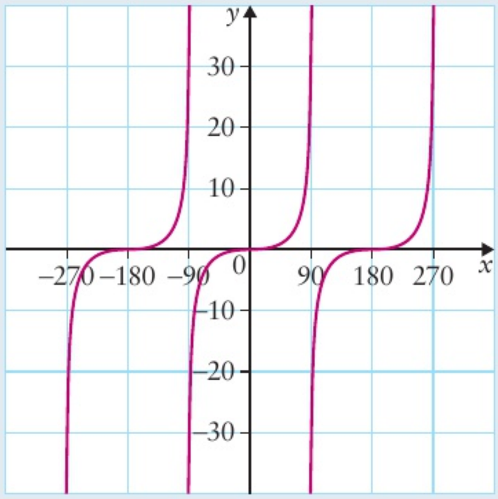
Challenge Q2c



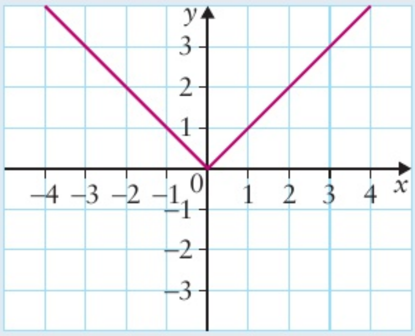
Suggest a smaller domain for which the function h has an inverse.

3 For each function illustrated by the graphs below, suggest a domain for which the function would have an inverse.

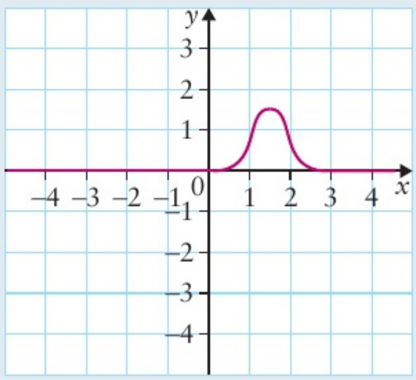
a



b



c



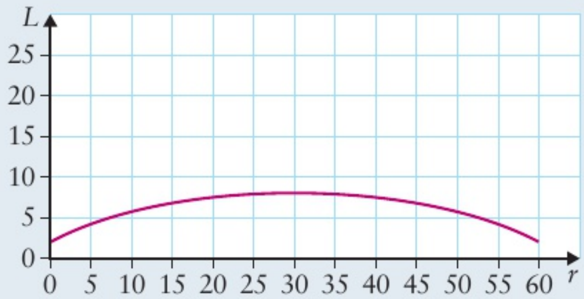
4 The pressure p (in Pascal) and volume V (in cubic metres) of a gas obey the relationship $pV = k$, where k is a constant.

a Find $p(V)$ and $V(p)$.

b Given that $p(3) = 4$, find k.

c Find $V(4)$.

5 The graph shows how performance, L (in metres), in the long jump varies with the length of the run-up, r (in metres), for an Olympic athlete.



a Does the function $L(r)$ have an inverse? Do you have an interpretation for this? What does the curve describe for $r > 30$ metres?

b For training purposes, the athlete's coach wants to graph how the performance, L varies with length of run up, r. Which part of the graph should she focus on?

c Describe the meaning of, and evaluate if possible: $L(10)$, $r(5)$, $L(0)$, $r(10)$, $r(0)$

6 Can you think of any functions that do not have an inverse and that are not mathematical functions? For example, consider the following:

a The function that associates each student of a particular class to their history teacher. Does this have an inverse?

b The relationship that associates words in English to their translation in another language you know. Does this have an inverse? Is it a function? What are some consequences?

13.2 Exponential functions $y = a^x$

13.2.1 Exponential growth

Explore 13.8

Your parents want to contribute to your pocket money. As a trial, they give you two options for the next month. The options are:

- receive 1 euro on the first day, 2 euros on the second day, 3 euros on the third day, increasing by 1 euro every day until the 30th
- receive 1 cent on the first day, 2 cents on the second day, 4 cents on the third day, doubling every day, until the 30th.

Which of the two options should you go for?

In Explore 13.8, you might have realised how rapidly the second option grew with time. In fact, on the last day of the month the second option would produce an income of more than 5 million euros, despite the fact that on the first day it was only worth one hundredth of the first option!

Connections

Exponential functions are the continuous version of geometric sequences.

Functions like the second option are called **exponential functions**. They are characterised by the fact that the variable x is the **exponent** in a power whose base is a constant number.

We can compare the linear function $y = 3x$, the power function $y = x^3$, and the exponential function $y = 3^x$, using x values from 0 to 6.

x	0	1	2	3	4	5	6
$3x$	0	3	6	9	12	15	18
x^3	0	1	8	27	64	125	216
3^x	1	3	9	27	81	243	729

From this, we see that the exponential function grows far more rapidly than the others.

Explore 13.9

Can the table of values for $y = 3^x$ be extended to negative values of x?

x	-5	-4	-3	-2	-1	0	1	...
3^x							3	...

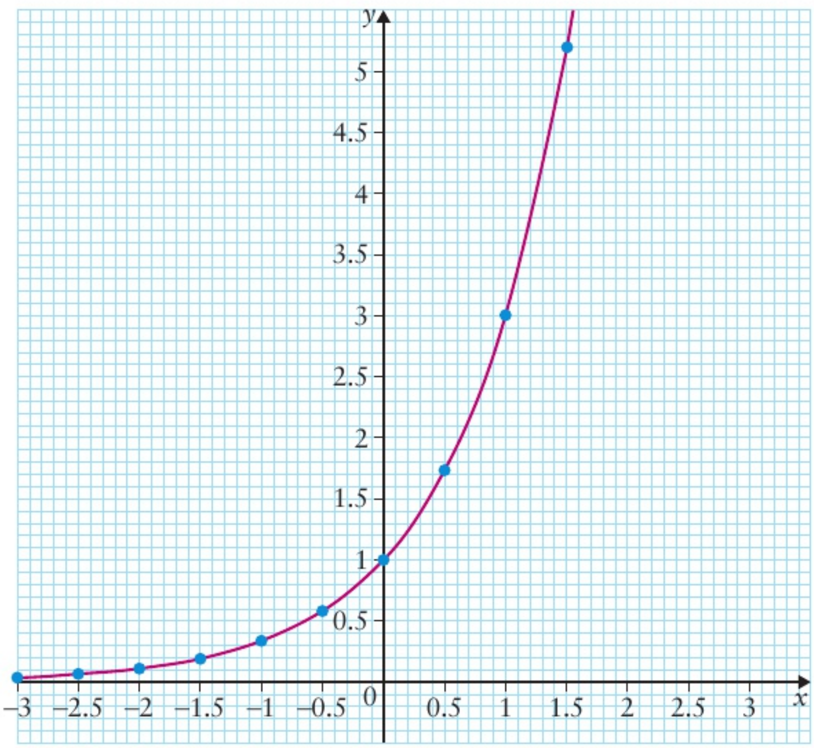
Can the table of values for $y = 3^x$ be extended to rational values of x?

x	...	-0.5	0.5	1.5	2.5
3^x	...				

Hint

Use technology for the evaluation of expressions like 3^{-4} or $3^{1.5}$.

The graph for the exponential function $y = 3^x$ over the set of real numbers looks like this. Use technology to compare this with the graphs of $y = 3x$ and $y = x^3$.



Exponential functions can be used to describe processes that grow or decay at an increasingly rapid rate.

Investigation 13.3

A piece of paper is 0.10 mm thick. How many times would you have to fold it in half so it becomes as thick as the distance between the Earth and the Sun (150 million km)?

A piece of thinner paper is only 0.05 mm thick. How many times do you have to fold this piece of paper so it becomes 150 million km thick?

Of course the model is theoretical. There is the objection that you can fold a piece of paper only a few times before it becomes too rigid, but that can be overcome by cutting and stacking instead of folding. A rather clearer objection emerges when you ask how wide the column of paper would be to reach such heights.

Thinking skills

13.2.2 The graph of $y = a^x$ and exponential decay

What features of the exponential curve remain the same if we compare the graphs of exponential functions with different bases?

Fact

For many applications, the most commonly used base for exponential functions is the number e. Can you use your GDC to find the approximate value of e?

Explore 13.10

This exploration requires a GDC or other graphing software.

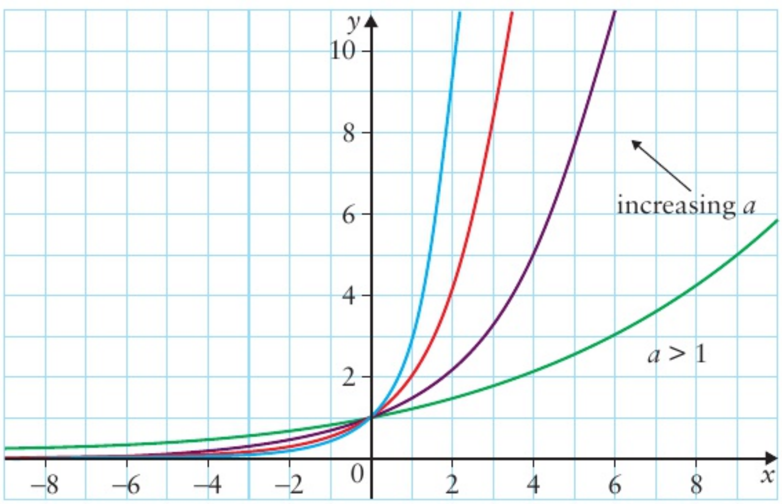
By graphing the functions $y = 2^x$ and $y = 3^x$ on the same set of axes, can you predict:

a the graph of $y = 10^x$ **b** the graph of $y = 1.5^x$

c the graph of $y = 1^x$?

How would this trend continue if the base of the exponential function became smaller than 1? Why?

If $a > 1$, the exponential function $y = a^x$ describes **exponential growth**. The x axis is an asymptote when x is large and negative.



Explore 13.11

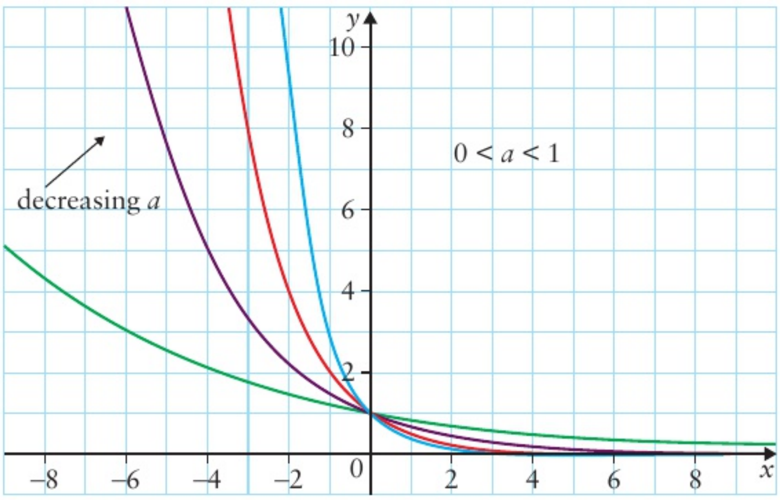
This exploration requires a GDC or graphing software.

By graphing the functions $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{3}\right)^x$ on the same set of axes, can you predict:

a the graph of $y = \left(\frac{1}{5}\right)^x$

b the graph of $y = \left(\frac{1}{10}\right)^x$?

If $0 < a < 1$, the exponential function $y = a^x$ describes **exponential decay**. The x-axis is an asymptote when x is large and positive.



Explore 13.12

Can you observe any special symmetry between the graphs of

$y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$?

Can you reconcile this result with the index laws you know?

Can you suggest two other exponential functions whose graphs behave in a similar way?

If $b > 1$, the exponential function $y = b^{-x}$ also describes exponential decay.

Investigation 13.4

In radioactive decay, an unstable substance like uranium transforms into another, more stable, element in a characteristic way. The number of remaining nuclei of uranium in a given sample always takes the same time to halve. This time is known as the **half-life** of the radioactive substance, and for uranium-232 it is 70 years.

For instance, if an experiment starts with 1 kg of uranium, after 70 years (after one half-life) there will be only 0.5 kg remaining.

1 Calculate how much uranium is still present after:

a 140 years **b** 210 years **c** 280 years.

2 Graph these values on a set of axes, with the remaining amount M in kilograms on the y-axis and time t in years on the x-axis.

3 How many half-lives does it take for the remaining uranium to become 15.625 g? How many years is that?

4 An expression for the remaining amount of uranium in kilograms as a function of the number, n, of half-lives that have passed since the beginning of the experiment is $M(n) = \left(\frac{1}{2}\right)^n$

Find an expression for $N(t)$, which is the remaining amount of uranium after t years.

All the function transformations we have learned about, vertical and horizontal stretches and shifts, can also be applied to exponential functions.

Worked example 13.8

a Describe the transformations that map the graph of $y = f(x)$ for the exponential function $f(x) = 3^x$ to the graph of $y = g(x)$, where $g(x) = 2(3^{x+1}) + 1$

b Graph both $f(x)$ and $g(x)$. What is the asymptote of $g(x)$?

Solution

a We need to identify the changes that map the graph of $y = f(x)$ into the graph of $y = g(x)$ in the correct order.

We have learned that transformations to the graph of a function relate to changes made to the expression for the function. So, we will work out how $f(x) = 3^x$ must be changed in order to obtain $g(x) = 2(3^{x+1}) + 1$

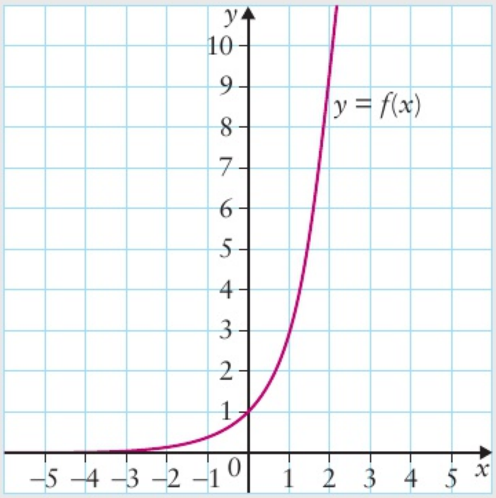
First, 3^x becomes 3^{x+1}. This is a horizontal shift of one unit to the left.

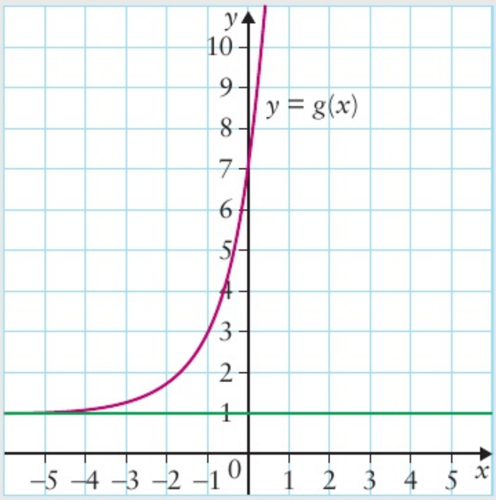
Next, 3^{x+1} becomes $2(3^{x+1})$. This is a vertical stretch by a factor of two.

Finally, $2(3^{x+1})$ becomes $2(3^{x+1}) + 1$. This is a vertical shift of one unit up.

Looking back, we could confirm of our findings by graphing the functions, as we will do for part b.

b The graphs of f and g are shown.





We can tell from the graph that the horizontal asymptote of $y = g(x)$ is the line $y = 1$. We shifted the whole graph one unit up, so the horizontal asymptote was also shifted up.

With the help of shifts, stretches and reciprocals, we can adapt the exponential function to describe **limited-growth** processes. Many processes are described by a growth that does not continue indefinitely in time but rather has an upper bound. Examples are the charging process of a battery, the number of rabbits on an island, or the population of planet Earth.

The charge stored in a battery cannot exceed the maximum capacity of the battery: charging a battery becomes harder and harder the fuller it is. An island has limited resources, so the more rabbits there are on the island the less food is available to each rabbit. In these cases, the charge in the battery and the number of rabbits do keep increasing as a function of time, but at smaller and smaller rates.

Fact

The Coulomb is a unit of electric charge.

Worked example 13.9

A rechargeable battery has a maximum capacity, M, of 10 Coulombs.

The battery is completely flat, so at $t = 0$ it is connected to a charger. $Q(t)$ represents the charge in the battery in Coulombs as a function of time measured in hours.

a i Explain why $M - Q(t)$ should decrease with time.

ii Suggest a function Q that would produce this effect.

b i Explain why $M - Q(t)$ should decrease with time at smaller and smaller rates.

ii Show that the function

$$Q(t) = 10\left(\frac{1}{3}\right)^t = 10(3^{-t})$$

has both the properties listed above.

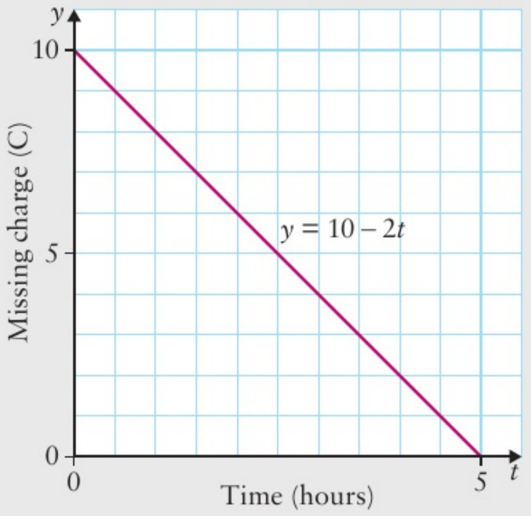
c Show that $Q(t)$has a horizontal asymptote at $Q = 10$

Solution

a i $M - Q(t)$ is the charge that is missing to reach full charge.

As the charging process goes on, the charge in the battery, $Q(t)$, goes up. Hence $M - Q(t)$ goes down.

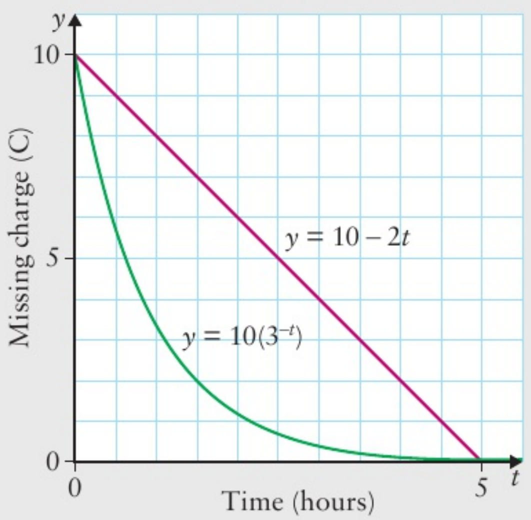
ii Any decreasing function would work. For example, the straight line shown here is $y = 10 - 2t$



This function would describe a charging process that takes 5 hours to complete.

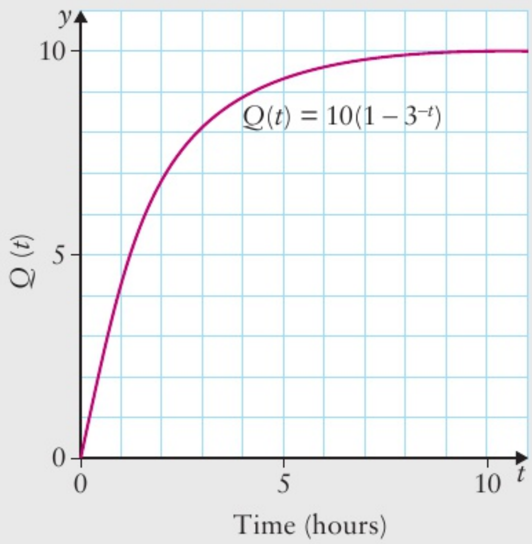
b **i** Since charge is building up in the battery, adding charge to it becomes more and more difficult. The missing charge should approach zero, but more and more slowly.

ii



The second graph represents a decreasing function that becomes less steep with time. This was not the case for the straight line $y = 10 - 2t$, which was decreasing but had a constant slope.

c We can now discuss the function $Q(t)$ that represents how the actual charge in the battery varies with time. From $M - Q(t) = 10(3^{-t})$ and $M = 10$, we obtain $Q(t) = 10(1 - 3^{-t})$.



The horizontal asymptote is $Q = 10$, which means that according to our model function the battery will never become fully charged.

Investigation 13.5

Consider the same battery as in Worked example 13.9

1 Calculate the charge in the battery after six hours.

2 Calculate the time required for the battery to reach:

a 90%

b 99% of its maximum capacity.

3 The next time the battery becomes flat, it is recharged with a more powerful charger. Suggest a new alternative definition for the function Q.

Investigation 13.6

Is there a similarity between stretching the graph of an exponential function vertically and shifting it horizontally?

1 For the function $f(x) = 3^x$, graph $y = f(x)$ using technology.

2 Consider the effect of stretching the graph of $y = f(x)$ vertically away from the x-axis by a factor of three.

a Given that the transformed graph has equation $y = g(x)$, find an expression for $g(x)$.

b Add the transformed graph of $y = g(x)$ on the same set of axes.

3 Consider the effect of shifting the graph of $y = f(x)$ horizontally to the left by one unit.

a Given that the transformed graph has the equation $y = h(x)$, find an expression for $h(x)$.

b Add the transformed graph of $y = h(x)$ on the same set of axes. What do you observe? Do you have an algebraic explanation for your observation?

Practice questions 13.2

1 For each of the following equations, graph the left hand side and the right hand side separately. For example, in part a, graph $y = 2^x$ and $y = 87$. Then use the graphs to help solve each equation.

a $2^x = 87$ b $3^x = 0.34$ c $5^x = 156$

d $10^x = 0.089$ e $\left(\frac{1}{2}\right)^x = 4$ f $\left(\frac{1}{7}\right)^x = 0.1$

2 Thorium is a radioactive element. It takes 24 seconds for 100 grams of thorium to reduce to 50 grams.

a How much thorium is still present:

i after 48 seconds

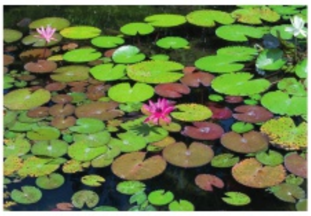
ii after 100 seconds?

b Draw a graph for the amount of thorium remaining as a function of time.

c After how many seconds will the amount of thorium remaining be:

i 12.5 g ii 10 g iii 0 g?

3 A water plant grows on the surface of a pond. The area covered by the plant doubles every three days. When the plant is first put in the water, it covers $\frac{1}{100}$ of the surface of the pond.



a What fraction of the surface of the pond will the plant cover after:

i 9 days ii 15 days iii 19 days?

b How many days will it take for the plant to cover half the pond?

c How many more days will it take for the plant to cover the whole pond?

Challenge Q4

4 When a fizzy drink is poured into a cylindrical glass, a layer of foam is formed. The width, W cm, of the foam layer decreases exponentially with time t seconds after pouring the drink according to $W(t) = 5 \times 2^{-1.5t}$.

a Calculate the width of the foam layer after:

i 1 second ii 5 seconds iii 10 seconds.

b When the drink is poured, the height H cm of the liquid from the bottom of the glass is 12 cm, but as the foam turns into a liquid, the height increases. The density of foam is 20 times smaller than the density of the liquid.

i Find an expression for $H(t)$.

ii Find an expression for $T(t)$, where T is the height of the top of the foam layer from the bottom of the glass.

Reflect

What types of equation can you solve if you have access to technology? Namely, what are the limitations of solving an equation graphically? Make up your own equation and try to solve it by looking for the intersections between the graphs of the left and of the right hand sides.

13.3 Logarithms and laws of logarithms

Explore 13.13

Consider the equality $8 = 2^3$, which involves the numbers 2, 3 and 8. The equality reads '8 is the number obtained when the base 2 is raised to the power of 3'.

Can you rearrange the equality above in the form $2 = \ldots$?

Complete the statement '2 is the number obtained when ...'

Complete the statement '3 is the number ...'

Can you rearrange the equality in the form $3 = \ldots$?

A power statement such as $a = b^c$ can be expressed in two more equivalent ways: we can make b the subject, or we can make c the subject.

Solving for b, we obtain a **root**: $b = \sqrt[c]{a}$

In fact, $\left(\sqrt[c]{a}\right)^c = a$

Solving for c instead, we obtain a **logarithm**:

$c = \log_b a$

In other words, $\log_b a$ is the **exponent** to which the base b has to be raised to obtain a:

$b^{\log_b a} = a$

> **Reminder**
>
> We are already familiar with roots and we can see them as powers with fractional exponents. In other words, $\sqrt[c]{a}$ is the base that, raised to the exponent c, gives a.

> **Fact**
>
> $\log_b a$ reads 'log base b of a' or 'log b of a'.

Worked example 13.10

Describe the role played by the exponent in each expression, then rewrite the expression as a logarithm.

a $2^3 = 8$ **b** $10^{-2} = 0.01$ **c** $16^{\frac{1}{2}} = 4$

Solution

a 3 is the exponent to which 2 has to be raised to obtain 8, therefore $3 = \log_2 8$

b -2 is the exponent to which 10 has to be raised to obtain 0.01, therefore $-2 = \log_{10} 0.01$

c $\frac{1}{2}$ is the exponent to which 16 has to be raised to obtain 4, therefore $\frac{1}{2} = \log_{16} 4$

> **Fact**
>
> $\log_a a = 1$

Logarithms are an equivalent way of writing powers:
$c = \log_b a$ means $a = b^c$

This is analogous to the relationship between fractions and products:
$z = \dfrac{x}{y}$ means $x = y \times z$

> **Connections**
>
> The two expressions are equivalent only when they are both defined: for instance, $4 = (-2)^2$ cannot be written as $2 = \log_{-2} 4$
>
> Can you relate this to the true statement $0 = 3 \times 0$ and its equivalent in fraction form?

Worked example 13.11

Describe the role played by the second factor in each expression, then rewrite as a fraction.

a $3 \times 4 = 12$ **b** $2 \times 5 = 10$

Solution

a 4 is the number that multiplied by 3 gives 12, therefore $4 = \frac{12}{3}$

b 5 is the number that multiplied by 2 gives 10, therefore $5 = \frac{10}{2}$

Reflect

Do all fractions evaluate to integers? What about the meaning of a fraction such as $\frac{2}{5}$? What does $\frac{2}{5} + \frac{1}{3}$ mean? What does $3 \times \frac{2}{5}$ mean? Why does $5 \times \frac{2}{5} = 2$?

As with fractions, which really become useful when they do not evaluate to integers, we need to understand the meaning of expressions such as $\log_3 5$, and how to work with them. For example, when adding logarithms, or when multiplying them by a number.

Reminder

$\mathbb{N}$ is the set of natural numbers 0, 1, 2, 3...

Explore 13.14

What is the meaning of $\log_3 5$? Can you give an estimate for $\log_3 5$ in the form $a < \log_3 5 < b$, where $a, b \in \mathbb{N}$?

Hint

Recall that $\log_b a$ is the exponent to which the base b has to be raised to obtain a.

Worked example 13.12

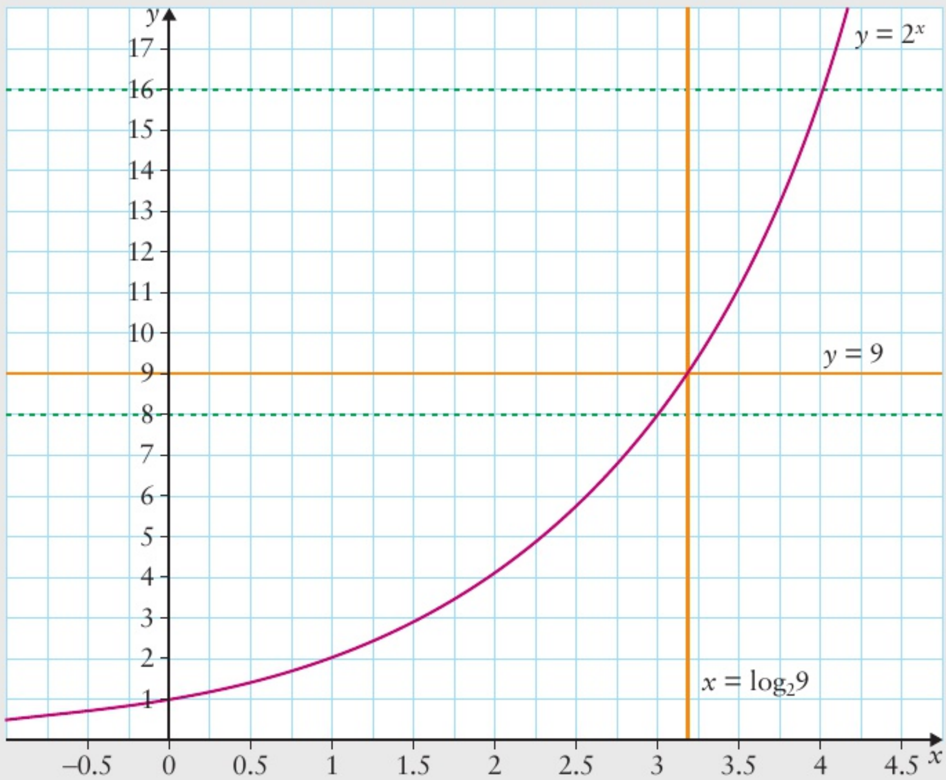
Estimate $\log_2 9$ in the form $a < \log_2 9 < b$, where $a, b \in \mathbb{N}$

Solution

To investigate the unknown meaning of $\log_2 9$, we start by calling it x. If we set $x = \log_2 9$, we can then rewrite this logarithmic expression as an equivalent power:

$x = \log_2 9$ becomes $2^x = 9$

The exponential equation for the number x can be solved by graphing $y = 2^x$ and $y = 9$, and looking for the intersection, as we did in Section 13.2.2.



We can see that $x = \log_2 9$ is somewhere between 3 and 4. This makes sense, because $2^3 = 8$ and $2^4 = 16$ (dashed lines).

The process we followed in Worked example 13.12 is precisely what we learned in Section 13.1 about inverse functions. We will exploit this link between exponential functions and logarithmic functions in the next section.

First, we will establish a few rules to help us work with logarithms.

Explore 13.15

Can you rewrite the index law $x^m \cdot x^n = x^{m+n}$ from the point of view of the exponents m and n?

The laws of logarithms, which can be derived from the index laws, are:

I $\log_x ab = \log_x a + \log_x b$

II $\log_x \frac{a}{b} = \log_x a - \log_x b$

III $\log_x a^n = n \log_x a$

Hint

Let $a = x^m$ and $b = x^n$

Rewrite these two statements as logarithms.

Continue this chain of equalities by making use of index laws: $\log_x ab = \log_x (x^m \times x^n) = \ldots$

Express the final result using logarithms.

Reflect

The first law of logarithms is expressed in words as: the logarithm of a product is the sum of the logarithms. How would you express the other two laws in words?

Can the third law of logarithms be derived directly from the first law of logarithms?

Hint

Multiplication is repeated addition, and powers are repeated multiplication.

There are four more results that we can derive from the definition of logarithms or from the three laws of logarithms discussed above.

1 $\log_a 1 = 0$

2 $\log_a a = 1$

3 $\log_a a^x = x$

4 $a^{\log_a x} = x$

Investigation 13.7

Thinking skills

Communication skills

1 Prove the first two results above by rewriting them in power form.

2 The meaning of $\log_a b$ is 'the exponent to which a has to be raised to obtain b'. Prove that $\log_a a^x = x$

3 The meaning of $\log_a x$ is 'the exponent to which a has to be raised to obtain x'. What happens if you raise a precisely to that exponent? Prove that $a^{\log_a x} = x$

Like all identities, the laws of logarithms can be used in either direction. In Worked examples 13.13 and 13.14 we will see examples of both uses.

Worked example 13.13

Write as a single logarithm:

a $\log_2 7 + \log_2 3$

b $\log_{10} 32 - \log_{10} 16$

c $\log_5 9 + \log_5 9$

Solution

a Using the first law:

$\log_2 7 + \log_2 3 = \log_2 3 \times 7 = \log_2 21$

b Using the second law:

$$\log_{10}32 - \log_{10}16 = \log_{10}\frac{32}{16} = \log_{10}2$$

c Using the third law:

$$\log_5 9 + \log_5 9 = 2\log_5 9 = \log_5 9^2 = \log_5 81$$

Worked example 13.14

Write as a linear combination of logarithms of x and y:

a $\log_2 xy$ b $\log_{10}\frac{y}{x}$ c $\log_5\frac{\sqrt{x}}{y^2}$

Solution

a Using the first law:

$$\log_2 xy = \log_2 x + \log_2 y$$

b Using the second law:

$$\log_{10}\frac{y}{x} = \log_{10}y - \log_{10}x$$

c Using the second law:

$$\log_5\frac{\sqrt{x}}{y^2} = \log_5\sqrt{x} - \log_5 y^2$$

Then, using the third law:

$$\log_5\sqrt{x} - \log_5 y^2 = \frac{1}{2}\log_5 x - 2\log_5 y$$

Hint

A linear combination of X and Y is an expression of the form $aX + bY$, where a, b are real numbers. For instance, both $2X + 3Y$ and $X - Y$ are linear combinations of X and Y.

Investigation 13.8

Logarithms are a convenient mathematical tool when discussing quantities that can vary by several orders of magnitude.

1 Can you complete a copy of the following table?

x	0.00001	0.0001	0.001	0.01	0.1	1	10	100	1000	10 000	100 000
x in scientific notation											
$\log_{10}x$											

2 Research 'SI prefixes'. Can you add a row to the table above?

Hint Q2

SI stands for Système International d'unités.

3 What are some quantities you have met in other subject areas that take values that can vary by several orders of magnitude?

Social skills

4 Research the following topics in small groups.

a Energy released by earthquakes (geology)

b Luminosity of stars (astronomy)

c pH of solutions (chemistry)

d Sound intensity level (physics)

e Light exposure and f-stops (photography)

f The slide rule (engineering)

g Log-log and log-linear scales (all sciences)

h Bach's equal temperament (music)

Practice questions 13.3

1 Write each expression in logarithmic form.

a $16 = 2^4$ b $5^2 = 25$ c $3^3 = 27$

d $5 = 5^1$ e $2^{-1} = \frac{1}{2}$ f $3^{-2} = \frac{1}{9}$

g $1 = 7^0$ h $(25)^{\frac{1}{2}} = 5$ i $\sqrt{9} = 3$

j $27^{\frac{4}{3}} = 81$ k $100^{-\frac{1}{2}} = 0.1$ l $\left(\sqrt{3}\right)^2 = 3$

2 Write each expression in power form.

a $\log_2 8 = 3$ b $\log_{10} 100 = 2$ c $\log_7 \frac{1}{49} = -2$

d $\log_6 6 = 1$ e $\log_6 1 = 0$ f $\log_{16} 4 = \frac{1}{2}$

g $\log_{27} 3 = \frac{1}{3}$ h $\log_{27} \frac{1}{3} = -\frac{1}{3}$ i $\log_2 \frac{1}{8} = -3$

j $\log_3 \sqrt{3} = \frac{1}{2}$ k $\log_{\sqrt{5}} 5 = 2$ l $\log_4 8 = \frac{3}{2}$

3 Evaluate the following.

a $\log_6 36$ b $\log_2 32$ c $\log_{10} 0.1$

d $\log_6 \frac{1}{36}$ e $\log_3 \sqrt{3}$ f $\log_5 1$

g $\log_3 \frac{1}{\sqrt{3}}$ h $\log_8 64$ i $\log_{64} 8$

j $\log_9 27$ k $\log_{27} \frac{1}{9}$

4 Evaluate the following.

a $\log_{10} 2 + \log_{10} 5$

b $\log_8 2 + \log_8 32$

c $\log_3 36 - \log_3 4$

d $\log_2 28 - \log_2 7$

e $\log_3 7 + \log_3 \frac{1}{7}$

f $\log_3 1 + \log_3 12 - \log_3 4$

g $2\log_3 6 - \log_3 4$

h $2\log_{10} 5 + \log_{10} 4$

5 Evaluate the following.

Challenge Q5

a $\log_b b^2$

b $\log_b 3b - \log_b 3$

c $7\log_3 a - \log_3 a^7$

d $\dfrac{\log_b x^5}{\log_b x}$

e $\log_a \frac{7}{3} + \log_a \frac{3}{7}$

f $\dfrac{\log_a x - \log_a x^2}{\log_a x}$

g $\log_3 9 \times \log_9 3$

h $\dfrac{\log_a b^2 + \log_a b}{\log_a \sqrt{b}}$

6 Write as a linear combination of logarithms.

Challenge Q6

a $\log_b \dfrac{xy}{z}$

b $\log_b 3b - \log_b 3$

c $\log_3 \dfrac{\sqrt{x}}{y^2 z}$

d $\dfrac{\log_b x^5}{\log_b x}$

e $\log_a x(x + 1)(x + 2)$

f $\log_b y\sqrt{y^2 + 1}$

13.4 Logarithmic functions and equations

13.4.1 Logarithmic functions

Explore 13.16

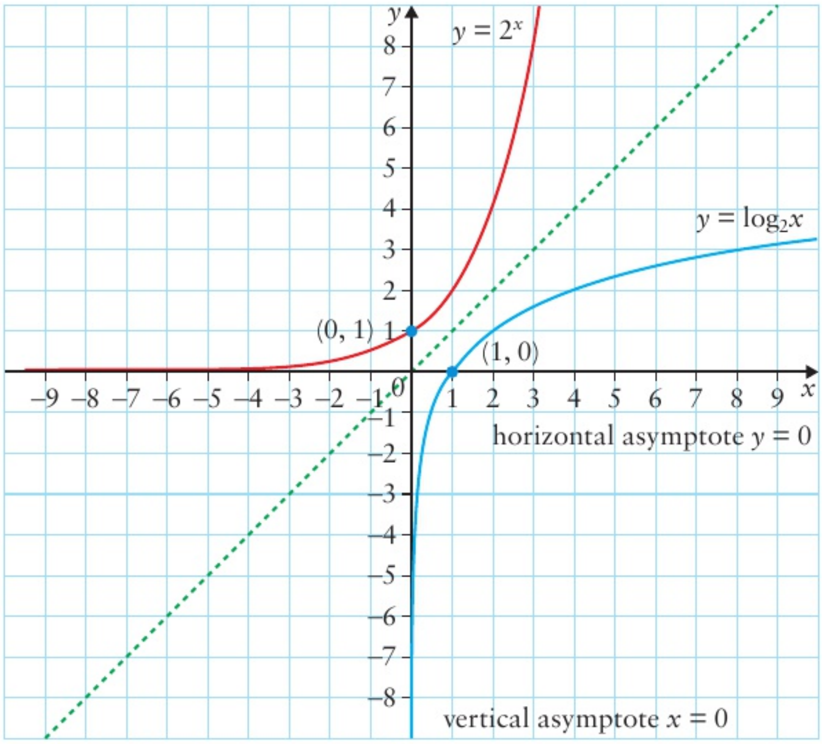
Can you confirm that the exponential function $f(x) = 3^x$ has an inverse and then find the equation of this inverse?

Reflect

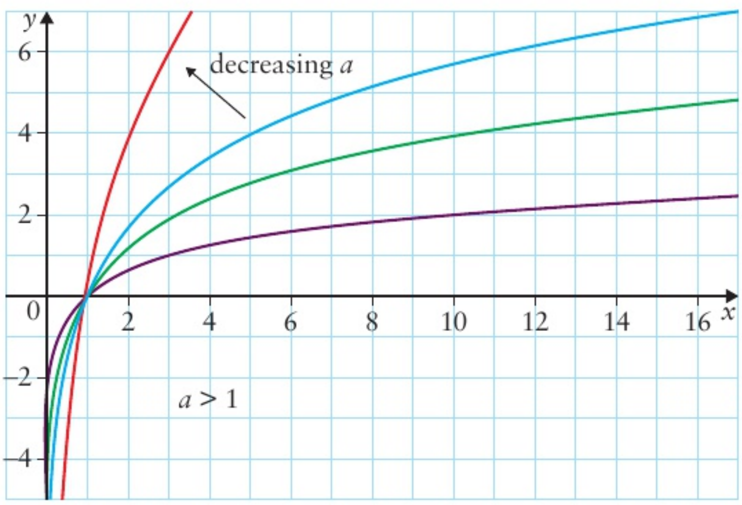
What strategies can we use if we now want to draw the graph of $y = f^{-1}(x)$?

Can we make use of the fact that $\log_a a^x = x$, or equivalently $a^{\log_a x} = x$?

Given that exponentials and logarithms are inverse functions of each other, the results we established in Section 13.1.2 allow us to, for example, draw the graph of $y = \log_2 x$ as the mirror image of the graph of $y = 2^x$ in the line $y = x$, without having to create a table of values.



Extending this result to other exponential graphs, we have the following graphs for the logarithmic functions $y = \log_a x$



Practice questions 13.4.1

1 a Sketch the graph of:

i $y = \log_4 x$ ii $y = \log_{0.25} x$

b Prove that $\log_a x = -\log_{\frac{1}{a}} x$

2 Based on the graphs for exponential decay in Section 13.2.2, sketch the graph of $y = \log_a x$ for $0 < a < 1$

Challenge Q1b

Challenge Q2

13.4.2 Exponential and logarithmic equations

In an exponential equation, the unknown, for example x, features as an exponent.

Two examples are $2^x = 8$ and $2^x = 9$. As for all equations, finding the solution means finding the value or values of x that satisfy the equation, that is, all those that make the left hand side equal to the right hand side.

Explore 13.17

How can you solve the equation $2^x = 8$?

What would change when trying to solve the equation $2^x = 9$?

Hint

The second equation relates to a problem we already discussed in Section 13.3.2.

Reflect

What makes solving an exponential equation simple? What makes it difficult?

Recall the important result we obtained in Section 13.1: the solution to the equation $f(x) = M$ is given by $x = f^{-1}(M)$, where M is a number and f is a one-to-one function.

Worked example 13.15

Solve:

a $9^x = 27$ b $8^x = 3$

Solution

a Since both 9 and 27 are powers of 3, we can write $9 = 3^2$ and $27 = 3^3$

The equation becomes $(3^2)^x = 3^3$

Using index laws, the left hand side can be written as 3^{2x}, so the equation becomes $3^{2x} = 3^3$

When the bases are the same, we can compare exponents:

$2x = 3$ hence $x = \dfrac{3}{2}$

b In this case, we can apply the same function (the inverse of $f(x) = 8^x$) to both sides of the equation. Taking logarithms base 8 of both sides we obtain:

$\log_8 8^x = \log_8 3$

$x = \log_8 3$

Fact

The inverse of the exponential function e^x is called the natural logarithm $\ln x$. This simply means $\ln x = \log_e x$.

We can evaluate any logarithm with the help of technology. All GDCs have a $\log_b$ key that allows you to choose the base.

Explore 13.18

With technology if needed, can you evaluate these, correct to three significant figures?

$\log_3 27$ $\quad$ $\log_9 27$ $\quad$ $\log_5 11$ $\quad$ $\log_8 3$

Worked example 13.16

The amount of fossil fuels available for energy production is decreasing, and in ten years' time only 50% of today's supply will be available.

If we reduce our dependence on fossil fuels for energy production in favour of more sustainable sources, we can assume that this trend will continue.

a Find a mathematical expression for the amount of fossil fuels available in t years.

b Based on your model, work out when the supply of fossil fuels will fall below these percentages of today's supply:

i 12.5% $\quad$ **ii** 10% $\quad$ **iii** 1%

Solution

a We need to find an expression for the function $f(t)$ that describes the amount of fossil fuels still available t years from now. What data do we have? The question gives us some qualitative information (this trend continues) and some quantitative information (50% will remain in 10 years).

What functions do we know that could display a continuing trend of 50% decrease every 10 years? Exponential decay, as we saw in the uranium investigation in Section 13.2.2, matches this description if we make the half-life equal to 10 years.

Next we need to turn this reasoning into an actual equation:

an exponential function of the form $y = \left(\frac{1}{2}\right)^n$ will halve whenever n increases by 1.

We want our function $f(t)$ to halve whenever t increases by 10, so we set

$$f(t) = \left(\frac{1}{2}\right)^{\frac{t}{10}}$$

Looking back, does this function make sense? Does it answer the question? What did we learn from this exercise that we will be able to use in other, analogous problems? Can we work out some extensions to this question?

We can check whether this function is consistent with the data we have.

Right now, at $t = 0$, we have 100% of today's supply. Does our function satisfy $f(0) = 1$?

In 10 years, at $t = 10$, we will have 50% of today's supply. Does our function satisfy $f(10) = 0.5$?

In 20 years, we will have 25% of today's supply. Does our function satisfy $f(20) = 0.25$?

Our function satisfies all these conditions, so it describes the trend and answers the question.

b **i** We set up the following equation:

$f(t) = 12.5\%$ or, more explicitly, $\left(\frac{1}{2}\right)^{\frac{t}{10}} = 0.125$

We can solve this equation by observing that $0.125 = \frac{1}{8} = \left(\frac{1}{2}\right)^3$

Comparing exponents in $\left(\frac{1}{2}\right)^{\frac{t}{10}} = \left(\frac{1}{2}\right)^3$, we obtain $\frac{t}{10} = 3$ and finally $t = 30$

Reminder

You can check this statement by using index laws: $\left(\frac{1}{2}\right)^{n+1} = \frac{1}{2} \times \left(\frac{1}{2}\right)^n$

Connections

After working this question out, can you see the answer to the following question at a glance?

The amount of oil barrels, o, in 15 years time will be 20% of today's supply. Assuming this trend continues, write an expression for $o(t)$, with t measured in years.

If we can answer $o(t) = \left(\frac{1}{5}\right)^{\frac{t}{15}}$ by analogy, then we have added a new tool to our mathematical toolbox.

ii We need to solve $\left(\frac{1}{2}\right)^{\frac{t}{10}} = 0.10$

There are no shortcuts here, since 0.10 cannot be easily written as a power of $\frac{1}{2}$.

Instead, we take the logarithm base $\frac{1}{2}$ of both hand sides:

$$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{t}{10}} = \log_{\frac{1}{2}} 0.10$$

The left hand side is $\frac{t}{10}$ (by definition of logarithm). Using technology, the right hand side evaluates to $\log_{\frac{1}{2}} 0.10 \approx 3.32$. The amount of fossil fuels will fall below 10% of today's supply in about 33 years.

iii We can repeat the same calculation above to solve $\left(\frac{1}{2}\right)^{\frac{t}{10}} = 0.01$

We can also observe that 1% is 10% of 10%. This tells us immediately that it is going to take 33 years to go from 100% to 10%, and another 33 years to go from 10% to 1%.

We can check with technology that the answer is ≈ 66.4 years.

Connections

We could say that 33 years is the 'tenth-life' for this decay process.

To conclude our discussion of applications of inverse functions, we are now going to investigate logarithmic equations.

Reflect

Go back to the beginning of this section, where we defined exponential equations. How would you define a logarithmic equation?

We have solved exponential equations by taking the logarithm of both sides. How could you solve a logarithmic equation?

Explore 13.19

Can you express in words what you are looking for when solving the logarithmic equation $\log_2 x = 4$?

Worked example 13.17

Solve the equation $\log_3 x = 2$

Solution

This logarithmic equation can be solved by applying the same function to both sides. The function we apply has to be the inverse of $\log_3 x$, so we perform the operation

$3^{\text{left hand side}} = 3^{\text{right hand side}}$

$3^{\log_3 x} = 3^2$

$x = 3^2 = 9$ Using the result $a^{\log_a x} = x$ from Section 13.2.2.

If a logarithmic equation contains more terms, we can make use of the laws of logarithms or of index laws in order to find its solution.

Worked example 13.18

Solve the equation $\log_4 x + \log_4 x^2 = 3$

Solution

Using the first law of logarithms:

$\log_4 x + \log_4 x^2 = 3$

$\log_4 x \times x^2 = 3$

$\log_4 x^3 = 3$

Finally, by applying the function 4^x to both sides:

$4^{\log_4 x^3} = 4^3 \Rightarrow x^3 = 4^3 \Rightarrow x = 4$

Looking back, we could also have transformed the left hand side of this equation into a single logarithm by using the third law of logarithms.

$\log_4 x + \log_4 x^2 = 3$

$\log_4 x + 2\log_4 x = 3$

Adding the two logarithms gives:

$3\log_4 x = 3$

and simplifying:

$\log_4 x = 1$

Finally, by applying the function f to both sides, where $f(x) = 4^x$:

$4^{\log_4 x} = 4^1 \Rightarrow x = 4$

Reflect

Look back at Practice questions 2, 3 and 4 from Section 13.2.
Can you answer them using a different method now?

Do all exponential and logarithmic equations have solutions?

Practice questions 13.4.2

1 Solve the following exponential equations.

a $2^x = 8$ b $3^x = 27$ c $5^x = 625$

d $10^x = 1000$ e $\left(\frac{1}{2}\right)^x = \frac{1}{16}$ f $\left(\frac{1}{3}\right)^x = \frac{1}{9}$

g $\left(\frac{1}{5}\right)^x = \frac{1}{25}$ h $\left(\frac{1}{10}\right)^x = 0.01$ i $2^x = \frac{1}{4}$

j $3^x = \frac{1}{27}$ k $5^x = \frac{1}{25}$ l $10^x = 0.001$

m $\left(\frac{1}{2}\right)^x = 32$ n $\left(\frac{1}{3}\right)^x = 81$ o $\left(\frac{1}{5}\right)^x = 125$

p $\left(\frac{1}{10}\right)^x = 100$

2 Identify the two integers that are nearest to the solution of each equation. For example, if $3^x = 11$ then $3^2 < 11 < 3^3 \Rightarrow 2 < x < 3$

a $2^x = 15$ b $3^x = 100$ c $5^x = 12$

d $10^x = 459$ e $2^x = \frac{1}{5}$ f $3^x = \frac{1}{10}$

g $5^x = \frac{1}{100}$ h $10^x = 0.0123$

Connections Q2d

Can you see a link with scientific notation? Is there a link between the number of digits of x and $\log_{10}x$?

3 Check your predictions in question 2 by solving each equation using logarithms.

4 Solve the following logarithmic equations.

a $\log_{10} x = 5$ **b** $\log_4 x = \frac{1}{2}$ **c** $\log_3 x = 1$

d $\log_5 x = 0$ **e** $\log_9 x = \frac{3}{2}$ **f** $\log_3 x = -3$

g $\log_{10} x = -2$ **h** $\log_8 x = \frac{1}{3}$ **i** $\log_5 x = -\frac{1}{2}$

j $\log_{16} x = \frac{3}{4}$

5 Write as power expressions and solve:

a $\log_x 8 = 1$ **b** $\log_x 8 = 3$ **c** $\log_x 27 = 3$

d $\log_x 4 = \frac{1}{2}$ **e** $\log_x \sqrt{3} = \frac{1}{2}$ **f** $\log_x \frac{1}{8} = -3$

6 Solve:

a $\log_5 2 + \log_5 x = 2$ **b** $\log_3 x - \log_3 2 = 3$

c $\log_2 x = \frac{\log_3 8}{\log_3 2}$ **d** $\log_2 x + \log_2 x^2 = 1$

e $\log_5 3x - \log_5 3 = 2$ **f** $\log_{100} 2x - \log_{100} x^2 + \log_{100} 5 = \frac{1}{2}$

7 The voltage V (in Volts) across an electronic measuring device varies with temperature T (in °C) as $V(T) = 1 + 2^T$. The voltage U (in Volts) across a different device varies with temperature as $U(T) = 9 - 2^T$. Find:

a the temperature at which the two devices give the same reading

b the common reading of the two devices at the temperature you found in part a.

13 Inverse functions, exponentials and logarithms

Self assessment

- I can distinguish functions that have an inverse from functions that do not.
- I can find the inverse a function in table and mapping diagram form.
- I can find the graph of $f^{-1}(x)$ from the graph of $f(x)$.
- I can find the expression for $f^{-1}(x)$ from the expression for $f(x)$.
- I can recognise the features of the graph of an exponential function.
- I can sketch the graph of exponential functions $y = a^x$
- I can relate exponential and logarithmic functions to each other through the process of function inversion.
- I can sketch the graph of logarithmic functions $y = \log_a x$
- I can apply index laws and laws of logarithms to simplify algebraic expressions.
- I can apply inverse functions to the solution of simple exponential and logarithmic equations.
- I can solve harder exponential and logarithmic equations using technology.

Check your knowledge questions

1 Consider the function $f(x) = 3x + 1$

Find:

a $f(2)$ **b** $f^{-1}(7)$ **c** $f^{-1}(x)$

2 For each of the following functions, sketch the graph and determine whether the function has an inverse. If it does, sketch the graph of the inverse function and find its equation.

a $f(x) = 6 - 2x$ **b** $g(x) = x^2 + 2x + 1$ for $x \geqslant 0$

c $h(x) = 2^x - 1$ **d** $s(x) = \log_3 x^2 + 1$ for $x > 0$

3 An object is dropped from a cliff at time $t = 0$.

The vertical distance d metres it travels as a function of time t seconds is given by:

$d(t) = \frac{1}{2}gt^2$, where $g = 9.81\,\mathrm{m\,s^{-2}}$

a Find the distance travelled by the object in:

i 1 second **ii** 2 seconds **iii** 4 seconds.

b Find an expression for the time required to travel a distance h m.

c It takes 5 seconds for the object to reach the bottom of the cliff. Find the height of the cliff.

4 Solve the following equations. Give your answer to three significant figures where necessary.

a $2^x = 128$ b $10 \times 10^x = 1000$ c $10^{x+1} = 1000$

d $(2^x)^2 = 16$ e $2^{2x} = 16$ f $3^{2x+1} = 27$

g $3^x = 14$ h $3^{1-x} = 9^{x+1}$

5 Solve the following equations. Give your answer to three significant figures where necessary.

a $\log_2 x - \log_2 x^2 = 4$ b $\log_2 x = 1.4$

c $\log_3 x^2 = -1$ d $\log_b 4x - \log_b 3 = \log_b(x-1)$

Challenge Q6

6 If $x = 10^p$, $y = 10^q$ and $z = 10^r$, express each of the following in terms of p, q and r.

a $\log_{10} xyz$ b $\log_{10} \frac{xy}{z}$

c $\log_{10} \frac{x^2}{yz}$ d $\log_{10} \frac{z^2\sqrt{x}}{y^3}$

e $\log_{10} \frac{y^2 z}{x} - \log_{10} \frac{x}{y}$ f $\log_{10} \frac{y}{x} + \log_{10} \frac{x}{y}$

g $\log_{100} \frac{y}{x} + \log_{100} \frac{x}{y}$ h $\log_{10} \frac{xy}{z} + \log_{10} \frac{yz}{x} + \log_{10} \frac{zx}{y}$

i $\log_{100} \frac{y^2 z^6}{x^4}$

Challenge Q7

7 Stars may be measured relative to our Sun by a value m according to the formula:

$m = \log_{10} \frac{P}{P_{Sun}}$ where P Watts is the power of the star and P_{Sun} Watts is the power of the Sun.

a Calculate the value of m for a star with $P = 3.4 \times 10^{33}$
Use $P_{Sun} = 3.8 \times 10^{26}$

b Determine the power emitted by a star with:

i $m = 3.5$ ii $m = -1.4$

Challenge Q8

8 Jack deposits \$500 in a bank account that pays an annual interest of 5%.

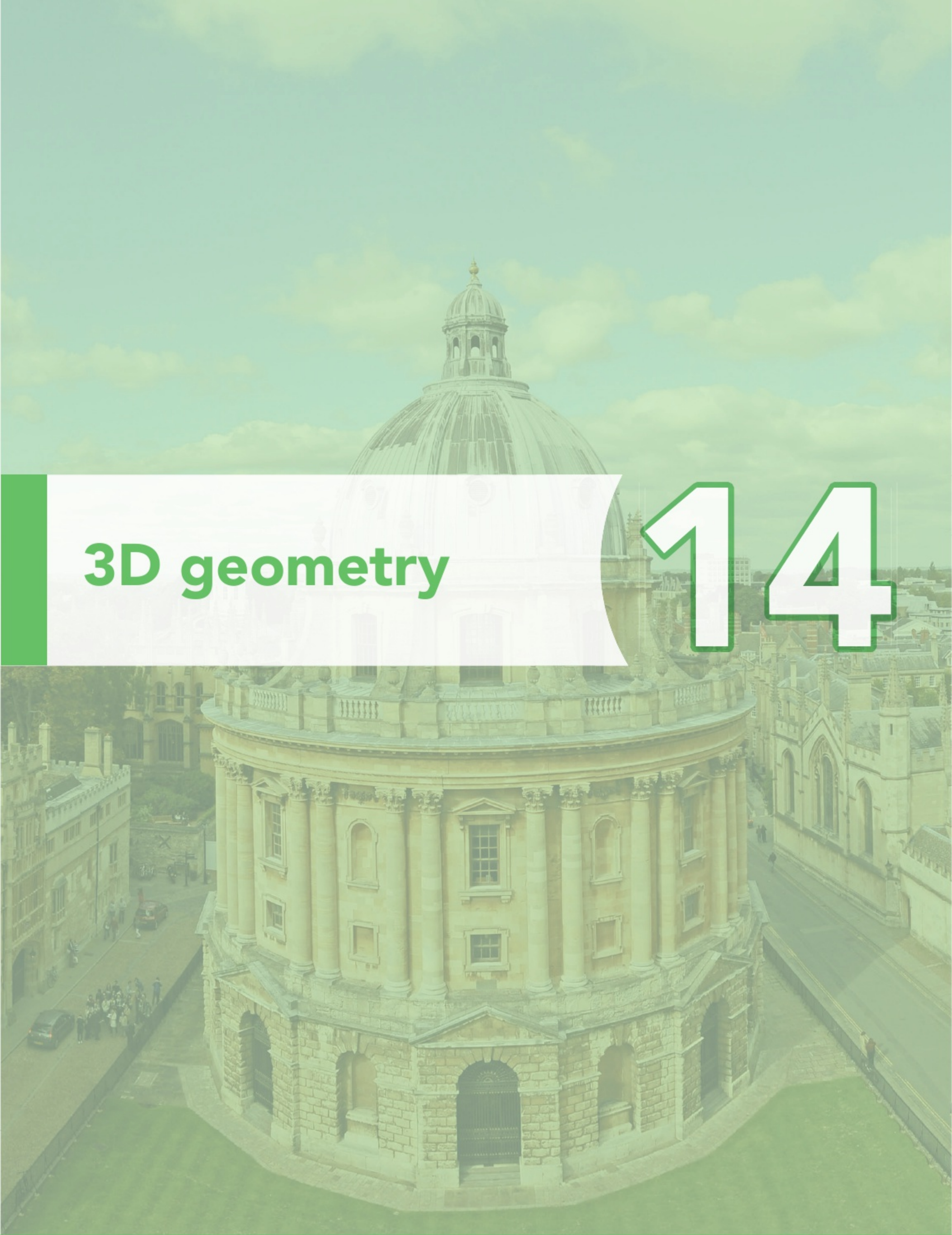
a Calculate the sum Jack will have in his account after:

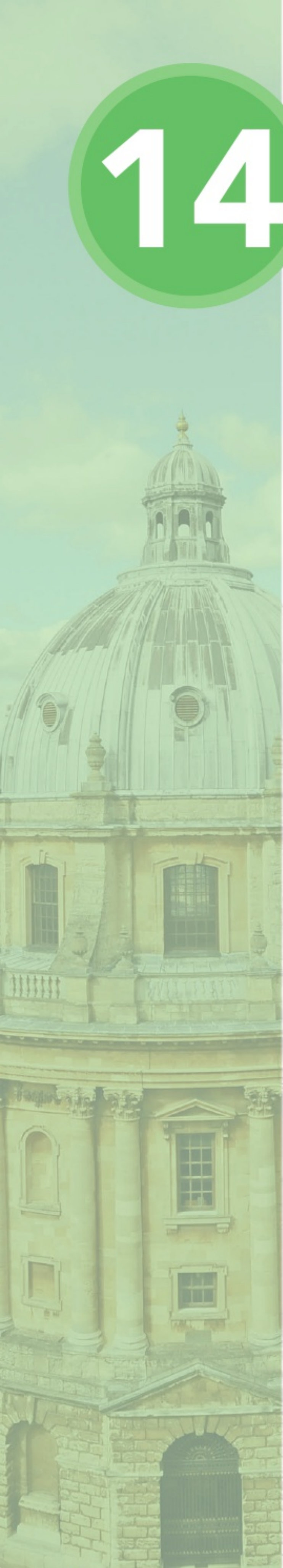
i 1 year ii 2 years.

b How long will it take for Jack's savings to exceed \$1000?

c What interest rate must Jack's bank pay if he wants his deposit to double in 10 years?

14 3D geometry





14 3D geometry

KEY CONCEPT

Relationships

RELATED CONCEPTS

Generalisation, Models, Representation, Space

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Exploring the relationship between objects in 2D and 3D allows us to generalise the orientation of immeasurable objects in space.

Factual

- What is the difference between the slant height and the height of a pyramid?
- How can we measure distances, surface areas and volumes in 3D objects?

Conceptual

- Can three planes intersect at one point?

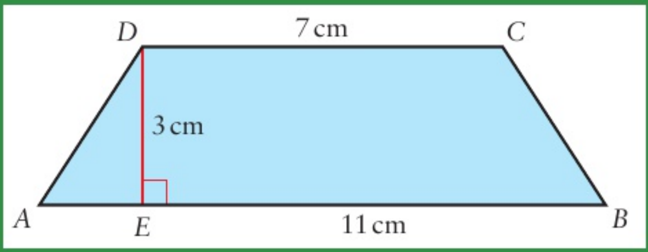
Debatable

- Can a solid have infinite surface area but finite volume?
- How many dimensions does the universe have?

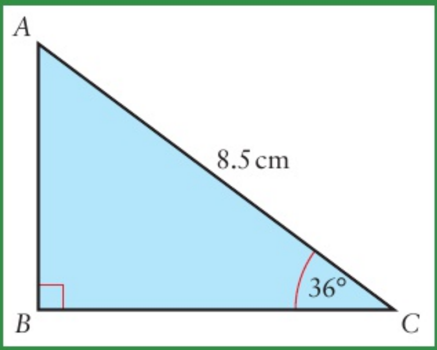
Do you recall?

1 One side of a right-angled triangle is 8 cm long. The hypotenuse is 12 cm long. Find the length of the other side.

2 Find the area of the trapezium shown below.



3 Find the length of side AB. Give your answer to an appropriate number of decimal places.

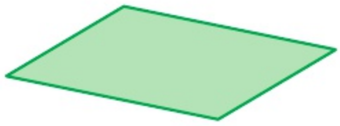


14.1 Points, lines and planes in 3D

In this chapter we will see how geometrical concepts, shapes and objects are used in representing the world around us.

We have already learned about the properties that lines and points satisfy in two dimensions. We will now explore some of the relationships that exist between points, lines and planes in three dimensions (3D).

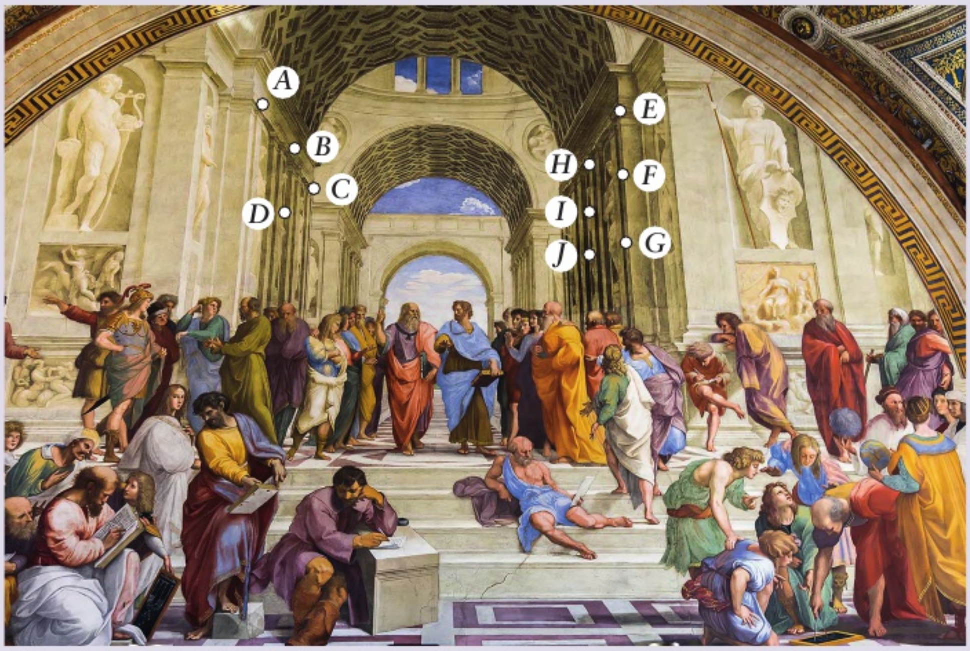
Besides points and lines, there are new objects to be considered in 3D space: planes. A **plane** is as a flat, unbounded surface such that the straight line joining any of its points lies entirely within that surface.



Fact

The illustration suggests that planes have edges, but planes are unbounded. They are infinite flat surfaces.

Explore 14.1



The fresco above is the famous *The School of Athens* by Raphael. Completed in 1511, it shows Raphael's exceptional ability to represent 3D space on the 2D surface of a canvas.

Using the points marked on the painting, can you tell whether:

- the points E, F, G and H are on the same plane
- the points D, E, F and H are on the same plane?

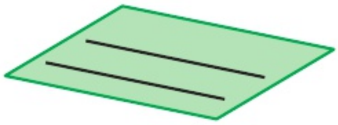
Extend the lines AB and EJ. Do they meet?

Reflect

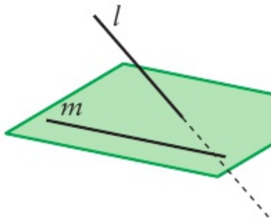
Describe, in your own words, the three dimensions that form the space around you.

In this section we will define the terms required to discuss objects in 3D space.

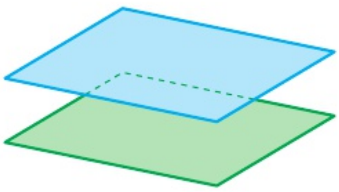
Two lines are **parallel** if there is a plane that contains them, and they do not intersect.



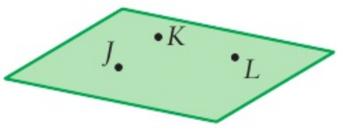
Two lines are **skew** if they are not parallel and they do not intersect. Lines l and m are skew.



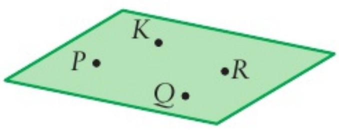
Two planes are **parallel** if they do not intersect.



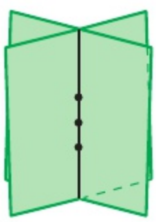
We can **determine** a plane from any three non-collinear points. For any three non-collinear points there is exactly one plane passing through them.



Four or more points are said to be **coplanar** if they lie on the same plane. Points K, P, Q and R are coplanar.



Any three collinear points have infinitely many planes passing through them.



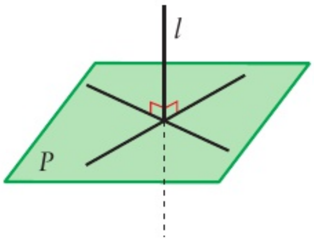
Fact

A plane can also be named by using three non-collinear points. We can call the plane shown JKL.

Fact

If a line is perpendicular to a plane then it is perpendicular to any line drawn in the plane.

A line l is **perpendicular to a plane** if it is perpendicular to two intersecting lines in the plane. Line l is perpendicular to plane p.



Reflect

Why do we need to use two lines on the plane for the definition of a line perpendicular to a plane? Wouldn't one line be enough?

Explore 14.2

Thinking skills

Social skills

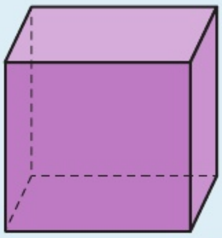
Communication skills

Copy the table and draw a diagram to show each property of objects in 3D space.

Property	Diagram
Two non-parallel planes intersect in a line.	
If two parallel planes are intersected by a third plane, then the lines of intersection of the planes are parallel.	
Two planes perpendicular to the same line are parallel.	
Given a line and a point not on the line, there is only one plane that both contains the point and is perpendicular to the line.	
Given two intersecting lines, there is always a plane that contains them. That is, two intersecting lines determine a plane.	
If two lines are both perpendicular to the same plane, then the lines are parallel.	

Hint

When drawing objects in 3D space it is sometimes convenient to use a cube to represent the 3D space. Use dashed lines to represent elements that are hidden from the view.



Practice questions 14.1

1 Write whether each statement is true or false. If the statement is false, sketch a counterexample.

a Two points are always collinear.

b Four points can be non-coplanar.

c If two parallel lines lie on two distinct planes, then the planes must be parallel.

d Two planes can intersect at a point.

e Three planes can intersect at a point.

f Two skew lines can intersect.

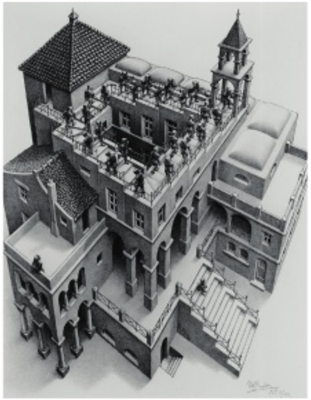
g If three lines intersect at a point then they must be coplanar.

h Three planes can intersect in a line.

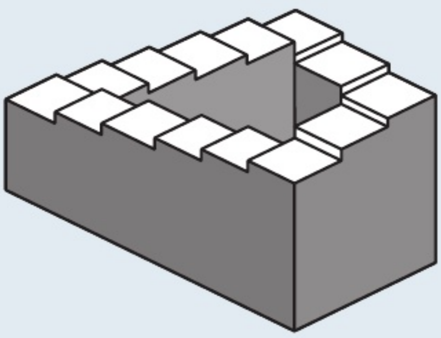
i If a line is perpendicular to a line in a plane, then it is perpendicular to the plane.

j If a line is perpendicular to one of two parallel planes, it must be perpendicular to the other plane.

2 The staircase on the roof of the building in the picture opposite is one of the impossible staircases drawn by MC Escher. The people on the outside are always ascending, while those on the inside are always descending. The paradox has been created using distorted proportions.



Here is a simpler representation of the impossible staircase.



a Identify two parallel planes.

b Identify two perpendicular planes.

c Identify a line and a plane perpendicular to it.

d Identify two parallel lines.

e Identify two skew lines.

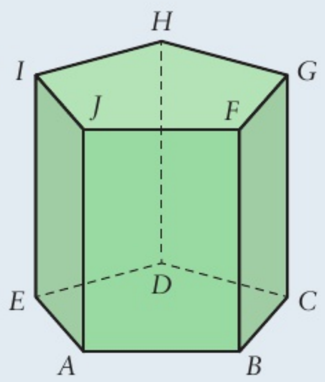
3 Sketch each situation, if possible. If it is not possible, say why.

a Plane F contains four non-collinear points A, B, C and D.

b Line h is skew to line k.

c Line m is perpendicular to planes N and O.

4



Use the diagram to:

a identify which plane is parallel to plane JIH

b identify all the lines that are skew to line AB

c identify all the lines that are parallel to line AJ

d identify four coplanar points (there is more than one answer)

e explain whether points I, J, F are collinear

f identify the intersection point of line FB with plane EAD

g identify the intersection of plane JFG with plane IHE

h identify the intersection of plane JFG, plane IHE and plane HGC.

Connections

In Picasso's painting *Brick factory at Tortosa* the buildings are represented as polyhedra. Cubism was an art movement of the late 19th and early 20th centuries that represented reality through a radical geometrisation of forms.

14.2 Polyhedra

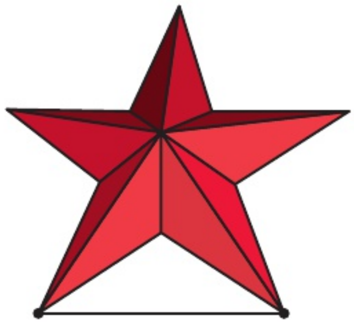
A **polyhedron** is a three-dimensional shape bounded by parts of intersecting planes. The intersecting planes form polygons that are called the **faces** of the polyhedron. The sides of the faces are called the **edges** of the polyhedron and their corners are called **vertices**.

Fact

The word polyhedron comes from the Greek words *poly*, meaning many, and *hedron*, meaning faces.

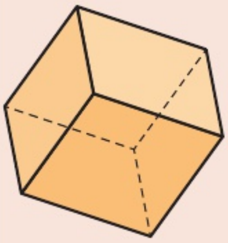
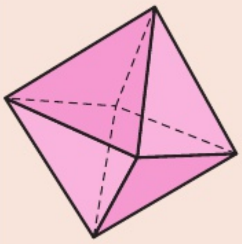
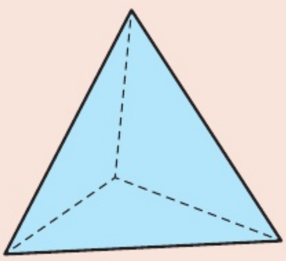
When all the faces of a polyhedron are congruent regular polygons, the polyhedron is said to be **regular**.

A polyhedron is **convex** if the line segments connecting *any* two points on its surface always within the polyhedron. The polyhedron shown on the right is not convex, because you can draw a line segment between two points that does not lie inside the polyhedron.



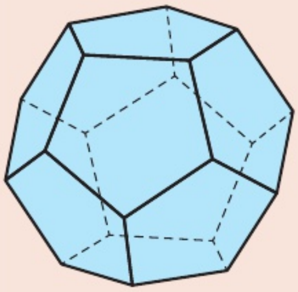
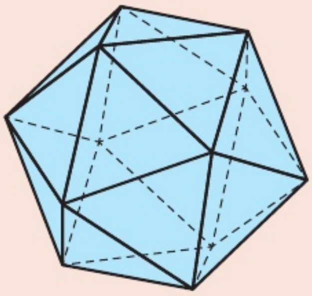
Investigation 14.1

Copy and complete the table below (continued on the following page).

Polyhedron	Number of faces, F	Number of edges, E	Number of vertices, V
 Hexahedron (Cube)			
 Octahedron			
 Tetrahedron			

Connections

There are five regular convex polyhedra, which are called the **Platonic solids**. Two of them are the tetrahedron and the cube. The philosopher Plato associated them with the classical elements (earth, fire, air, water) and the constellations. In the 16th century Kepler tried to relate the platonic solids to the orbits of the planets known at the time. Geometrical objects or shapes often have a philosophical meaning and represent the expression of a culture.

 Dodecahedron			
 Icosahedron			

Can you relate the number of faces (F), vertices (V) and edges (E) in a convex polyhedron?

Research skills

The number of faces (F), vertices (V) and edges (E) in a convex polyhedron are related by Euler's polyhedral formula. Find out what the formula is and verify that it is true using your table results.

The number of faces (F), vertices (V) and edges (E) in a convex polyhedron are related by a formula, called Euler's formula: $F + V - E = 2$

Worked example 14.1

A three-dimensional figure has 12 vertices, 6 faces and 14 edges. Is it a convex polyhedron?

Solution

We need to determine whether a figure with 12 vertices, 6 faces and 14 edges is a convex polyhedron. We can use Euler's formula to see if these properties match those of a convex polyhedron.

$F + V - E = 2$

$6 + 12 - 14 = 4$

Since $4 \neq 2$, we can conclude that, there is no convex polyhedron with 12 vertices, 6 faces, and 14 edges. The figure is not a convex polyhedron.

Practice questions 14.2

1 For each polyhedron:

i count the number of edges that meet at vertex A

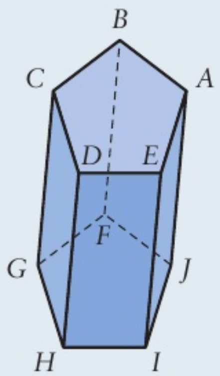
ii count the number of faces that meet at edge AB

iii name two skew edges

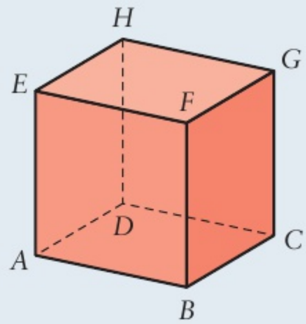
iv name two parallel edges

v count the number of faces, vertices and edges and verify Euler's formula.

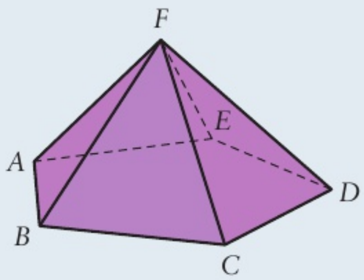
a



b



c



2 Identify the following platonic solids by first finding the number of faces.

a 12 edges, 8 vertices

b 30 edges, 12 vertices

c 6 edges, 4 vertices

d 30 edges, 20 vertices

3 A three-dimensional figure has 5 faces, 9 edges and 11 vertices. Is it a convex polyhedron?

4 A pyramid has 7 faces and 12 edges.

a Find the number of vertices.

b Identify what type of pyramid it is. Explain your answer.

Challenge Q5

5 Amber, a mathematical ant, is at a vertex of a cube and decides to go for a walk along the edges of the cube. She wants to visit each vertex once and only once and then to arrive back where she started.

a If each edge of the cube is 3 cm, what distance does she travel?

b Sketch the cube and show one of Amber's possible routes by numbering the vertices in order that Amber meets them.

c How many possible paths are there? Justify your answer.

Explore 14.3

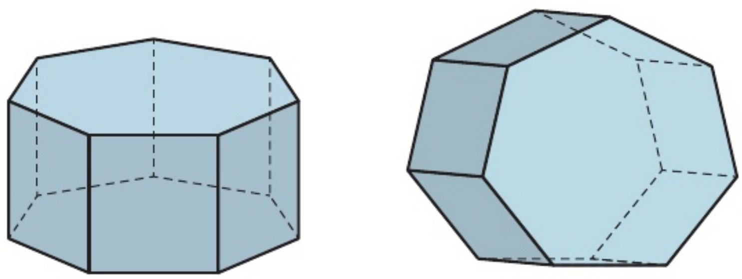
Use graphing software, such as GeoGebra, create your own cubist art work with polyhedra.

14.3 Surface areas of prisms and cylinders

14.3.1 Prisms

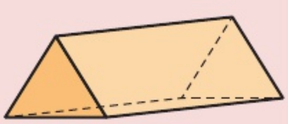
A **prism** is a polyhedron with two congruent faces lying on two parallel planes. The congruent faces are called the **bases**. The other faces of the prism are called the **lateral faces** and are formed by the edges that connect the corresponding vertices of the two bases. In this book we will look at **right prisms**. These are prisms where the edges of the lateral faces are perpendicular to the bases.

Prisms are named according to the shape of the bases. This is a heptagonal prism.



Fact

The base of a triangular prism is a triangle, even if the prism is lying on one of its lateral faces.

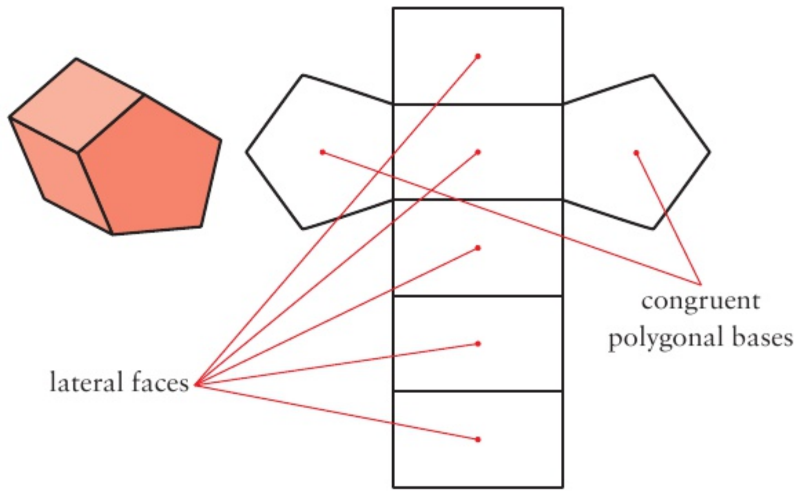


Reflect

What type of polygons are the lateral faces? Are the lateral faces all congruent polygons?

How many faces does a triangular prism have? What about a pentagonal prism, or a prism whose base is an n-sided base polygon?

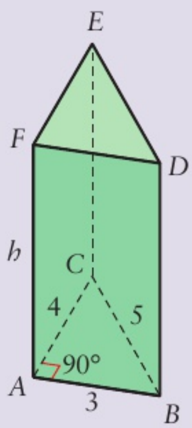
The **lateral area** of a prism is the sum of the areas of its lateral faces. The total **surface area** of a prism is the sum of its lateral area and the areas of its bases. The surface of a prism is made of rectangles (the lateral faces) and two congruent polygons (the bases).



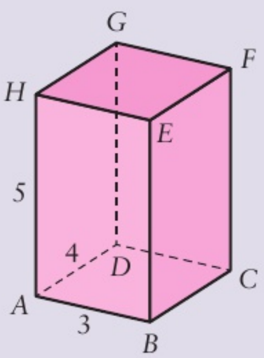
Explore 14.4

Look at the triangular prism. The bases are congruent with sides 3 cm, 4 cm and 5 cm. The height, h, is 5 cm.

Can you find the lateral area of this prism?



Can you find the lateral area of a cuboid, with length 3 cm, width 4 cm and height 5 cm?



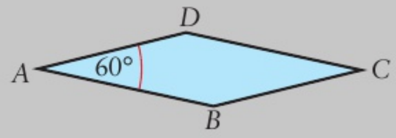
The total surface area of a prism is the sum of the areas of the two bases and the lateral area. Thus, the total surface area of a prism can be written as

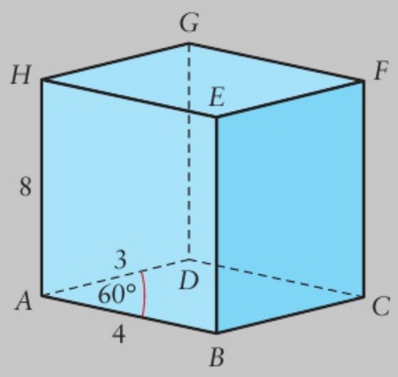
$S = 2B + P \times h$

where B is the area of the base, P is the perimeter of the base and h is the height of the prism.

Worked example 14.2

The bases of a prism are parallelograms with sides 4 cm and 3 cm and smaller angle 60°, as shown in the diagram. The height of the prism is 8 cm. Find the surface area of the prism. Give your answer in exact form.





Solution

Understand the problem

To find the surface area we need to calculate the lateral area and the area of the two bases and add them up.

Make a plan

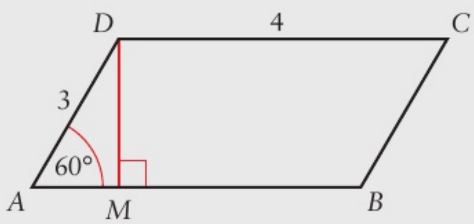
First we calculate the area of the two parallelograms. Then we find the area of the lateral faces. Then we add up all these quantities.

The formula for the area of a parallelogram is $B = bh$, where b is the length of the base and h is the height. We are not given the height, but we know the measure of the angle DAB. We can use trigonometry to find the height.

The lateral faces together form a big rectangle whose length is the perimeter of the parallelogram base and whose height is 8 cm. We can find the lateral area, L, by multiplying the perimeter of the parallelogram $ABCD$ by the height of the prism.

Carry out the plan

The sine ratio of an angle A is defined as $\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$



In the parallelogram this is $\sin 60° = \frac{\text{height}}{\text{oblique side}} = \frac{DM}{AD}$

So

$$\sin 60° = \frac{DM}{3}$$

$$\frac{\sqrt{3}}{2} = \frac{DM}{3}$$

$$DM = \frac{3\sqrt{3}}{2}$$

The area of the parallelogram is $B = bh = 4 \times \frac{3\sqrt{3}}{2} = 6\sqrt{3}\ \text{cm}^2$

The perimeter of the parallelogram is $(4 + 4 + 3 + 3)$.
So, $L = (4 + 4 + 3 + 3) \times 8 = 112\ \text{cm}^2$

The total surface area is $S = 2B + L = (12\sqrt{3} + 112)\ \text{cm}^2$

Look back

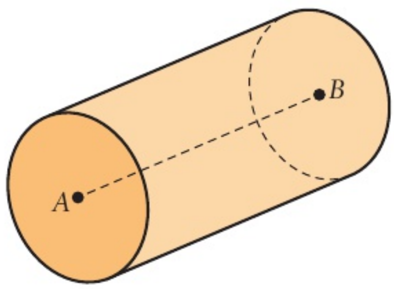
The question asks for the answer in exact form, so we do not calculate the value of $\sqrt{3}$

Reminder

The sine, cosine and tangent ratios of 30°, 45°, 60° can be calculated exactly using the special triangles.

14.3.2 Cylinders

If we use a circle instead of a polygon as the base of a prism, we get another solid. A **cylinder** is a three-dimensional figure bounded by two congruent circular regions lying on two parallel planes, the bases, connected by a curved surface. The **axis** of a cylinder is the segment joining the centres of the two bases.



Fact

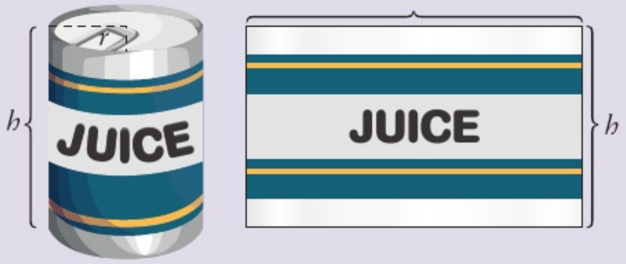
The cylinder is not a polyhedron, since it is formed by a curved surface. The bases of a cylinder are the two congruent circles.

The lateral area of the cylinder is the area of the curved surface. The total surface area of the cylinder is the sum of the areas of the two circular bases and the lateral area.

Explore 14.5

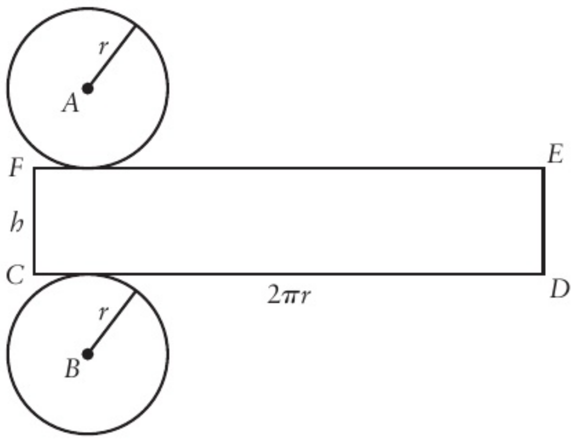
Take a juice can and trace the top and the bottom of it on paper.

Cut out from the paper a long rectangle with width equal to the height of the can and wrap the paper rectangle around the can. Cut off the excess paper so the edges just meet, and then unwrap it as shown.



Work out the area of the top and bottom circles as well as the rectangle you formed.

This is a net for a cylinder with radius r and height h.



Similarly to a prism, the surface area of a cylinder is given by the sum of the area of the two circles and the lateral area. The lateral area is the area of a rectangle whose length is $2\pi r$ (the circumference of the base) and whose width is the height, h, of the cylinder:

$S = 2\pi r^2 + 2\pi rh$

Worked example 14.3

A tin of tomato soup has a circular base with radius 3 cm and height 10 cm. How much paper is needed for the labels of a box of 12 tins? Give your answer rounded to the nearest integer.

Solution

The label of a tin completely covers the lateral area of a cylinder. Therefore, we need to find the lateral areas of 12 cylinders.

We will use the formula for the lateral area of one tin and multiply by 12.

The lateral area, L, of a cylinder is the area of the rectangle with length $2\pi r$ and width the height of the cylinder, $L = 2\pi rh$

Each can has a lateral area of:

$L = 2\pi rh = 6\pi \times 10 = 60\pi \approx 188.5\,\text{cm}^2$

So the lateral area of 12 cans is:

$12 \times 60\pi \approx 2261.9\,\text{cm}^2$

The paper needed for the labels of 12 tins is $2262\,\text{cm}^2$, correct to the nearest integer.

Worked example 14.4

A recycling company pays £0.02 for an empty tin of soup with a base of radius 3 cm and a height of 10 cm. A tin of beans can has a base of radius 4 cm and a height of 8 cm.

Assume that the recycle value is proportional to the surface area. How much would the company pay for an empty tin of beans? Give your answer correct to three significant figures.

Solution

We need to find the surface area of a tin of beans and compare it to the surface area of a tin of soup.

The surface area of a tin of beans is:

$S_b = 2\pi r^2 + 2\pi rh$

$= 32\pi + 64\pi$

$= 96\pi\,\text{cm}^2$

The surface area of a tin of soup is:

$S_t = 2\pi r^2 + 2\pi rh$

$= 18\pi + 60\,\pi$

$= 78\pi\,\text{cm}^2$

Since the recycle value is proportional to the surface area we need to solve the following proportion:

$$\frac{78\pi}{0.02} = \frac{96\pi}{x}$$

Solving for x:

$$x = \frac{96\pi}{78\pi} \times 0.02 \approx 0.0246$$

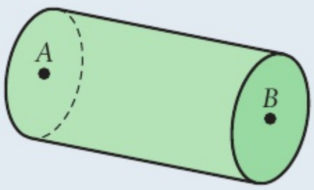
The company will pay £0.0246 for each tin of beans.

This is a sensible answer. The recycle value is higher for a tin of beans than a tin of soup. A tin of beans has a larger surface area than a tin of soup.

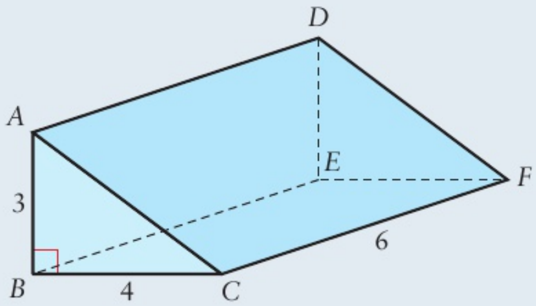
Practice questions 14.3

1 Calculate the surface area of each solid. Give your answer in exact form. All measurements are in centimetres.

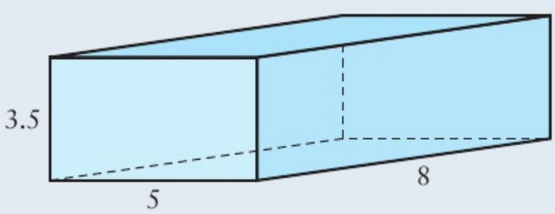
a A cylinder of radius 1 cm and height 4 cm.



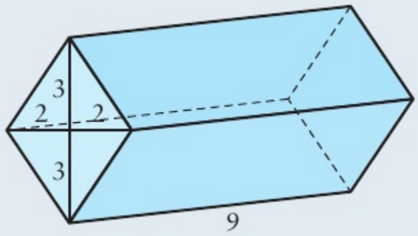
b A prism with right-angled triangle bases as shown.



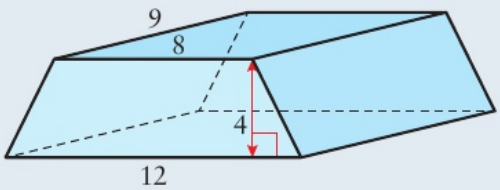
c A cuboid as shown.



d A prism with rhombus bases as shown.



e A prism with trapezium bases as shown.



2 The dimensions of a rectangular hall are 11 m × 9 m and height 3 m. The hall has a door of 2.30 m × 1.50 m and two windows of 1.50 m × 1.50 m. Calculate the cost of two coats of white paint of the walls and ceiling (excluding the windows and the door) at the rate of £7.50 per m^2.

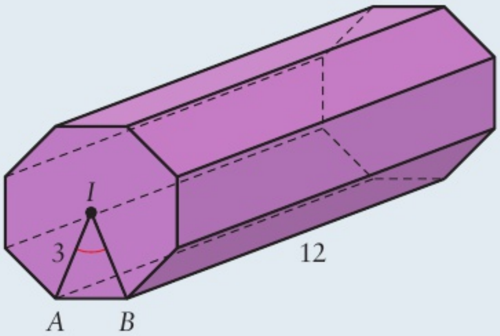
3 How much paper is needed for the label of a tin of tomatoes that has a circular base of radius 4 cm and a height of 9 cm? Give your answer rounded to one decimal place.

4 You earn £0.01 for recycling a tin of tomatoes with base of radius 3 cm and height 10 cm. A tin of beans has a circular base of radius 3 cm and height 12 cm. How much can you expect to earn for recycling a tin of beans? Assume that the recycle value is proportional to the surface area. Give your answer correct to three significant figures.

5 The lateral surface area of a cylinder is 200 cm^2 rounded to the nearest integer. The height is 9 cm. What is the surface area of the cylinder? Give your answer rounded to the nearest integer. Explain how you found your answer.

Challenge Q6

6 The prism below is an octagonal prism where the base is a regular octagon.



a Calculate the angle AIB.

b Work out the height of the isosceles triangle AIB.

c Work out the base of the isosceles triangle AIB.

d Work out the area of the regular octagon.

e What is the perimeter of the regular octagon?

f Work out the surface area of the octagonal prism.

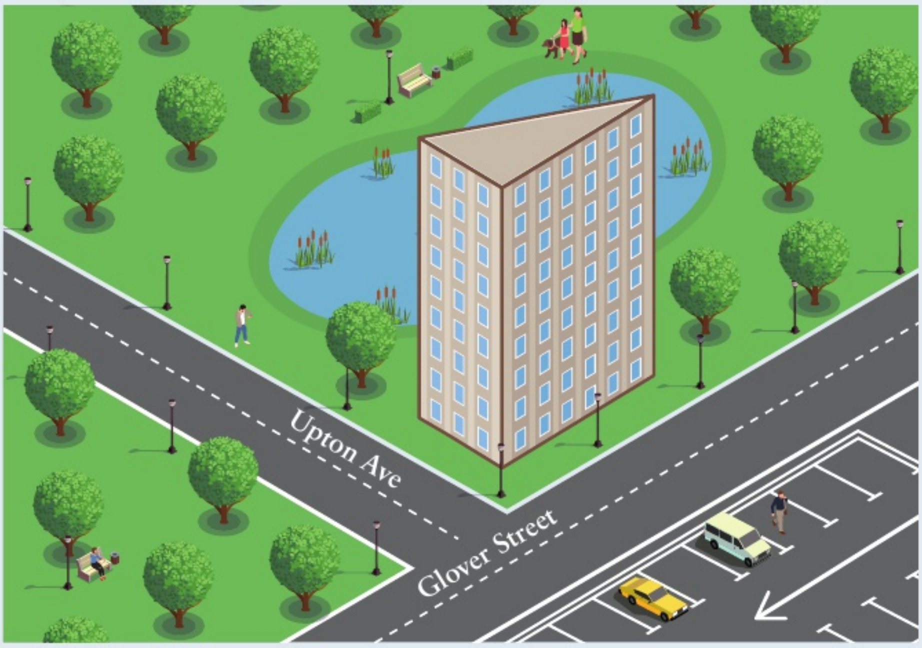
7 The cheese in the picture is modelled by a cylinder of radius 7 cm and height 4 cm. The cut wedge represents one-seventh of the cheese.

a Calculate the surface area of the cheese before it is cut.

Challenge Q7b

b Calculate the surface area of the remaining cheese after the wedge is removed. Did the surface area increase, decrease, or remain the same?

8 Look at the buliding in the image below.



a What type of prism is the building?

b The building is 131 feet high. The side of the building that faces Upton Avenue measures 87 feet. The side that faces Glover Street measures 173 feet, and the back side measures 190 feet. Calculate the lateral area.

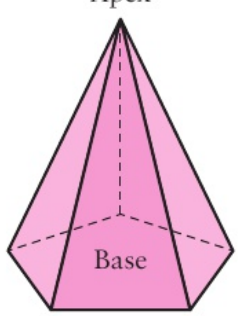
c There are nine storeys in the building. Assuming that all the storeys have the same height, how high is each storey?

d Assume that Upton Avenue and Glover Street are perfectly perpendicular. Estimate the surface area of the building.

14.4 Surface areas of pyramids and cones

14.4.1 Pyramids

A (geometric) **pyramid** is a polyhedron formed by connecting the vertices of a polygonal base with a point outside the base called an apex. The lateral faces of a pyramid are formed by triangles, where one of the vertices is the apex. The pyramids are named by the shape of the polygonal base. We will consider only **right pyramids**, where the line segment joining the apex to the centroid of the base (or the height of the pyramid) is perpendicular to the base of the pyramid.



Connections

The complex of the Egyptian Pyramids of Giza is one of the Seven Wonders of the Ancient World and the only ancient wonder still in existence. As the name indicates, these edifices have the shape of geometric pyramids. It is now thought that they were representative of the descending rays of the sun.

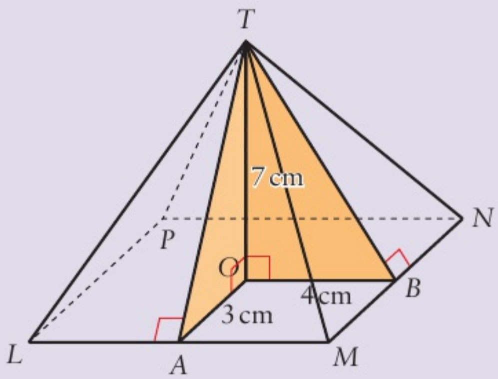
Fact

A **centroid** of a geometric figure is its geometric centre. Physically, if an object's density is homogeneous, then the centroid is its centre of gravity. In a triangle, the centroid is where the medians meet. In a parallelogram it is where the diagonals meet.

As with prisms and cylinders, the surface area of a pyramid is the total area given by the sum of the areas of the lateral faces and the area of the base.

Explore 14.6

Look at the rectangular pyramid.



Can you describe the lateral faces of the pyramid? Be as specific as possible.

Do you have enough information to calculate the area of the lateral faces?

Work out the slant heights of the pyramid.

Can you work out the areas of the faces?

Can you work out the surface area of the pyramid?

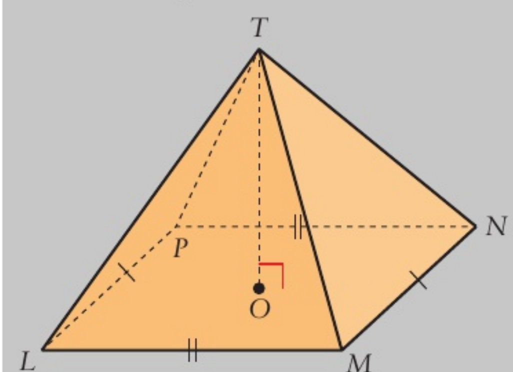
Hint

The **slant height** of a lateral face (triangle) is the perpendicular height going from the apex to the midpoint of the base of the triangle.

Worked example 14.5

In the pyramid, $LM = 8$ cm, $MN = 6$ cm and $TO = 7$ cm

Calculate the surface area of the pyramid. Give your answer rounded to one decimal place.



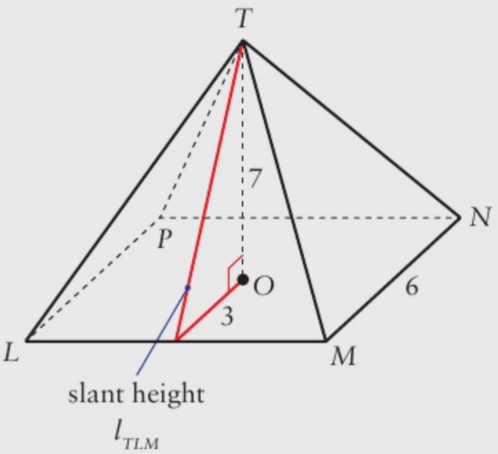
Solution

The base of the pyramid is a rectangle so it is a rectangular pyramid. Its surface area is the sum of the area of the base rectangle and the areas of the lateral faces. The lateral faces are not all congruent, but opposite faces are congruent since opposite sides of a rectangle are congruent.

We need to find the area of the base rectangle and the areas of the lateral faces. To calculate the area of the lateral faces, we need to find their slant heights. The slant height is the hypotenuse of the right-angled triangle formed by the height of the pyramid and the line segment joining O (the centroid of the rectangle) to the midpoint of the side of the base.

The length of the line segment joining O to the midpoint of the side LM is half the length half of the side MN. So, it is 3 cm.

The slant height l_{TLM} of the triangular face TLM is the hypotenuse of the triangle with sides of length 7 cm and 3 cm.



Therefore,

$l_{TLM} = \sqrt{7^2 + 3^2} = \sqrt{49 + 9} = \sqrt{58}$

The area of the triangle TLM is

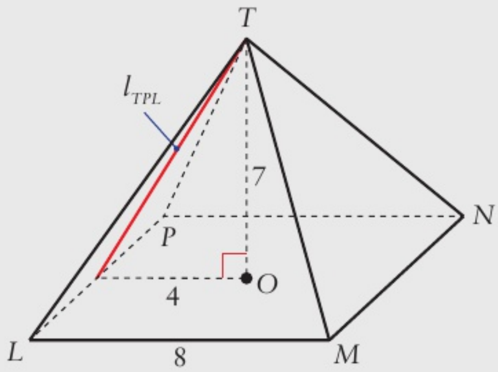
$A_{TLM} = \left(\frac{1}{2}\right)(8)\sqrt{58} = 4\sqrt{58}$

Triangle TPN is congruent to triangle TLM. Therefore,

$A_{TPN} = 4\sqrt{58}$

The length of the line segment joining O to the midpoint of the side PL is half the length of the side LM. So it is 4 cm.

The slant height l_{TPL} of the triangular face TPL is the hypotenuse of the triangle with sides of length 7 cm and 4 cm.



Therefore,

$$l_{TPL} = \sqrt{7^2 + 4^2} = \sqrt{49 + 16} = \sqrt{65}$$

The area of the triangle TPL is

$$A_{TPL} = \left(\frac{1}{2}\right)(6)\sqrt{65} = 3\sqrt{65}$$

Triangle TMN is congruent to triangle TPL. Therefore,

$$A_{TMN} = 3\sqrt{65}$$

The area of the base is the area of the rectangle

$$B = 8 \times 6 = 48\,\text{cm}^2$$

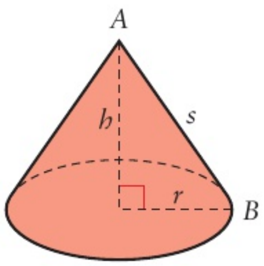
The surface area is

$$\begin{aligned} S &= B + 2A_{TLM} + 2A_{TPL} \\ &= 48 + 2\left(4\sqrt{58}\right) + 2\left(3\sqrt{65}\right) \\ &= 157.3\ \text{cm}^2\ (1\ \text{d.p.}) \end{aligned}$$

The total surface area is 157.3 cm². For accuracy, round only at the end of the exercise. For the intermediate calculations we either use the exact value or we store the value of the square roots and use the stored value in further calculations.

14.4.2 Cones

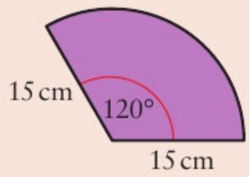
A **cone** is a solid formed by a circular base and a point outside the base called the apex or vertex. A cone is not a polyhedron, since its lateral face is curved.



Like the pyramid, the cone has a slant height, s. The slant height, the height, h, and the radius, r, form a right-angled triangle. The total surface area of a cone is the sum of the area of the circular base and the area of its curved lateral surface.

Investigation 14.2

1 On a sheet of paper draw a sector of radius 15 cm and angle 120°.



2 Cut out the sector and join the two straight edges, securing them with tape.

3 You should have made a cone. Complete the following sentences:

a The centre of the sector corresponds to the _______ of the cone.

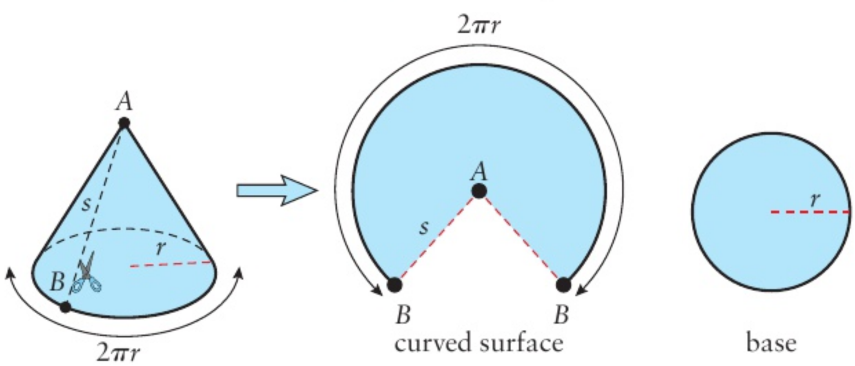
b The slant height of the cone corresponds to the _________of the sector.

c The circumference of the base of the cone is equal to the__________ of the sector.

To develop our understanding of the surface area of a cone, we will reverse the steps from Investigation 14.2.

Consider a cone of slant height s and base radius r.

We can cut along a straight line joining the vertex to a point on the base, which is the slant height. This produces a net of the curved surface. The net of the curved surface is a sector of a circle, radius s.



To calculate the area of a sector, we must find what fraction it is of a complete circle. Normally this is done by looking at the sector angle and comparing it to 360°, but we can also do this by comparing the length of the sector's arc, $2\pi r$, to the circumference of the whole circle, $2\pi s$. For example, if the sector's arc length is half the circumference of the circle, then the sector's area is half the area of the circle.

Therefore,

$$\text{area of sector} = \frac{\text{length of sector's arc}}{\text{circumference of circle}} \times \text{area of circle}$$

$$= \frac{2\pi r}{2\pi s} \times \pi s^2 = \pi rs$$

area of the sector = area of the curved surface

So, curved surface area = πrs

Therefore, the total surface area, S, of a cone of radius r and slant height s is:

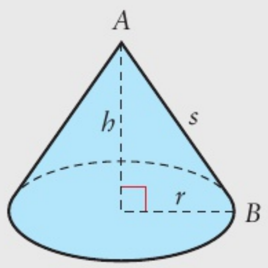
$S = \pi rs + \pi r^2$

Worked example 14.6

A cone has height 8 cm and base radius 6 cm. Calculate the surface area of the cone in terms of π.

Solution

We start by making a sketch of the cone. The slant height, the base radius and the height of the cone form a right-angled triangle.



The formula for the surface area is in terms of the slant height and the base radius, $S = \pi rs + \pi r^2$. We have the height, h, and the base radius, r, and we need to find the slant height, s.

The slant height is the hypotenuse of the right-angled triangle shown in the diagram. We can use Pythagoras' theorem and then the formula for the surface area.

$s = \sqrt{6^2 + 8^2} = \sqrt{100} = 10\text{ cm}$

$S = \pi rs + \pi r^2 = 60\pi + 36\pi = 96\pi\text{ cm}^2$

The total surface area is $96\pi\text{ cm}^2$

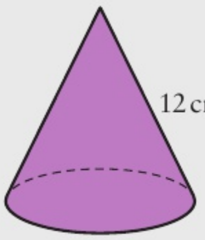
Worked example 14.7

Challenge

A cone has a surface area of $13\pi\,\text{cm}^2$ and slant height of 12 cm. Find the radius of the circular base.

Solution

We start by making a sketch of the cone.



The formula for the surface area of a cone is in terms of the slant height s and the radius of the base r. The surface area and the slant height are given. We can form and solve an equation for r using the formula for the surface area.

We substitute the values for S and s to get the following equation, and then solve for r.

$13\pi = 12\pi r + \pi r^2$

$\pi r^2 + 12\pi r - 13\pi = 0$

$r^2 + 12r - 13 = 0$

This quadratic equation has two solutions: $r = -13$ or $r = 1$

But since the radius must be positive, the solution is $r = 1$ cm

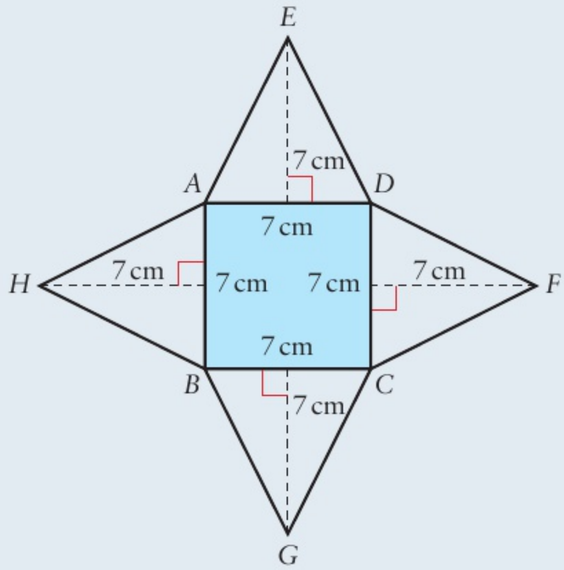
We can check our solution by calculating the surface area of a cone with radius 1 cm and slant height 12 cm.

Surface area $= \pi rs + \pi r^2 = 12\pi + \pi = 13\pi\,\text{cm}^2$, which agrees with the given area.

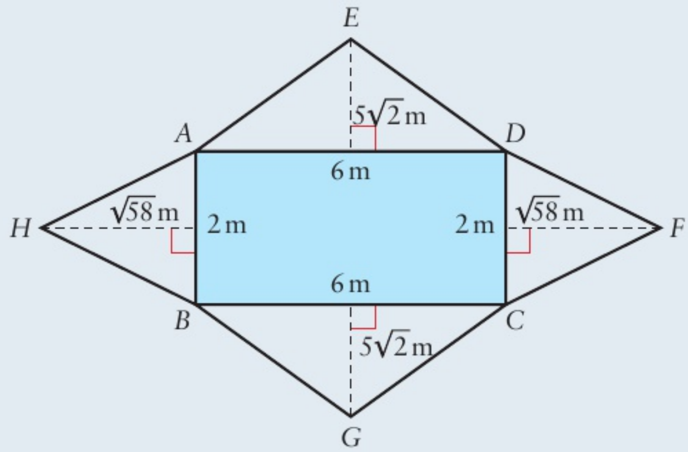
Practice questions 14.4

1 Work out the surface area of each pyramid given its net.

a

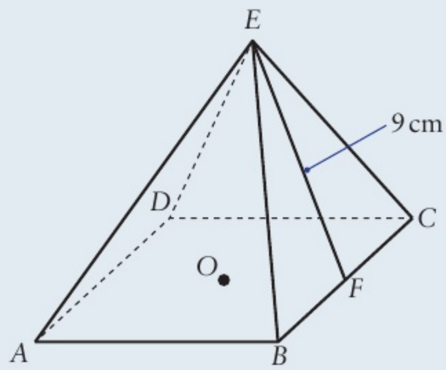


b

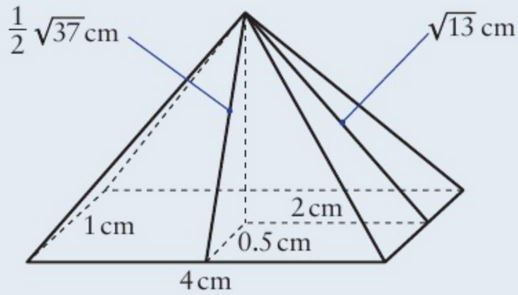


2 Calculate the surface area of each pyramid. Give your answers in exact form.

a Square pyramid

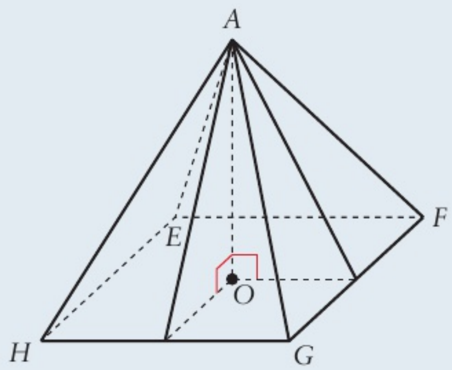


b Rectangular pyramid

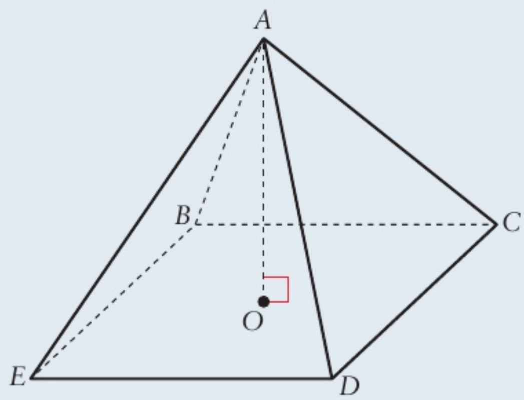


3 Use Pythagoras' theorem to calculate the slant height of each face. Then calculate the surface area of each pyramid. Give your answers rounded to one decimal place where necessary.

a *EFGH* is a rectangle. $HG = 4$ cm, $GF = 3$ cm, $AO = 4$ cm



b *BCDE* is a rectangle. $ED = 9$ cm, $DC = 10$ cm, $AO = 8$ cm



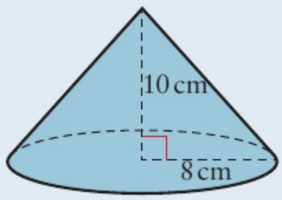
4 The Louvre Pyramid has a vertical height of 21.60 m and a square base of sides 35 m. Calculate the glass lateral area of the Louvre Pyramid.

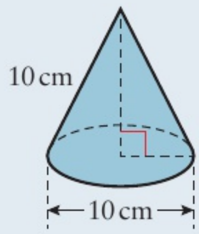
5 The Great Pyramid of Giza was a royal tomb of queens and pharaohs in ancient Egypt. It is thought that the architects wanted to relate the measures of the pyramid to the golden ratio (rounded to 1.618). The base of the pyramid is a square of sides 230.4 m and the vertical height of the pyramid is 146.5 m.

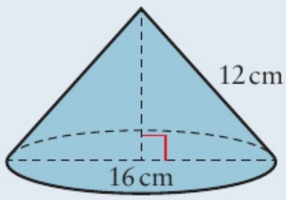
a Calculate the ratio of the base side to the vertical height. How close is this ratio to the golden ratio?

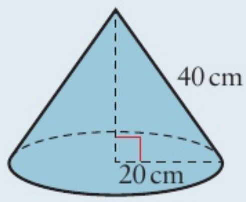
b Archaeologists say that the pyramids were originally covered in highly polished limestone. Calculate the area of the pyramid that is covered with limestone.

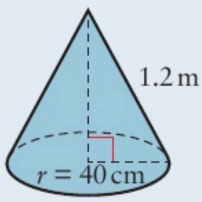
6 Find the surface area of the following cones. Give your answers in terms of π.

a


b


c


d


e


7 A cone has base radius 8 cm and height 15 cm.

a Find the slant height.

b Find the curved lateral area in terms of π.

c Find the surface area in terms of π.

8 A cone has base radius $\sqrt{5}$ cm and height 12.5 cm.

a Find the slant height.

b Find the curved lateral area in terms of π.

c Find the surface area in terms of π.

9 A cone has base diameter 14 cm and slant height 25 cm. Find the surface area in terms of π.

10 Find the surface area of cones with the following dimensions. Give your answers rounded to two decimal places.

a Radius 14 cm and height 18 cm

b Diameter 35 cm and slant height 26 cm

11 Christmas trees can be modelled by cones. The 2020 Rockefeller Center Christmas Tree was 75 feet tall and 45 feet wide (the diameter of the base). Find the surface that is available for Christmas decorations.

12 Michael wants to sew a toy circus tent for his son Joel. The tent is made of two parts, a cylinder and a cone, as shown in the diagram. The diameter of the base is 1.60 m, the height of the cylinder is 1 m and the vertical height of the cone is 50 cm. Michael intends to buy 20% more fabric than is needed to form the tent because he knows that some will be lost in the sewing process. How many square metres of fabric will Michael buy?

13 A cone has a surface area of $69\ \text{cm}^2$ and a radius of 3 cm. Find the slant height. Give your answer rounded to one decimal place.

14 The curved lateral area of a cone, when unfolded, is a sector of radius 10 cm and angle 180°.

Challenge Q14

a Find the base radius of the cone.

b Find the surface area of the cone.

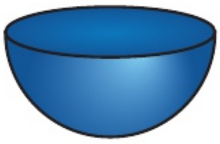
15 A cone has a surface area of $35\pi\ \text{cm}^2$ and slant height of 2 cm. Find the radius of the circular base.

Challenge Q15

14.5 Surface area of a sphere

A **sphere** is a solid in which every point on the surface is an equal distance from the centre of the solid.

A **hemisphere** is half a sphere.



So far, we have derived the formula for the surface areas of solids from their nets. This is not possible for a sphere. In Explore 14.7 you will look at the surface of a sphere in a way that lets you estimate a formula for the surface area.

Connections

It is not possible to map a portion of a sphere or an entire sphere onto a plane without introducing some distortion. So how can we create maps for navigation? There are special projections, such as the Mercator projection or the stereographic projection, that allow us to draw maps of the Earth on a flat page. Some of them preserve the area; others do not. There is a quantitative method to measure the amount of distortion and how it changes from place to place on the sphere.

Look through an atlas and find the name of the projection used for each chart. The variety of projections used might surprise you.

Explore 14.7

Imagine that you remove the covering from a baseball. You will end up with two pieces of material as shown below.

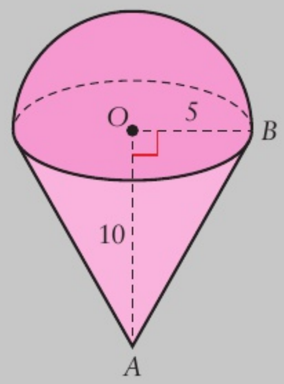
If r is the radius of the baseball, can you draw any conclusions about the surface area of the baseball?

For a sphere of radius r, the surface area is given by the formula:

$S = 4\pi r^2$

Worked example 14.8

Work out the surface area of this solid.
Give your answer correct to three significant figures. All measurements are in centimetres.
O is the centre of the base circle of the hemisphere.



Solution

The solid can be considered as two solids: a hemisphere and a cone. Measurements are given in the diagram.

We work out the lateral surface area for each and then add them up. We do not need to include the base of either solid, as these do not form part of the surface area of the combined solid.

The lateral surface area of the hemisphere is half of that of a sphere.

area $= 2\pi r^2 = 2\pi \times 5^2 = 50\pi \text{ cm}^2$

For the cone, we need to find the slant height first. The slant height, AB, of the cone is the hypotenuse of right-angled triangle AOB.

Using Pythagoras' theorem, $AB = \sqrt{10^2 + 5^2} = \sqrt{125} = 5\sqrt{5}$

lateral area of cone $= \pi rs = \pi \times 5 \times 5\sqrt{5} = 25\sqrt{5}\,\pi$

So the required surface area $= 50\pi + 25\sqrt{5}\,\pi = 25\pi(2 + \sqrt{5}) = 333 \text{ cm}^2$ (3 s.f.)

Practice questions 14.5

1 Find the surface area of a sphere with:

a radius 5 cm b diameter 12 cm.

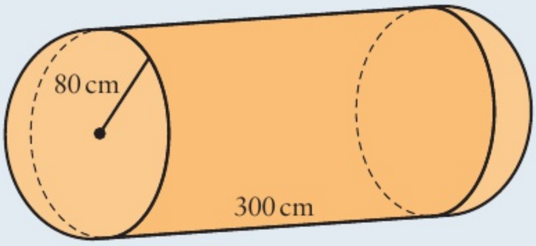
Give your answers in terms of π.

2 Find the surface area of a hemisphere with:

a radius 10 cm b diameter 10 cm.

Give your answers rounded to three significant figures.

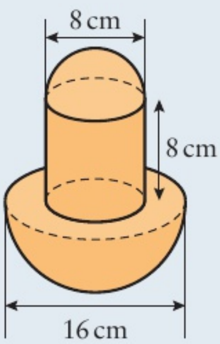
3 The solid in the diagram is made of a cylinder 300 cm long, with a radius of 80 cm. A hemispherical cap is placed on each end of the cylinder. Calculate the surface area of the solid. Give your answer correct to four significant figures.



4 A sphere has a surface area of 200 cm². Find its radius. Give your answer correct to one decimal place.

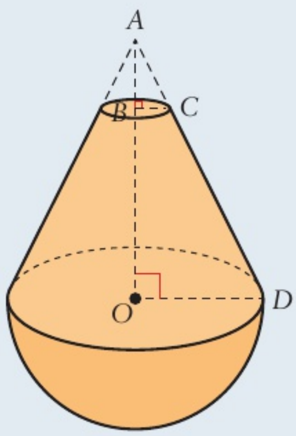
5 Work out the surface area of each compound solid. Give your answer correct to three significant figures.

a



Challenge Q5b

b $BC = 0.6$ m, $AB = 0.8$ m, $OD = 3.0$ m, $AO = 4.0$ m



6 The Earth is an almost perfect sphere flattened at the poles. The radius of the Earth is roughly 6371 km. Estimate the surface area of the Earth.

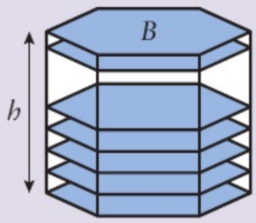
7 Gilda buys an ice cream. It is formed of a cone with slant height 12 cm and base radius 5 cm with a ball of ice cream on top, whose visible part is a hemisphere of radius 5 cm. Calculate the surface area of the ice cream.

14.6 Volumes of prisms and cylinders

The volume of a solid is the amount of space, measured in cubic units, needed to fill the solid.

Explore 14.8

The prism, with base B, has been sliced into layers of 1 unit height.



All cross-sections are congruent to the base.

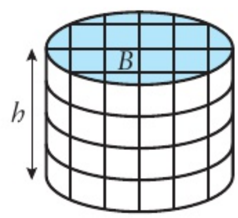
Can you work out what the volume of one layer is?

If you have h layers, can you find the volume of the prism?

Can you generalise your conclusion to any prism with base B and height h?

As we saw in Explore 14.8, the volume of a prism can be calculated by the formula:

volume = area of base × height



If we think of a cylinder as a circular prism, then the volume of a cylinder is:

$V = \pi r^2 h$

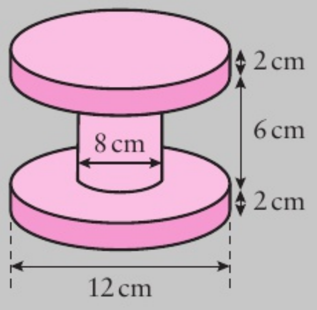
Reflect

Can you verify the formula for the volume of a cylinder? Research using available resources to confirm your discoveries.

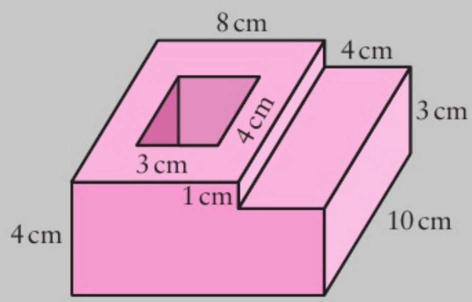
Worked example 14.9

Find the volumes of each compound solid.

a



b



Note that the hole goes all the way through the solid.

Solution

a The solid consists of three cylinders. The top and bottom cylinders are identical.

We apply the formulae for the volume of a cylinder to each one and add the results:

volume of the top/bottom cylinder $= \pi r^2 h = \pi \times 6^2 \times 2 = 72\pi$

volume of the middle cylinder $= \pi r^2 h = \pi \times 4^2 \times 6 = 96\pi$

So the volume of the whole solid $= 2 \times 72\pi + 96\pi = 240\pi\,\text{cm}^3$

b The solid consists of a rectangular prism which has had two rectangular prisms removed. The removal of one of these prisms has formed a hole.

volume of rectangular hole $= Bh = (3 \times 4) \times 4 = 48\,\text{cm}^3$

volume of rectangular prism removed from side

$= Bh = (4 \times 10) \times 1 = 40\,\text{cm}^3$

volume of the original prism $= Bh = (12 \times 10) \times 4 = 480\,\text{cm}^3$

So the volume of the solid $= 480 - 48 - 40 = 392\,\text{cm}^3$

Worked example 14.10

Lake Superior, between the USA and Canada, is the largest of the Great Lakes by volume. It has a surface area of $82\,100\text{ km}^2$ and an average depth of 147 m.

a Assuming that the lake basin is a 3D solid with uniform cross-section, estimate the volume of water of the lake in km^3.

b In 2020 the actual volume of water was $12\,000\text{ km}^3$. How close is your estimate to the actual volume?

Solution

a We need to find the volume of Lake Superior in km^3. We can model the lake as a prism. The depth of the lake is the height of the prism and the area of the uniform cross-section is the area of the base of the prism. We know that the volume of a prism is the area of the cross-section multiplied by the height.

We need to convert both measurements to the same units, then use the formula for the volume of a prism and substitute the area of the base and the height.

$147\text{ m} = 0.147\text{ km}$

The volume of the lake is

$V = Bh = 82\,100 \times 0.147 = 12\,068.7\text{ km}^3$

b The depth of the lake is not realistically uniform. We have used the average depth in our calculations. Thus some error can be expected.

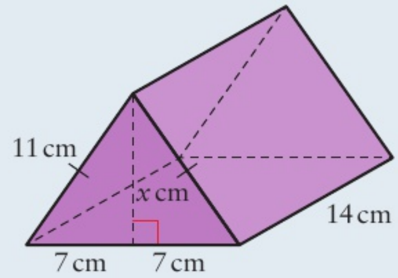
$12\,068.7\text{ km}^3$ differs from the given volume by 68.7 km^3. The difference is small in comparison to the size of the lake, so our estimate is good.

Practice questions 14.6

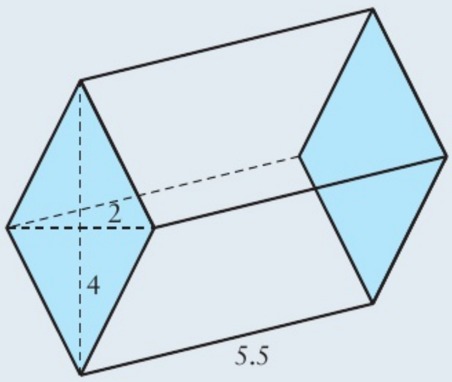
1 The base parallelogram of a prism has height 6 cm and base 11 cm. The height of the prism is 20 cm. Sketch the prism and then calculate its volume.

2 The base of a triangular prism is a right-angled triangle with two short sides of 6 cm and 9 cm. The height of the prism is 4 cm. Sketch the prism and then calculate its volume.

3 Calculate the volume of the prism.
Give your answer in exact form.



4 The base of the prism is a rhombus with major diagonal 4 cm and minor diagonal 2 cm. The height of the prism is 5.5 cm. Find the volume.



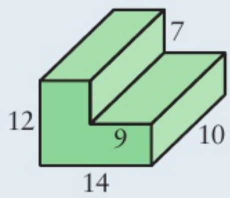
Reminder

The volume of compound solids can be found by decomposing the solids into simpler ones and then adding or subtracting volumes.

Challenge Q6

5 Calculate the volume of each solid. All measurements are given in cm.

a



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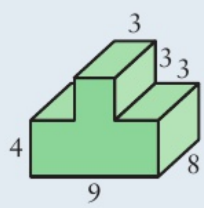

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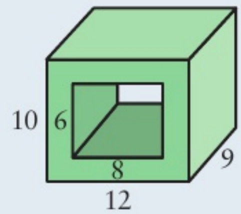
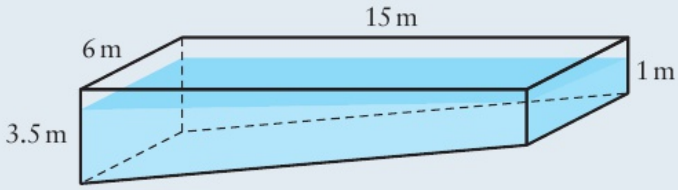
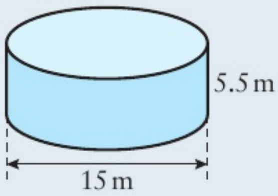


6 The diagram shows a swimming pool. The top of the pool is a rectangle measuring 15 m by 6 m. The depth of the pool changes, as shown in the diagram. The water is 3.5 m deep at the deep end and 1 m at the shallow end. Find the volume of water in the swimming pool.

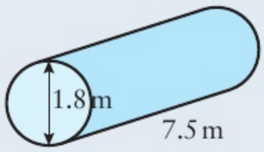


7 Calculate the volume of each cylinder. Give your answer rounded to one decimal place where necessary.

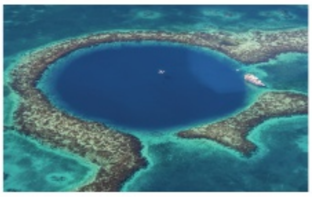
a



b



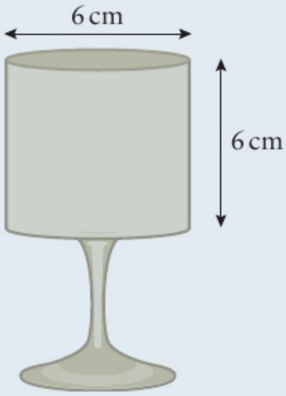
8 The Great Blue Hole is a giant sinkhole off the coast of Belize with a uniform cross-section that is an almost perfect circle of diameter 300 m. It is a World Heritage site. Its average depth is 143 m. How much water is contained in the Great Blue Hole?



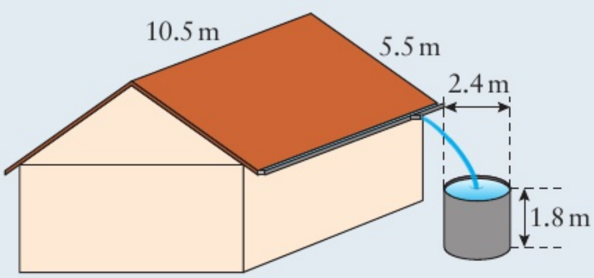
Challenge Q9

9 A tape factory needs to roll a length of double-sided tape 5.4 m long and 1.9 cm wide around a ring of radius 3 cm. The tape has a thickness of 0.5 cm. Find the volume of the tape. Give your answer correct to the nearest integer.

10 Tony and Dahlia are going to be married and would like to offer a lunch to their guests at a fancy restaurant. There will be 98 guests. The restaurant has proposed to set the table with glasses as shown in the figure. The bowl of each glass is a cylinder of diameter 6 cm and height 6 cm. Supposing that each guest will drink at least five glasses of juice during the lunch, what is the minimum number of litres of juice that will be consumed during the lunch?



11 When rainwater falls on the roof of a garage it is drained by the gutter into a tank. Each side of the roof is drained into a separate, identical tank.



Hint Q11a

A rainfall of 100 mm means the rain would cover the area on which it fell to a depth of 100 mm.

a If 100 mm of rain falls, find how many litres of water are drained into each tank.

b By how much would the water level in the tank rise when the water from part a was added?

c How many millimetres of rain would need to fall on the roof to fill the tank?

12 Bales of hay come in two shapes as shown on the left: round and square. The bales come in different sizes. If a round bale has a diameter of 1.2 m and height of 1.5 m, and a square bale has dimensions of 0.6 m by 0.6 m by 1.2 m, how many square bales contain the same amount of hay as one round bale?

14.7 Volumes of cones, pyramids and spheres

Explore 14.9

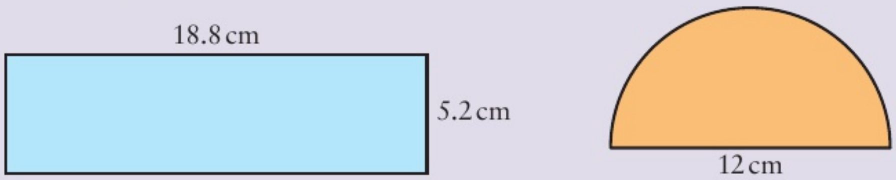
Thinking skills

Social skills

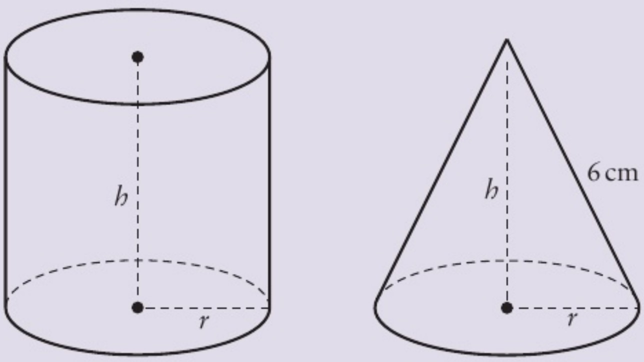
Communication skills

To complete this activity, you will need sticky tape, two pieces of heavy paper (at least 19 cm by 6 cm), and some sand, salt, sugar or lentils.

1 Cut out a rectangle of dimensions 18.8 cm and 5.2 cm and a semicircle of diameter 12 cm as shown in the diagram.



2 Tape the short sides of the rectangle and the edges of the semicircle, without overlapping, to make a cylinder and a cone.



3 Compare the heights of the cone and the cylinder. Now compare the base radius of each shape. What do you notice?

4 Place the cylinder on a plate. Fill the cone with sand (or other small particles) and then pour the sand from the cone into the cylinder. Repeat the process until the cylinder is full.

5 What can you deduce about the relationship between the volume of a cone and the volume of a cylinder that have the same height and base radius? Explain.

Challenge Q3

Reflect

Use a similar method to that used in Explore 14.9 to compare the volume of a pyramid with the volume of a prism with the same base and height. Research using available resources to confirm the formulas you established.

The volume of a cone of radius r and height h is given by:

$$V = \frac{1}{3}\pi r^2 h$$

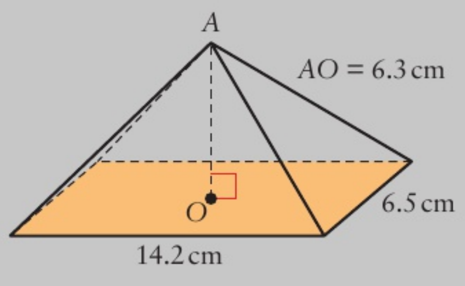
The volume of a pyramid of base area A and height h is given by:

$$V = \frac{1}{3}Ah$$

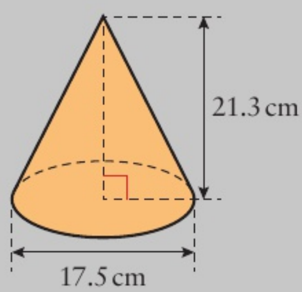
Worked example 14.11

Calculate the volume of each solid. Give your answers to two decimal places.

a A rectangular pyramid



b A cone



Solution

a The base of the pyramid is a rectangle of 14.2 cm by 6.5 cm. Its height is 6.3 cm.

First, we calculate the area of the base. Then we use the formula for the volume of a pyramid to calculate its volume.

The area of the base rectangle is $B = 6.5 \times 4.2 = 27.3\,\text{cm}^2$

The volume of the pyramid is $V = \frac{1}{3}Bh = \frac{1}{3} \times 27.3 \times 6.3 = 57.33\,\text{cm}^3$

b The cone has a base diameter of 17.5 cm and a height of 21.3 cm.

First we calculate the radius of the base. Then we use the formula for the volume of a cone to calculate its volume.

The radius is $r = 17.5 \times \frac{1}{2} = 8.7\,\text{cm}$

The volume of the cone is:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(8.75^2)(21.3) = 1707.75\,\text{cm}^3 \text{ (2 d.p.)}$$

Fact

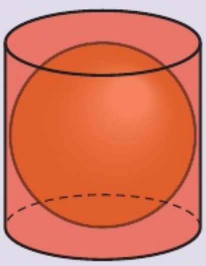
The standard measure of the diameter of a tennis ball is 6.86 cm. The standard measure of the diameter of a table tennis ball is 3.80 cm.

Explore 14.10

For this activity you will need a plastic tube of tennis or table tennis balls, a measuring jug and some water.

Consider the tube of tennis balls. We can assume that the tube has the same diameter as the tennis balls inside.

1 Cut the tube so that the height of the new cylinder is equal to the diameter of one of the balls. Such a cylinder is said to be **circumscribed** about the ball.



2 Can you find the volume of the new cylinder? Show how.

3 Keep the tennis ball inside the cylinder. Pour water into the cylinder until it is completely full.

4 Pour the water into the measuring jug.

5 Can you tell what fraction of the cylinder volume was filled with water?

6 What can you deduce about the volume of the ball?

Reflect

Research using available resources to confirm the formula you discovered for the volume of the sphere.

The volume of a sphere of radius r is: $V = \frac{4}{3}\pi r^3$

Worked example 14.12

The volume of Mars is $1.631 \times 10^{11}\,\text{km}^3$. Modelling Mars as a sphere, find its surface area.

Solution

We can model planets as spheres. The measure of Mars' volume is written in scientific notation. To find the surface area of Mars, we need to find the radius.

We can use the formula for the volume of a sphere to find the radius. Then we substitute the value for the radius in the formula of the surface area of a sphere.

volume $= V = \frac{4}{3}\pi r^3 = 1.631 \times 10^{11}$

Solving for r gives $r^3 = \frac{3}{4\pi} \times 1.631 \times 10^{11}$

So, $r = \sqrt[3]{\frac{3}{4\pi} \times 1.631 \times 10^{11}} \approx 3389.4\,\text{km}$

Now we substitute r into the formula for the surface area.

$S = 4\pi r^2 \approx 1.4 \times 10^8\,\text{km}^2$

We can check that the radius is correct by comparing it to the established radius in related literature. An established estimate is $r = 3389.5\,\text{km}$, which is very close to our calculation.

Reflect

Compare the formulas for the volume and surface area of a cone with the formulas for the volume and surface area of a cylinder.

Practice questions 14.7

1 Sketch each cone or pyramid and find its volume.

a A cone with base radius 4 cm and height 3 cm.

b A square pyramid with height 15 cm and base side 18 cm.

c A cone with slant height 13 cm and base radius 5 cm.

d A triangular pyramid where the base is a right-angled triangle with sides of 2 cm and 5 cm and the height of the pyramid is 10 cm.

2 Suppose that the slant height of a square-based pyramid forms an angle of 45° with the base. The sides of the square base measure 7 cm.

a What is the height of the pyramid?

b Find the volume of the pyramid.

3 The slant height of a cone forms an angle of 35° with the height of the cone. The base radius is 3 cm.

a Calculate the height of the cone. Give your answer rounded to two decimal places.

b Find the volume of the cone. Give your answer rounded to two decimal places.

4 A teepee can be modelled by a cone. Originally the teepee tents of Native Americans were 12 feet high and had a diameter of 10 feet. To build a tent, long wooden sticks were tied as shown in the photograph and opened in a circle at the bottom to form a wooden frame in the shape of a cone. Then the wooden frame was covered by buffalo hide.

a Calculate the minimum length of each wooden stick. Do you think the measure you got is accurate? Justify your answer.

b Calculate the volume of the cone-shaped part of the teepee.

5 a The Pyramid of the Sun in Mexico is one of the largest pyramids in Mesoamerica. The base of the pyramid is a rectangle of dimensions 220 m by 224 m. Its height is 65.6 m. Estimate its volume.

b The Great Pyramid is the largest pyramid in Giza, Egypt, and it is a square pyramid with base side 230 m and height 147 m. Find the ratio of its volume to the volume of the Pyramid of the Sun.

6 A white paper cup in the shape of a cone is designed to hold popcorn. The diameter of the cup is 12 cm and the height is 17 cm. What is the volume of popcorn the cup could hold?

7 The volume of a cone is $100\pi\,\text{cm}^3$. If the area of the base is $25\pi\,\text{cm}^2$, find the base radius, the slant height and the height of the cone.

8 A cube of side 10 cm has the same volume as a square-based pyramid with height 10 cm. Find the length of the sides of the base of the square-based pyramid.

9 Calculate the volume of a sphere with each of the given measurements. Give your answers correct to three significant figures.

a radius = 3.1 cm

b diameter = 4.6 cm

c radius = 500 m

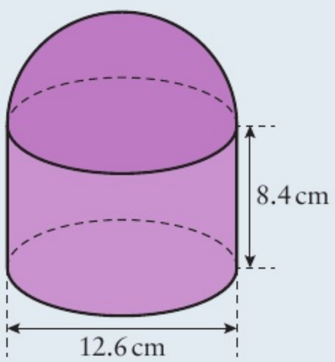
d diameter = $4\sqrt{2}$ cm

10 A sphere has diameter 12 cm. Find its volume. Give your answer in exact form.

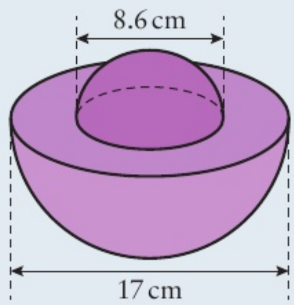
11 A hemisphere has radius 4.5 cm. Find its volume and surface area, including the base. Express your results in exact form.

12 Calculate the volume of each compound solid. Give your answers correct to three decimal places.

a



b



13 A sphere has volume $1200\pi\,\text{cm}^3$. Find its radius.

14 A gas tank has the shape of a sphere with diameter 15 m. How many m^3 of gas will fit in it?

15 Assume that the Earth is a sphere with volume $1.08\,321 \times 10^{12}\,\text{km}^3$.

a Find its radius.

b Find its surface area.

c 70% of the surface area of the Earth is covered by water. Find the measure of the surface area covered by water.

Research skills

16 Which planet in the solar system has the greatest volume?

17 The circumference of a soccer ball is 71 cm.

a Find its volume and surface area.

b Compare the volumes of a volleyball, a basketball and a soccer ball. Which ball's volume is the smallest?

Challenge Q18

Hint Q18

When a sphere is inscribed in a cube, it means that the sphere touches the inside of each face of the cube.

18 A sphere of radius 10 cm is inscribed in a cube. Calculate the length of a diagonal of the cube.

19 A soap bubble of 6 cm diameter is inflated until its diameter is 8 cm. What is the ratio between the new volume and the old volume?

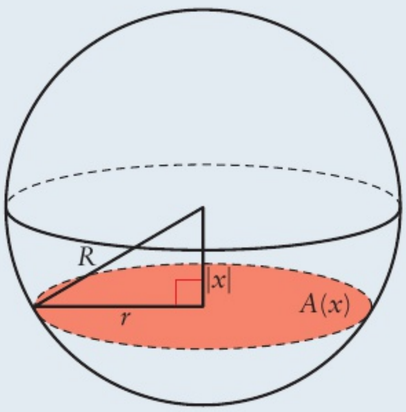
Challenge Q20

20 A plane cuts a sphere of radius $R = 12$ cm, as shown in the diagram.

a Find an expression for the radius, r, of the cut area $A(x)$ in terms of the distance $|x|$

b Find an expression for the cut area $A(x)$ in terms of the distance $|x|$

c Suppose that the cut area $A(x)$ is half of the area of a circle of radius R. What is the distance $|x|$ from the centre of the sphere to the centre of the cut area?

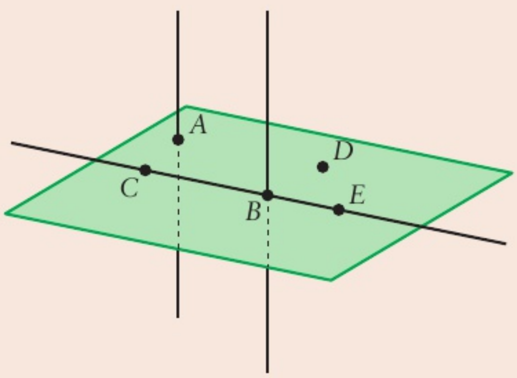


Self assessment

- I can identify two skew lines in 3D space.
- I can draw two skew lines in 3D space.
- I can explain whether two lines in 3D space are parallel.
- I can determine whether a set of points is coplanar.
- I can describe two parallel planes.
- I can sketch a line perpendicular to a plane.
- I can explain what the possible intersections of two planes are.
- I can define what a polyhedron is.
- I can identify edges, faces and vertices of a polyhedron.
- I can use Euler's formula to determine whether or not a solid is a polyhedron.
- I can use Euler's formula to find the number of faces, vertices or edges of a polyhedron.
- I can explain what the slant height of a pyramid or a cone is.
- I can explain the relationship between the slant height and the height of a pyramid or cone.
- I can find the lateral area and surface area of prisms, cylinders, pyramids and cones.
- I can unfold a cone and describe what shape it is.
- I can use the formula for the surface area of a sphere to find the radius.
- I can explain the relationship between the volume of a prism and the volume of a pyramid with the same base and height.
- I can explain the relationship between the volume of a cylinder and the volume of a cone with the same base and height.
- I can find the volume of a sphere.
- I can use the correct units of measure for surface areas and volumes.

Check your knowledge questions

1



a Identify two skew lines.

b Identify two parallel lines.

c List four coplanar points.

d List three collinear points.

e Identify two intersecting lines.

2 Sketch each situation.

a Lines l and g are intersecting.

b Plane P contains four coplanar points A, B, C and D.

c Line m is skew to line n.

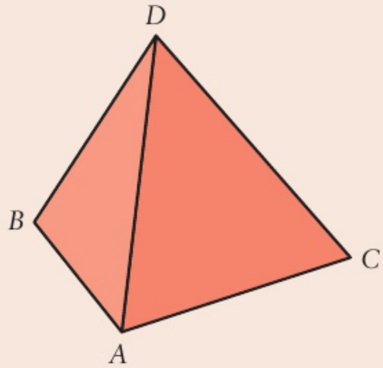
d Line f is parallel to line h.

e Points G, H, I and L are collinear.

f The intersection of planes Q and R is a line.

g Line s is perpendicular to planes N and O.

3



a Is this solid a polyhedron? Explain.

b List all the faces of the solid.

c List all the edges of the solid.

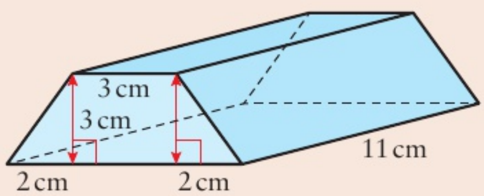
d List all the vertices of the solid.

e Verify that the solid satisfies Euler's formula.

4 A three-dimensional figure has 6 faces, 8 edges and 11 vertices. Is it a convex polyhedron?

5 Find the surface area and the volume of each three-dimensional shape. Give your answers rounded to the nearest whole number.

a



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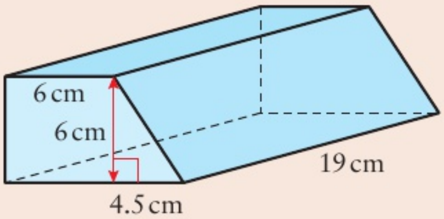

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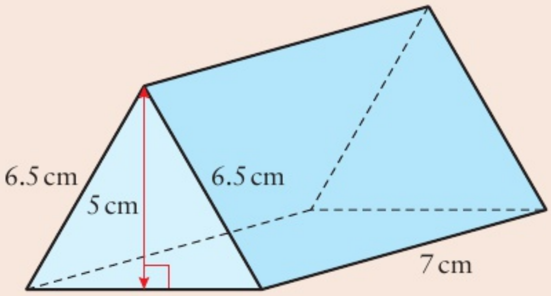
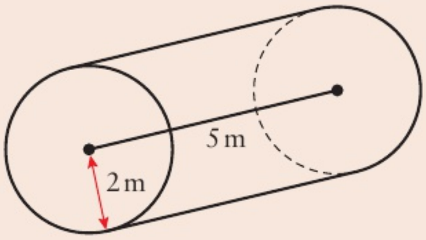
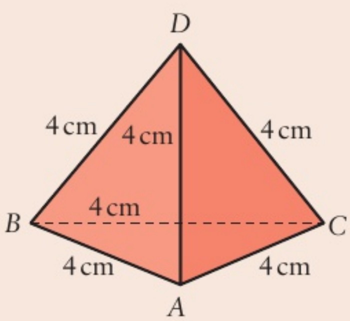


d



6 Find the surface area of the pyramid. Give your answer rounded to three decimal places.



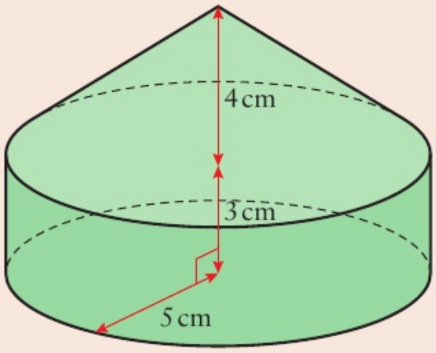
7 Find the volume and surface area of a square-based pyramid with height 11 cm and base side 4 cm. Give your answer rounded to one decimal place.

8 A cone-shaped silo contains grain. The radius is 5 m and the height is 7 m. If the silo can release grain at the rate of 12 m^3 per minute, how long would it take for the silo to empty fully? Round your answer to the nearest minute.

9 The top of a tower has the shape of a regular hexagonal pyramid. The base side has a length of 1.5 m and the pyramid height is 1.8 m. How many square metres of sheet metal is needed to cover the top of the tower if 20% extra sheet metal is needed for joints and overlaps?

10 Find the surface area and the volume of a sphere of radius 6 m. Give your answer in terms of π.

11 A cone is placed on top of a cylinder as shown. The height of the cone is 4 cm, the radius is 5 cm and the height of the cylinder is 3 cm. Find the volume and surface area of the composite solid. Round your answer to two decimal places.



15 Trigonometric equations and applications

15 Trigonometric equations and applications

KEY CONCEPT

Relationships

RELATED CONCEPTS

Approximation, Change, Generalisation

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Understanding the relationships between quantities and being able to generalise them can help us build better models and work with approximate measurements of non-standard shapes.

Factual

- How are the sine, cosine and tangent ratios of an obtuse angle defined?
- What information is needed to apply the formula for the area of a triangle?

Conceptual

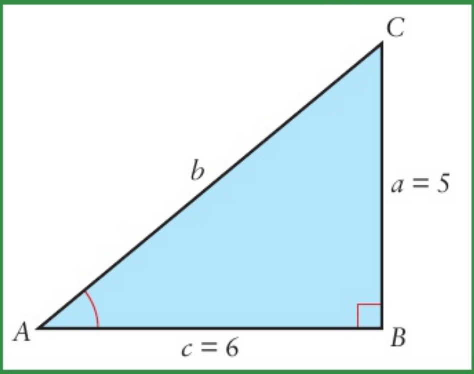
- How can the unit circle represent the trigonometric ratios of any angle?
- Can the sine of two different angles be the same?

Debatable

- How exact are the values of the sine and cosine of an angle calculated by a scientific calculator?
- How many different solutions does a trigonometric equation have?

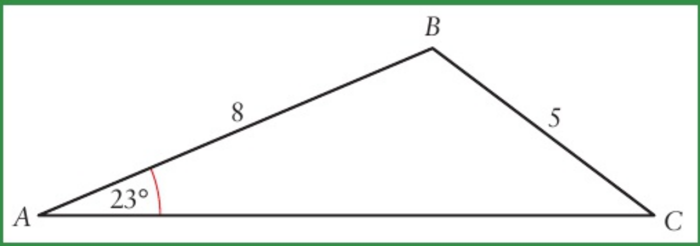
Do you recall?

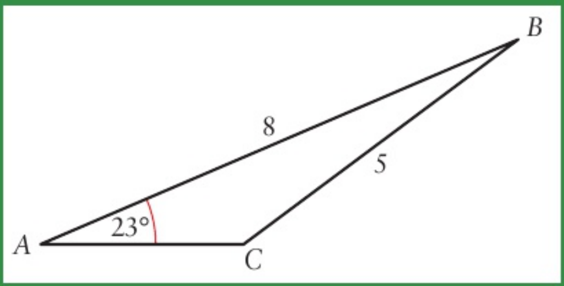
1 What is the value of $\sin A$, $\cos A$ and $\tan A$ in this triangle? What is the size of the angle at A?



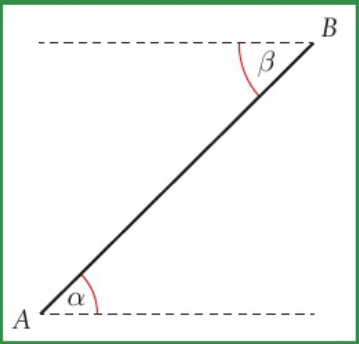
2 What is the equation of a circle of radius 1 and centre (0, 0)?

3 Are these two triangles congruent? State your reasoning.





4 Name the angle of elevation in the diagram. How are the angles of elevation and depression related?



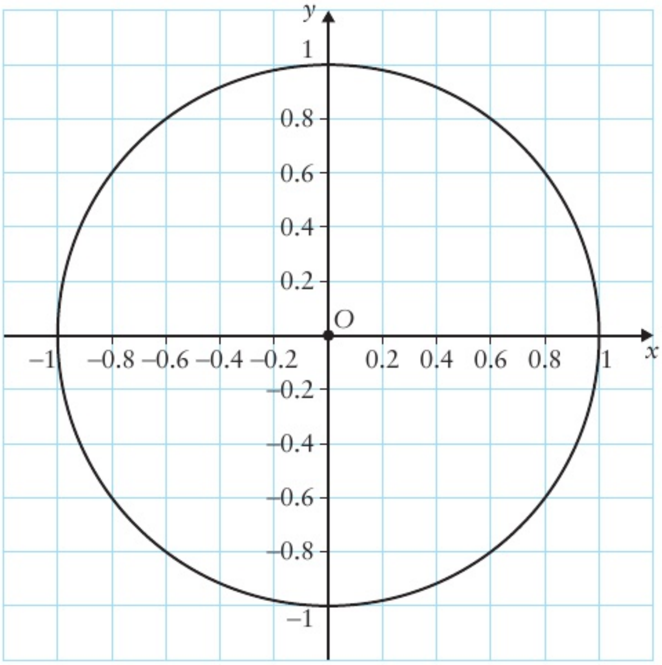
15.1 Trigonometric ratios on the unit circle

In Chapter 12, we saw the definition of the trigonometric ratios in right-angled triangles. In this chapter we will extend the ratios to any angle and we will explore the relationships between angles and sides in *any* triangle.

Consider a circle of radius 1 unit, with centre at the origin O of a coordinate plane. This circle is called the **unit circle.**

Reminder

The equation of the unit circle is $x^2 + y^2 = 1$

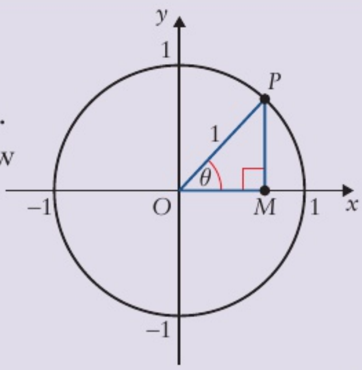


Thinking skills

Communication skills

Explore 15.1

Take a point $P(x, y)$ in the first quadrant of the unit circle. Mark the angle that the positive x-axis forms with the radius OP as θ, as shown. Mark the point M on the x-axis vertically below P. A right-angled triangle OMP is formed.



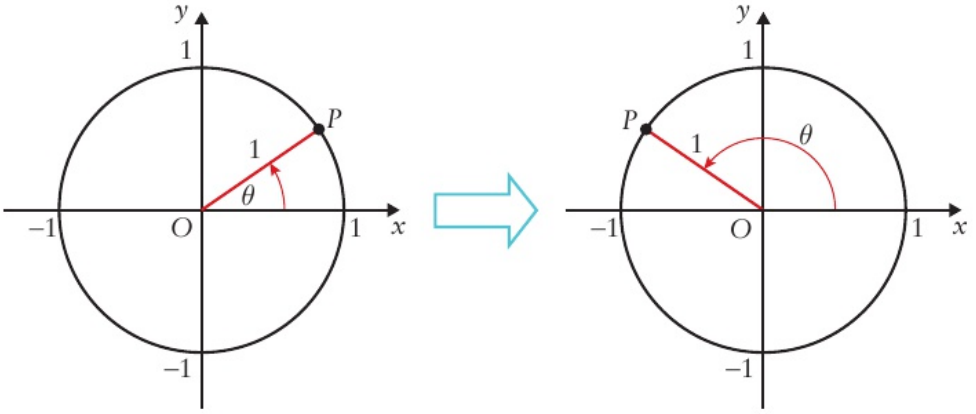
Copy and complete the table below and find the sine, cosine and tangent of θ in the right-angled triangle OMP.

Opposite side to θ	Adjacent side to θ	Hypotenuse	$\cos\theta$	$\sin\theta$	$\tan\theta$

Can you see why the unit circle is a useful tool?

The unit circle offers a new way of defining the trigonometric ratios for any angle, whether acute or non-acute.

In the diagram, starting from the positive x-axis, any angle θ on the unit circle corresponds to a point $P(x, y)$ on the circle, given by the intersection of the terminal side of θ and the unit circle. If we take a radius OP and rotate it anticlockwise about O, point P will still lie on the circle.



Fact

Positive angles θ are measured in the anticlockwise direction.

Looking at triangle OMP in the figure below, we can see the following:

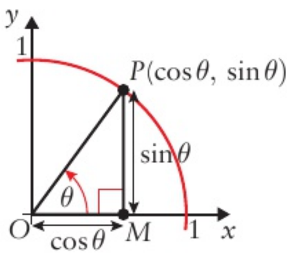
$$\cos\theta = \frac{OM}{OP} = \frac{x}{1} = x \qquad \sin\theta = \frac{MP}{OP} = \frac{y}{1} = y$$

Thus, we can say that P has coordinates $(x, y) = (\cos\theta, \sin\theta)$. This observation will result in a new definition of the trigonometric ratios.

For a point P on a unit circle:

$\cos\theta$ = the x-coordinate of P

$\sin\theta$ = the y-coordinate of P



Fact

Note that in triangle OMP $\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

In the right-angled triangle OMP we can use Pythagoras' theorem to establish a fundamental trigonometric identity.

$$OM^2 + MP^2 = OP^2 \Rightarrow \cos^2\theta + \sin^2\theta = 1$$

Fact

We write $(\cos\theta)^2$ as $\cos^2\theta$ and $(\sin\theta)^2$ as $\sin^2\theta$

Reflect

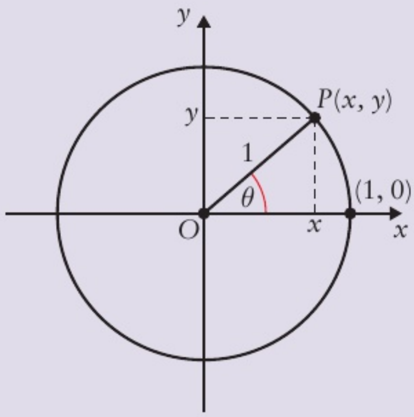
1 Can an angle measure more than 360°? Can an angle measure less than 0°? What about the corresponding point $P(x, y)$ in these cases?

2 Explain why the equation $\cos^2\theta + \sin^2\theta = 1$ is true when θ is of any size.

Explore 15.2

In which quadrants are $\sin\theta$, $\cos\theta$ and $\tan\theta$ negative?

Refer to the diagram to help you complete the table.



θ	Quadrant	$\cos\theta = x$	$\sin\theta = y$	$\tan\theta = \frac{y}{x}$
$0 < \theta < 90°$	1st quadrant	**positive**		**positive**
$90° < \theta < 180°$	2nd quadrant			
$180° < \theta < 270°$	3rd quadrant		**negative**	
$270° < \theta < 360°$	4th quadrant			

Worked example 15.1

Without using a calculator, write down the values of:

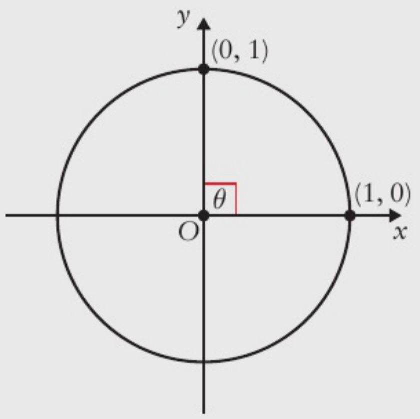
a $\cos 90°$ **b** $\sin 180°$.

Solution

If a question asks us to 'write down' an answer, it usually means that little or no calculation is needed.

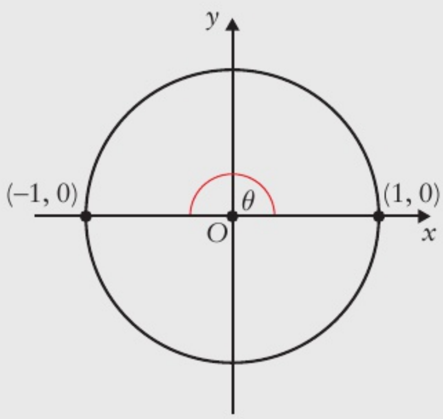
We can draw the required angle on a unit circle and identify the corresponding point $P(x, y)$ on the unit circle. Then we can apply the definition of the trigonometric functions.

a



The cosine of an angle θ is the x-coordinate of the corresponding point $P(\cos\theta, \sin\theta)$ on the unit circle. Since the x-coordinate is 0, $\cos 90° = 0$

b



The sine of an angle θ is the y-coordinate of the corresponding point $P(\cos\theta, \sin\theta)$ on the unit circle. Since the y-coordinate is 0, $\sin 180° = 0$

We can use a calculator to verify the values we found.

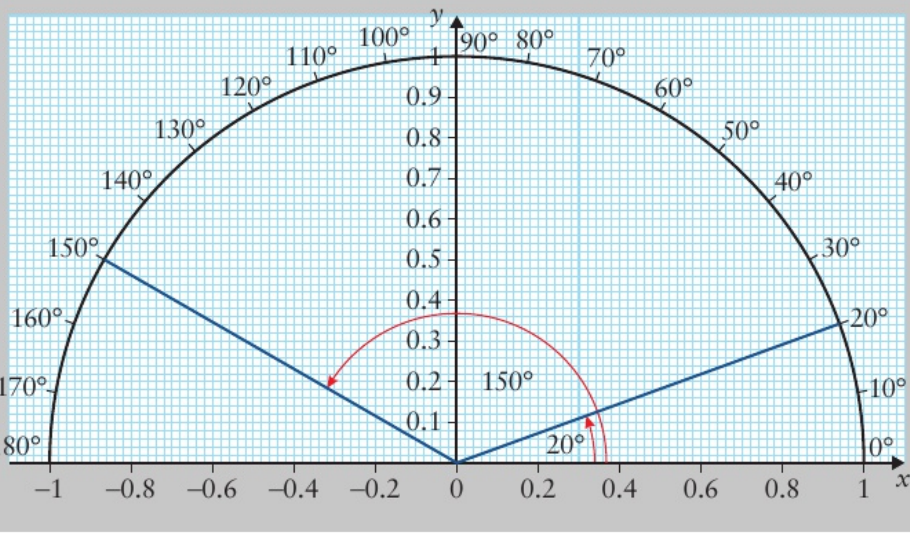
Reflect

Can you use the method from Worked example 15.1 to find trigonometric ratios for the other angles?

Worked example 15.2

Use the graph to find each value. Give your answers correct to two decimal places.

a $\sin 20°$ **b** $\tan 150°$



Solution

We can read the coordinates of the point $P(\cos\theta, \sin\theta)$ for the points where $\theta = 20°$ and $\theta = 150°$ on the graph.

a The sine of an angle θ is the y-coordinate of the point P. Reading it off the graph:

$\sin 20° = 0.34$

b The tangent of an angle θ is the ratio $\frac{y}{x}$. Reading the values off the graph:

$\tan 150° = \frac{0.50}{-0.87} = -0.57$

We can verify these values with a calculator: $\sin 20° = 0.342...$ and $\tan 150° = -0.577...$

Reflect

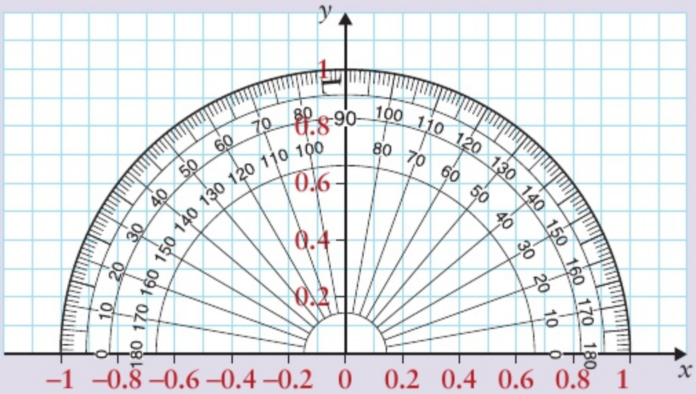
Suppose you want to find $\cos 20°$ and you already know $\sin 20°$.
What methods can you use?

Explore 15.3

How many angles θ, with $0 \leqslant \theta \leqslant 180°$, are there such that $\sin\theta = 0.5$? Use the graph to help.

How are the angles related?

Can you generalise your reasoning for any real value $0 \leqslant a \leqslant 1$, with $\sin\theta = a$?



Can $\tan 90°$ be defined? Explain your answer.

Practice questions 15.1

1 By referring to the graph in Worked example 15.2, find each trigonometric ratio. Give your answers correct to two decimal places.

a $\sin 30°$ b $\cos 30°$ c $\tan 30°$ d $\sin 60°$

e $\cos 60°$ f $\tan 60°$ g $\sin 150°$ h $\cos 150°$

Verify your results with a calculator. Do you notice any pattern?

2 By referring to the graph in Worked example 15.2, find one angle θ, to the nearest degree, such that:

a $\cos\theta = 0.64$ b $\cos\theta = -0.5$ c $\cos\theta = 0.5$

d $\cos\theta = -0.76$ e $\cos\theta = 0$ f $\cos\theta = -1$

g $\cos\theta = 1$ h $\cos\theta = -0.3$ i $\cos\theta = -0.94$

j $\cos\theta = 0.4$ k $\cos\theta = 0.18$

3 By referring to the graph in Worked example 15.2, find two angles $0 \leqslant \theta \leqslant 180°$, to the nearest degree, such that:

a $\sin\theta = 0.71$ b $\sin\theta = 0.94$ c $\sin\theta = 0.18$

d $\sin\theta = 0.34$ e $\sin\theta = 0.64$ f $\sin\theta = 0.78$

g $\sin\theta = 0.98$

4 State the quadrant in which θ lies in each case.

a $\sin\theta < 0$ and $\cos\theta > 0$ b $\sin\theta < 0$ and $\cos\theta < 0$

c $\sin\theta > 0$ and $\cos\theta < 0$ d $\sin\theta > 0$ and $\cos\theta > 0$

5 a Use a calculator to complete the table.

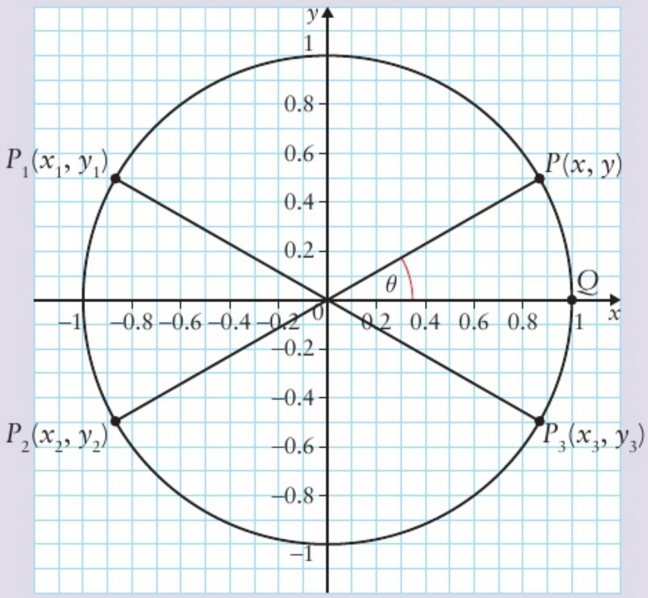
θ	32°	45°	107°	164°
$\sin\theta$				
$\sin(180° - \theta)$				
$\cos\theta$				
$\cos(360° - \theta)$				

b Make a conjecture about the relationship between $\sin\theta$ and $\sin(180° - \theta)$

c Make a conjecture about the relationship between $\cos\theta$ and $\cos(360° - \theta)$

15.2 Trigonometric relationships between acute and non-acute angles

Explore 15.4



Communication skills

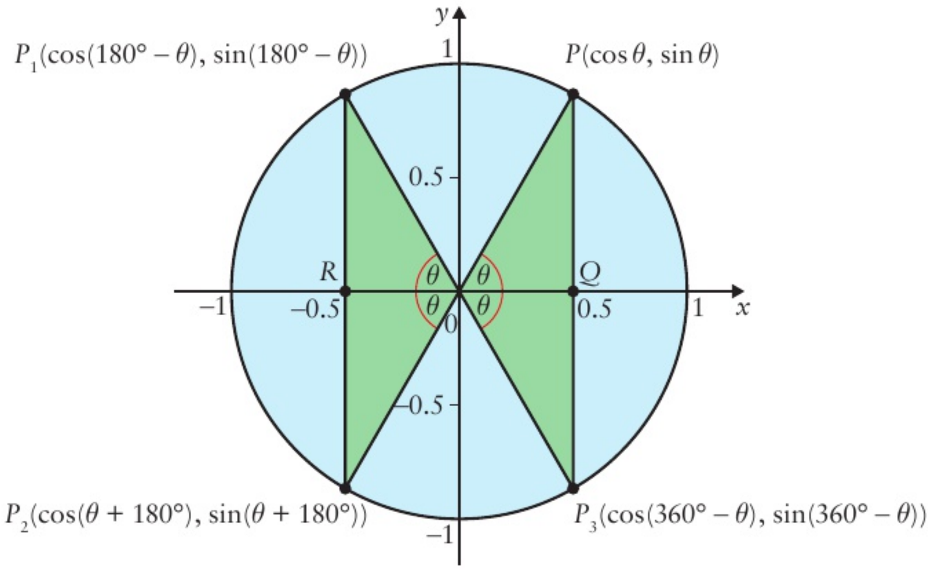
On the unit circle, point P_1 is the reflection of point P in the y-axis.

Can you express the coordinates of P_1, P_2 and P_3 in terms of x and y?

If angle $QOP = \theta$, how can you express the following angles in terms of θ?

a Angle QOP_1 b Angle QOP_2 c Angle QOP_3

The diagram shows that different angles can have the same sine, cosine or tangent ratios due to symmetries in the unit circle and congruent criteria for triangles.



Triangles OQP, ORP_1, ORP_2, OQP_3 are all congruent triangles since they have two corresponding congruent angles (90° and θ) and one congruent corresponding side (the radius of the unit circle). Using the AAS criterion and the signs of the x- and y-coordinates in the different quadrants, we can establish the following identities.

1 $\cos(180° - \theta) \equiv -\cos\theta$

2 $\sin(180° - \theta) \equiv \sin\theta$

3 $\tan(180° - \theta) \equiv -\tan\theta$

4 $\cos(180° + \theta) \equiv -\cos\theta$

5 $\sin(180° + \theta) \equiv -\sin\theta$

6 $\tan(180° + \theta) \equiv \tan\theta$

7 $\cos(360° - \theta) \equiv \cos\theta$

8 $\sin(360° - \theta) \equiv -\sin\theta$

9 $\tan(360° - \theta) \equiv -\tan\theta$

Connections

You learned about the AAS criterion in Chapter 9.

Reflect

Can you justify identities 1, 2, 5 and 7 above?

Worked example 15.3

Find the obtuse angle that has the same sine ratio as 31°.

Fact

The angles $180° - \theta$ and θ are supplementary angles.

Solution

Understand the problem

An angle θ is obtuse if $90° < \theta < 180°$. Two angles have the same sine ratio if they add up to 180° (if they are supplementary).

Make a plan

We need to calculate the supplementary angle of 31°.

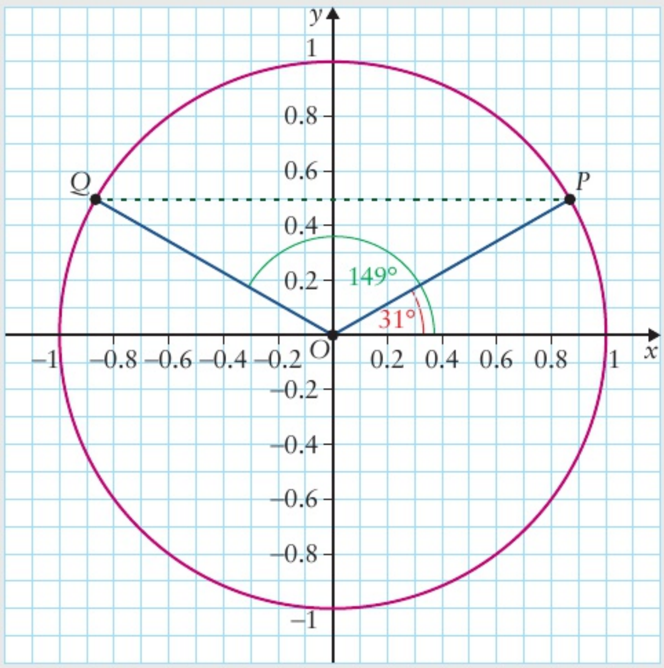
Carry out the plan

$180 - 31 = 149°$

Look back

We can use dynamic geometry software to check that the *y*-coordinates of the points on the unit circle determined by terminal sides of the angles 31° and 149° are equal. Alternatively, we can use a calculator.

$\sin 31° \approx 0.52$ and $\sin 149° \approx 0.52$



Worked example 15.4

a Find two distinct angles θ between 0° and 180° such that $\sin\theta = 0.62$

b Find an angle θ between 0° and 180° such that $\tan\theta = -7.03$

Give your answers to the nearest degree.

Reminder

Make sure that your calculator is set up in degree mode.

Solution

a A calculator gives $\sin^{-1}(0.62) \approx 38°$

This is an acute angle. The other angle that has the same sine ratio is its supplementary angle, $180 - 38 = 142°$

b $\tan(180° - \theta) = -\tan\theta = 7.03$

$180 - \theta = \tan^{-1}(7.03) = 82°$ (to the nearest degree)

$\theta = 180 - 82 = 98°$ (to the nearest degree)

Practice questions 15.2

1 For each set of angles, draw a unit circle and use a protractor to mark the angles.

a $20°$, $(180 - 20)°$, $(180 + 20)°$, $(360 - 20)°$

b $50°$, $(180 - 50)°$, $(180 + 50)°$, $(360 - 50)°$

c $70°$, $(180 - 70°)$, $(180 + 70)°$, $(360 - 70)°$

d $80°$, $(180 - 80)°$, $(180 + 80)°$, $(360 - 80)°$

2 **a** Copy and complete the table.
Give your answers correct to two decimal places.

	$\theta = 34°$	$\theta = 117°$
$\sin\theta$		
$\sin(180° - \theta)$		
$\sin(180° + \theta)$		
$\sin(360° - \theta)$		
$\cos\theta$		
$\cos(180° - \theta)$		
$\cos(180° + \theta)$		
$\cos(360° - \theta)$		

b Verify for each value of θ that $\cos^2\theta + \sin^2\theta = 1$

3 **a** Copy and complete the table. Give the exact values of the sine and cosine ratios of the given angles.

	$\theta = 30°$	$\theta = 45°$	$\theta = 60°$
$\sin\theta$			
$\sin(180° - \theta) = \sin\square = \sin\square$			
$\sin(180° + \theta) = \sin\square = -\sin\square$			
$\sin(360° - \theta) = \sin\square = -\sin\square$			
$\cos\theta$			
$\cos(180° - \theta) = \cos\square = -\cos\square$			
$\cos(180° + \theta) = \cos\square = -\cos\square$			
$\cos(360° - \theta) = \cos\square = \cos\square$			

Reminder

In Chapter 12 you learned the exact values of the trigonometric ratios of 30°, 45° and 60°.

b For each value of θ find $\tan\theta$ and verify that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

4 Find the obtuse angle that has the same sine ratio as:

a 27° b 42° c 70° d 86°

5 Find the acute angle that has the same sine ratio as:

a 97° b 124° c 148° d 172°

6 If θ is an angle between 0 and 180°, find θ, rounded to the nearest degree, such that:

a $\cos\theta = -0.27$ b $\cos\theta = 0.27$

c $\cos\theta = 0.8724$ d $\cos\theta = -0.8724$

e $\tan\theta = 0.417$ f $\tan\theta = -0.417$

g $\tan\theta = 29.228$ h $\tan\theta = -29.228$

7 Find two distinct angles θ, rounded to one decimal place, between 0 and 180°, such that:

a $\sin\theta = 0.27$ b $\sin\theta = 0.41$

c $\sin\theta = 0.5632$ d $\sin\theta = 0.616$

e $\sin\theta = 0.8201$ f $\sin\theta = 0.9$

8 Find two distinct angles θ, rounded to one decimal place, between 0 and 360°, such that:

a $\cos\theta = 0.27$ b $\cos\theta = -0.27$

c $\cos\theta = -0.41$ d $\cos\theta = -0.5632$

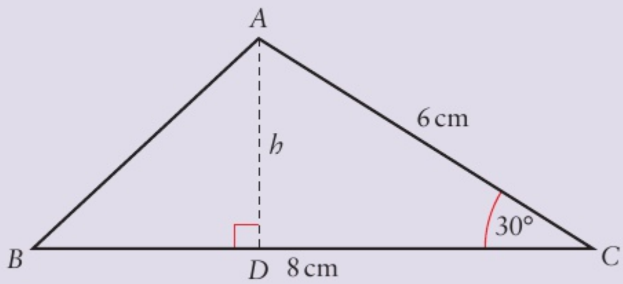
e $\tan\theta = 43.616$ f $\tan\theta = 95.8201$

15.3 The area of a triangle

We can find the area of a triangle using trigonometry if two sides and the included angle are known.

Explore 15.5

Thinking skills



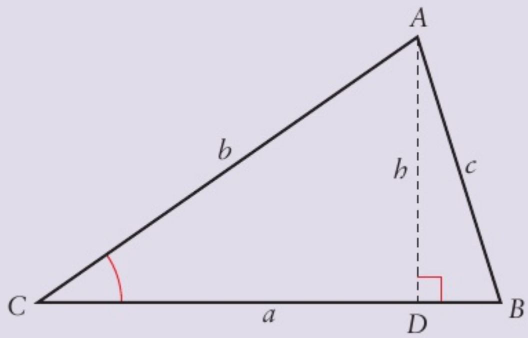
In triangle ABC, $AC = 6$ cm, $BC = 8$ cm, and angle $ACB = 30°$. AD is the height perpendicular to BC.

Can you work out the length, h, of the height AD?

Can you use the value you found for h to find the area of triangle ABC?

Explore 15.6

Consider a triangle ABC with sides a, b, c (opposite respectively to angles A, B, C). Suppose h is the height of triangle ABC perpendicular to side a.



In the right-angled triangle ADC, can you justify why $h = b\sin C$?

Hence, can you justify why the area of triangle ABC is:

area of triangle $ABC = \frac{1}{2}ab\sin C$?

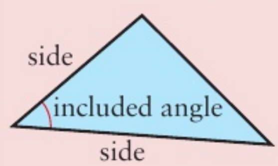
By similar arguments to those used in Explore 15.6, we can develop the formulas: area$=\frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

Reflect

If angle C is a right angle, how does the formula area $= \frac{ab\sin C}{2}$ change?

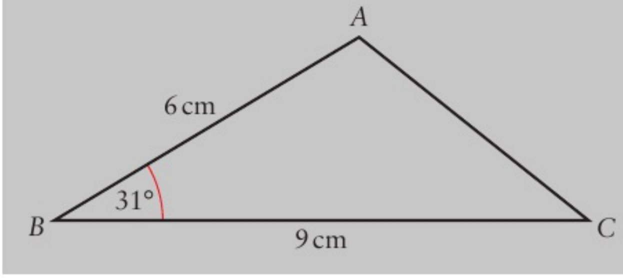
Fact

To use this formula, you need to know two sides and the included angle.



Worked example 15.5

Calculate the area of the triangle ABC. Give your answer correct to one decimal place.



Solution

We know two sides, $a = 9\,\text{cm}$ and $c = 6\,\text{cm}$, and the included angle $B = 31°$

We can apply the formula for area by substituting the given values.

$$\begin{aligned}\text{area} &= \frac{ac\sin B}{2}\\ &= \frac{(9)(6)\sin 31°}{2}\\ &= \frac{54 \times 0.515}{2}\\ &= 13.9\,\text{cm}^2\end{aligned}$$

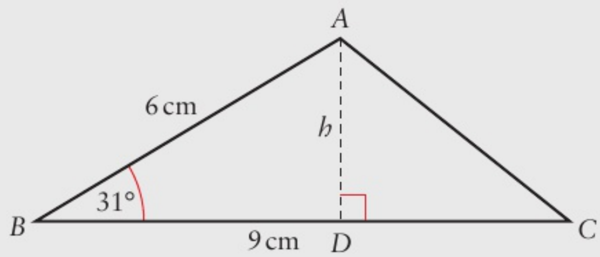
We can check our answer by using the area to calculate the height, h, from vertex A to side BC and comparing this to the value obtained by using trigonometry to calculate h.

$$\text{area} = \frac{\text{base} \times \text{height}}{2}$$

$$13.9 = \frac{9 \times h}{2}$$

$$h = \frac{2 \times 13.9}{9} = 3.1 \text{ (1 d.p.)}$$

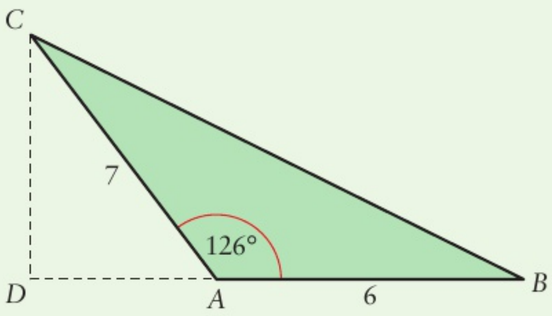
Using trigonometry: $h = 6\sin 31° = 3.1$ (1 d.p.)



The answers are the same, so we can be confident that our value for the area is correct.

Reflect

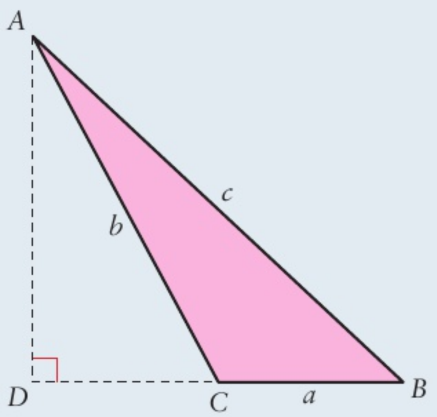
Why is the height CD of the obtuse triangle ABC equal to $7 \sin 126°$?



Challenge

Practice questions 15.3

1 Use the diagram to prove that the formula for the area of triangle ABC is true for obtuse triangles as well.



Hint Q1

Supplementary angles have the same sine ratio.

2 Find the area of each triangle. Give your answers correct to one decimal place.

a $C = 122°$, $a = 6\,\text{cm}$, $b = 11\,\text{cm}$

b $B = 130°$, $a = 83\,\text{m}$, $c = 31\,\text{m}$

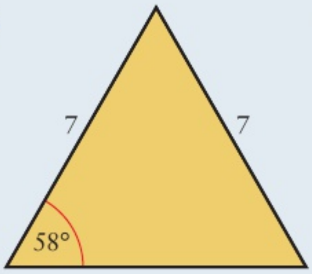
c $A = 10°$, $b = 3.5\,\text{cm}$, $c = 22\,\text{cm}$

d $B = 75.4°$, $a = 104\,\text{cm}$, $c = 49\,\text{cm}$

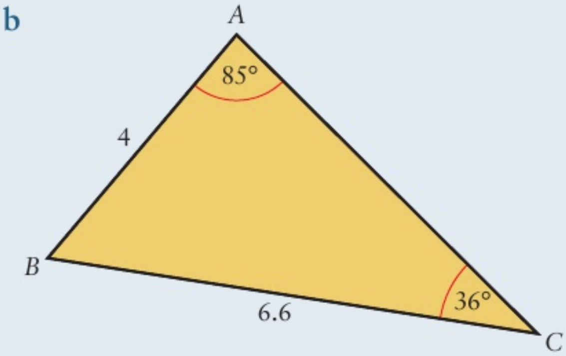
e $A = 83.3°$, $b = 16\,\text{km}$, $c = 21\,\text{km}$

3 Calculate the area of each shape. Give your answers to the nearest square unit.

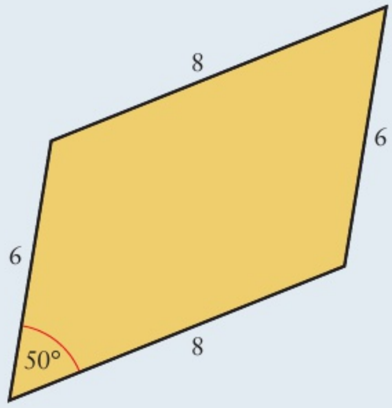
a



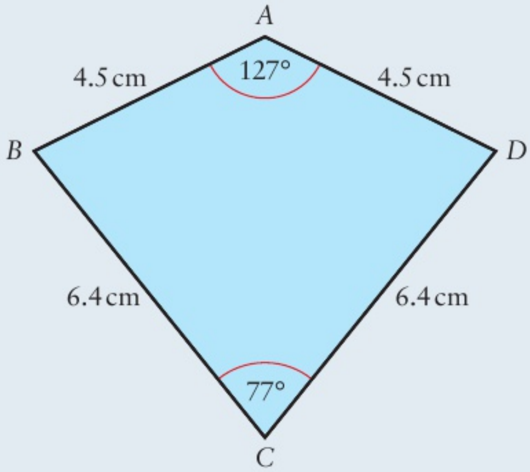
b



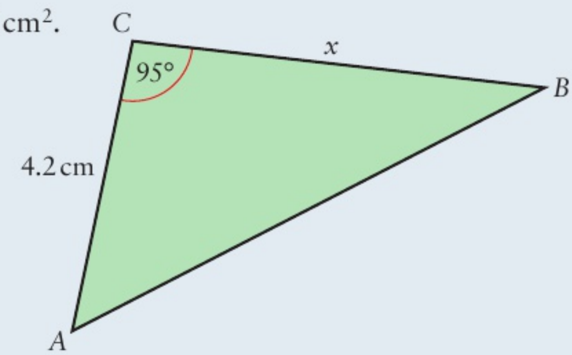
c



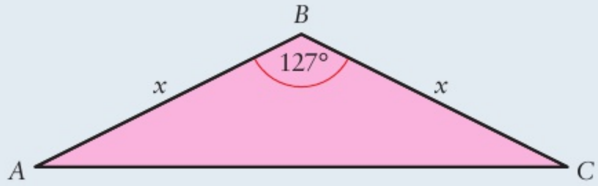
4 Calculate the area of the kite. Give your answer correct to one decimal place.



5 Triangle ABC has area $160\,\text{cm}^2$. Calculate the value of x.



6 Triangle ABC has area $82\,cm^2$. Calculate the value of x.



7 Using the exact value of sin 60°, find a formula for the area of an equilateral triangle of side a.

8 The Bermuda triangle is a triangular area in the Atlantic Ocean. It is famous because several aircrafts and ships have allegedly disappeared while crossing it. The vertices of the triangle are Miami, Bermuda and San Juan in Puerto Rico. The distance between Miami and San Juan is 1660 km and the distance between Miami and Bermuda is 1665 km. The angle at Miami measures 55.4°. Calculate the area of the Bermuda triangle.

9 Most sailing boats have two sails: the mainsail, and the headsail (which is bigger).

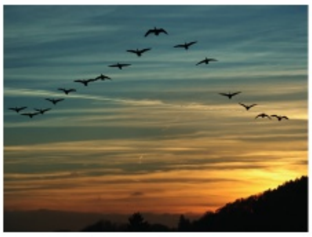
The mainsail is a right-angled triangular sail. The shortest side measures 2.43 m and the next side measures 7.5 m.

The headsail has a roughly triangular shape, but not right angled. The dimensions of the two longest sides of the sail are 12.5 m and 10 m. The included angle of the two longest sides is 25.2°

Calculate the area of the mainsail and the headsail.

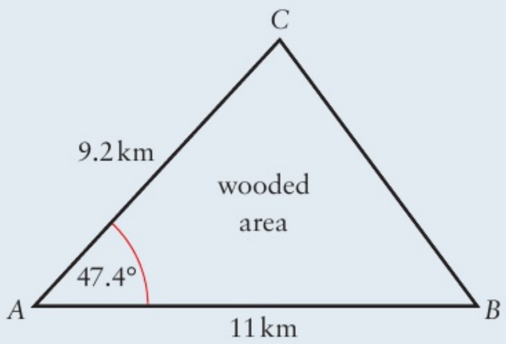
Hint Q10

The V formation can be modelled by an isosceles triangle.



10 Some migratory birds fly in a V formation. Scientists believe that this is for two reasons: to optimise the visual positioning and to catch the air from the bird in front to save energy in long flights. The angle of the V depends on each flock. Scientists have observed that the angle of the V formation of Canada Geese is about 58°. Suppose that the sides of the V formation of a particular flock of Canada Geese measure 12 m. Calculate the area of the isosceles triangle in the sky that the V formation covers.

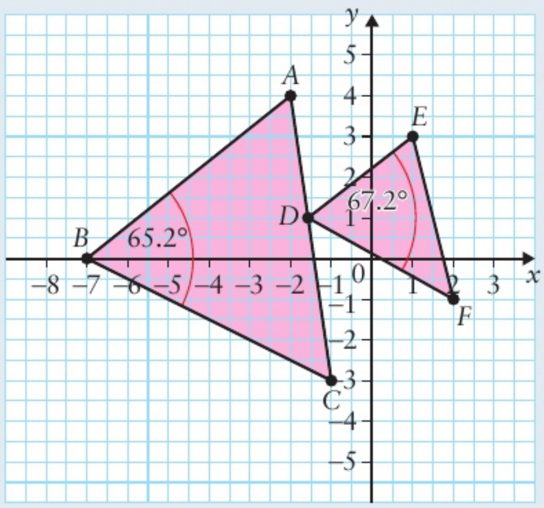
11 Jenni's dog got lost in a wooded area enclosed by three straight roads. One side of the area measures 9.2 km, another side measures 11 km and the included angle measures 47.4° Calculate the area Jenni must search to find her dog.



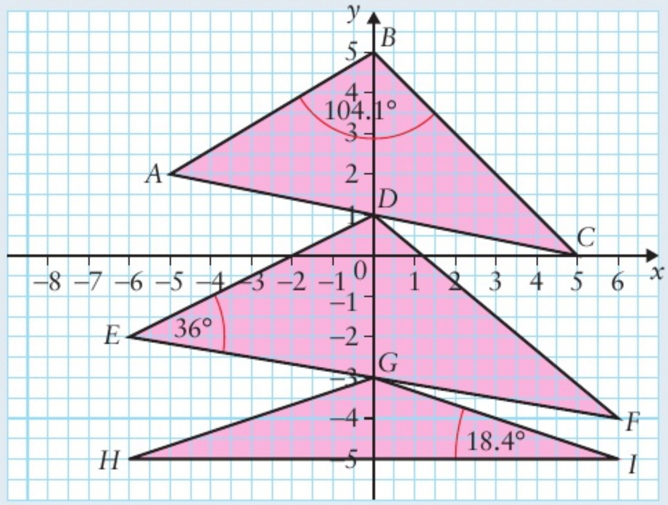
Challenge Q12

12 Use the distance formula in the coordinate plane and the provided angles to calculate the area of each shaded shape. Measurements are in centimetres.

a A fish



b A tree



15.4 The sine rule

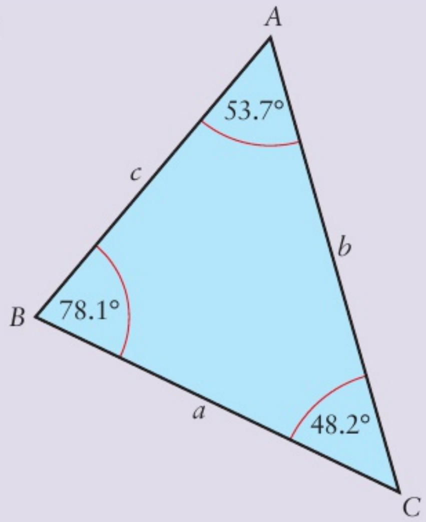
The sizes of the angles and sides in a triangle are closely related to each other.

Explore 15.7

What is the longest side in this triangle?

What is the shortest side?

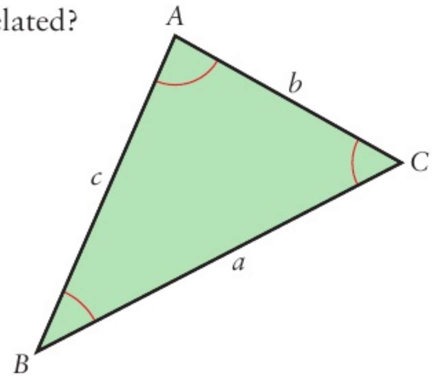
How are the angles and sides related?



Explore 15.8

How many triangles exist that have sides 2 cm and 7 cm and the included angle 115°? How do you know?

Think about all the methods you can use to find the third side.



How are the angles and sides in a triangle related? Take any triangle *ABC*.

The area of *ABC* is equal to:

$$\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

Dividing each expression by $\frac{1}{2}acb$ we get:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This set of equations is known as the **sine rule.** The sine rule is used to find unknown angles and sides.

It can also be written as:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Reflect

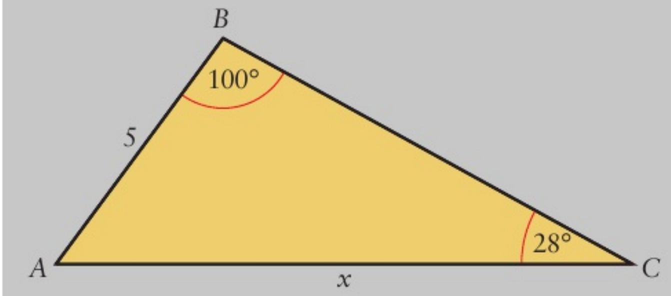
Is the sine rule true for right-angled triangles? Explain.

What information do we need to know in a triangle in order to use the sine rule?

If we know two angles and one side, we can find the other side by using the sine rule.

Worked example 15.6

Calculate the value of x in this triangle.



Solution

We are given angle *C* and its opposite side and angle *B*. We need to find the side opposite to angle *B*.

We can use the second form of the sine rule: $\frac{b}{\sin B} = \frac{c}{\sin C}$. We will make an equation and solve for x.

$$\frac{x}{\sin 100°} = \frac{5}{\sin 28°}$$

By cross-multiplying we get:

$$x \sin 28° = 5 \sin 100°$$

$$x = \frac{5 \sin 100°}{\sin 28°}$$

$$= 10.4884...$$

$$= 10.5 \text{ cm (1 d.p.)}$$

We can check our answer by substituting the value of x back into the sine rule and confirming that the ratios are the same.

$$\text{LHS} = \frac{10.4884}{\sin 100°} = 10.65 \text{ (2 d.p.)}$$

$$\text{RHS} = \frac{5}{\sin 28°} = 10.65 \text{ (2 d.p.)}$$

Reminder

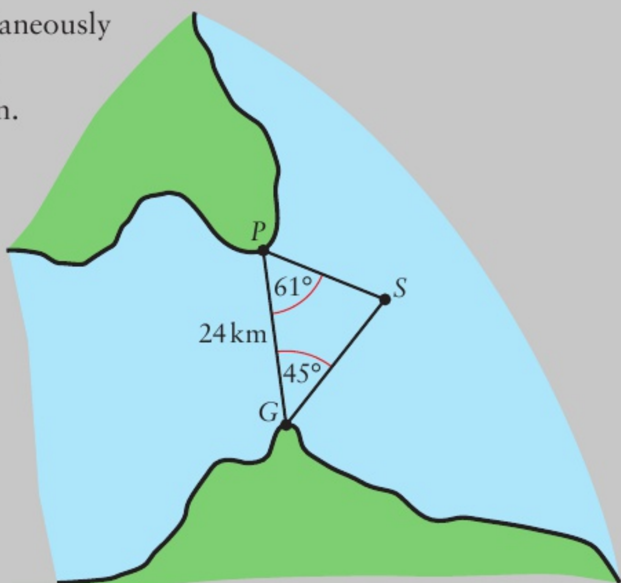
Do not round off calculations at an intermediate stage. Use the full solution in your calculator to reach an accurate final answer.

Reflect

In Worked example 15.6, we were given two angles and one side. How many congruent triangles can we expect that have corresponding congruent angles and sides? Why? Does our solution in Worked example 15.6 match the theory for congruent triangles?

Worked example 15.7

A ship, S, is sighted simultaneously from Pascal Point at P and Gauss Rock at G, as shown. Find the ship's distance from Gauss Rock.



Fact

Trigonometry made a fundamental contribution to the development of human history. For example, it made navigation possible for long and short distances. In case of long navigation, the only landmarks that could be used for calculations were the position of the sun during the day and the stars by night. Those angles and distances were measured accurately using devices such as the marine sextant (for angles) and the chronometer (for the exact time that measurements were taken).

Solution

The ship, S, Pascal Point, P, and Gauss Rock, G, form a triangle. We know the angles P and G and the included side PG. We need to find side SG. Since we know two angles and one side we can use the sine rule.

To apply the sine rule, we need to have an angle and its opposite side. Here we have the side PG, but we do not have angle S. So, first we find angle S and then we can apply the sine rule.

The sum of the interior angles in a triangle is 180°

$S = 180 - 61 - 45 = 74°$

We can now apply the second form of the sine rule:

$$\frac{24}{\sin 74°} = \frac{SG}{\sin 61°}$$

$$SG = \frac{24 \sin 61°}{\sin 74°} = 21.8\,\text{km (3 s.f.)}$$

We can check our answer by substituting our value of x back into the sine rule and confirming that the ratios are the same.

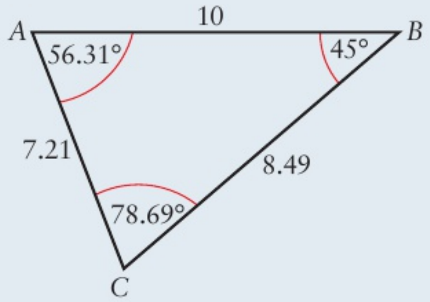
$$\text{LHS} = \frac{24}{\sin 74°} \approx 24.967$$

$$\text{RHS} = \frac{21.8367}{\sin 61°} \approx 24.967$$

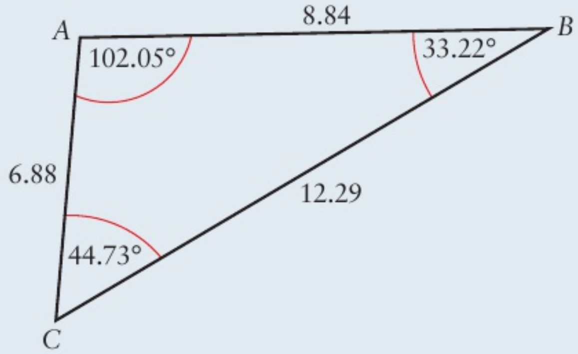
Practice questions 15.4

1 Verify that the sine rule is valid in each triangle. All side measurements are in cm. Round all ratios to two decimal places.

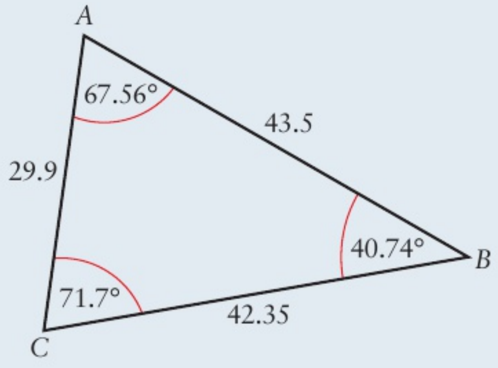
a



b

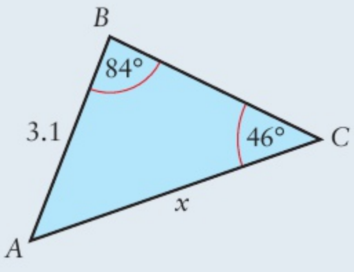


c

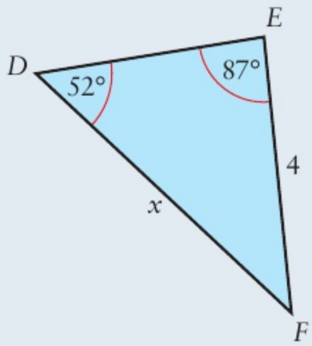


2 Find the value of x in each triangle. Give your answers rounded to two decimal places. All side measurements are in metres.

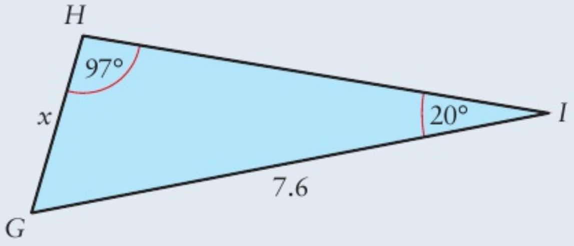
a



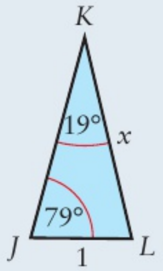
b



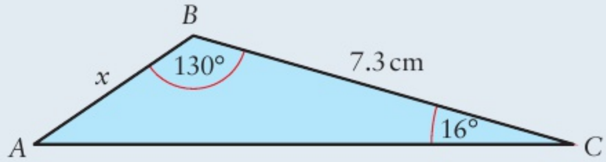
c



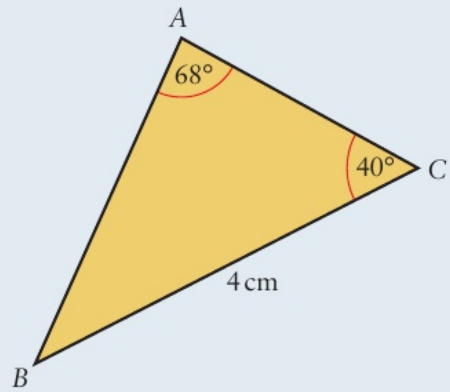
d



e



3 Find the area of the triangle.



Challenge Q4

4 Two fire-fighting aeroplane stations are 50 km apart, with station A directly west of station B. Both stations spot a fire. The bearing of the fire from station B is 304° and the bearing of the fire from station A is 039°. How far is the fire from station A?

Challenge Q5

5 In triangle ABC, side $AB = 10$ cm and side $BC = 7$ cm. Angle A measures 44°.

a Find two possible values for angle C.

b Draw a diagram that represents the two triangles.

6 In triangle ABC, side $AB = 10$ cm and side $BC = 13$ cm. Angle A measures 44°.

a Draw a diagram of triangle ABC.

b Calculate the measure of angle C.

Challenge Q6c

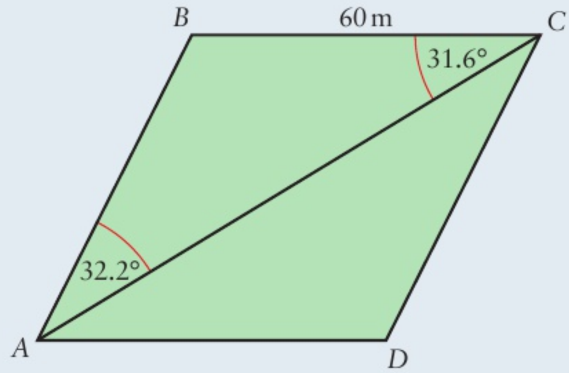
c Show that there is only one possible value for angle C.

7 Two boats, S and T, are 7 km apart when they pick up a distress call from a yacht. Boat S estimates that the angle between the line from S to the yacht and the line from S to T is 27°. Boat T estimates that the angle between the line from T to the yacht and the line from T to S is 35°. What are the distances, to the nearest km, from S to the yacht and from T to the yacht?

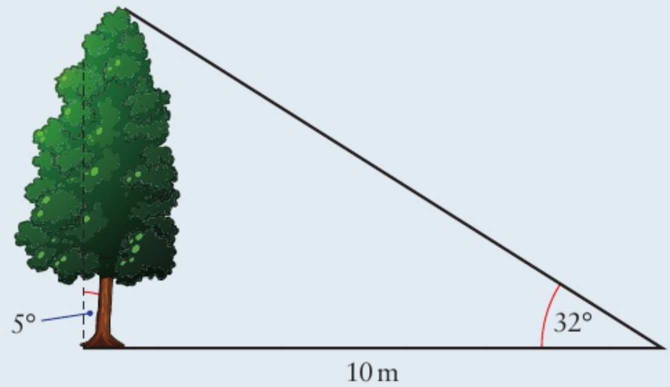
Hint Q7

Draw a diagram to help you visualise the situation.

8 A car park is in the shape of a parallelogram. The longest side measures 60 m. The diagonal, along with two adjacent sides of the parallelogram, creates two angles that measure 32.2° and 31.6°. Find the area of the car park.



9 A tree is inclined 5° from the vertical. At a distance of 10 m from the base of the tree the angle of elevation to the top is 32°. Estimate the height of the tree.



10 A water molecule is made of one atom of oxygen (O) and two atoms of hydrogen (H). The angle formed by the two bonds is 104.45° and the bond is 95.84 pm (picometers).

a Find the two angles OHH

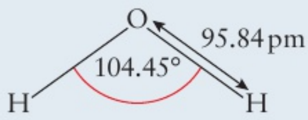
Hint Q10

1 picometre = 10^{-12} metres

Hint Q10a

The bonds of the molecule of water and the atoms form an isosceles triangle.

b Calculate the distance between the two atoms of hydrogen.

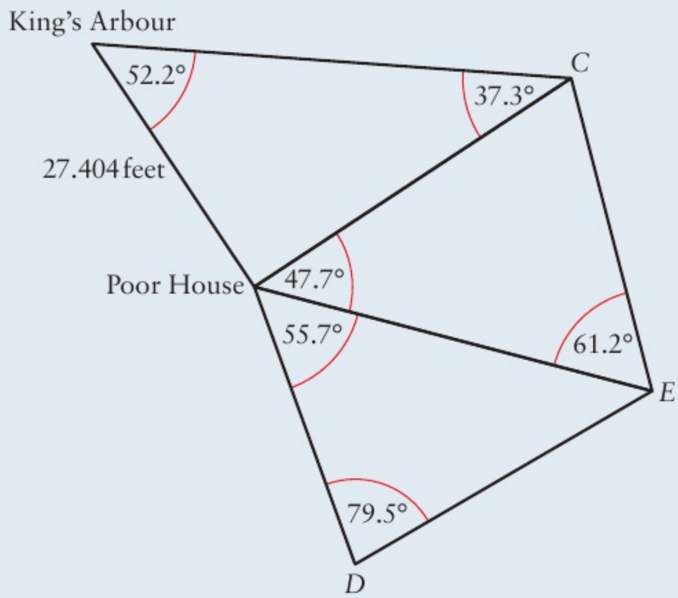


c As water freezes and turns into ice, the shape of the triangle changes. The bonds become 101 pm long and the bound angle measures 110.14°. Calculate the distance between the two hydrogen atoms.

Fact

The Principal Triangulation of Britain was the first trigonometric survey that aimed to provide precise coordinates for almost 300 landmarks, from which maps could be drawn. It was carried out between 1791 and 1853.

11 The first stage in the triangulation of Great Britain was the accurate measurement between King's Arbour and the Poor House at Hampton. Surveyors used iron bars and deal rods and then re-measured using glass tubes. Then they used triangle trigonometry to find other distances. Use the scale drawing below to answer the questions and find out how the surveyors measured the other distances.



a Find the distances from the Poor House to point C and from King's Arbour to point C.

b Find the distances from the Poor House to point E and from C to point E.

c Find the distance from the Poor House to point C using the triangle with vertices Poor House, C and E. Compare it to the distance you found in part a.

d Find the distances from the Poor House to point D and from E to point D.

e Estimate the shortest distance from point D to King's Arbour.

Challenge Q11e

Reflect

In question 10 of Practice questions 15.4, you looked at the angles and distance between hydrogen and oxygen atoms in water. What happens to the volume of the water when it freezes? Do your results match with your experience? Research the density of water, ice and steam and their molecular structure.

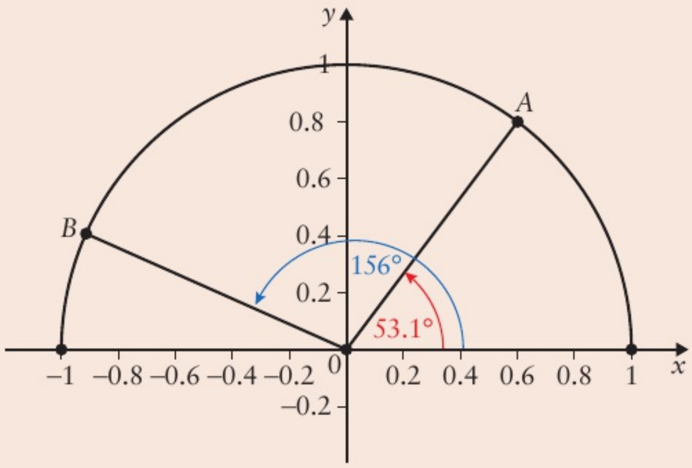
Research skills

Self assessment

- I can define the sine, cosine and tangent of any positive angle using the unit circle.
- I can explain the relationship between the equation $\cos^2\theta + \sin^2\theta = 1$ and the equation of the unit circle.
- I can give the correct sign to the trigonometric ratios on the unit circle depending on the quadrant where they lie.
- I can find all the solutions of the equation $\sin\theta = a$, for $0 \leqslant a \leqslant 1$ and $0 \leqslant \theta \leqslant 180°$.
- I can find the values of the trigonometric ratios of a non-acute angle using the symmetries on the unit circle and the values of the trigonometric ratios of the corresponding acute angle.
- I can find the values of the trigonometric ratios of an obtuse angle using the values of the trigonometric ratios of the corresponding acute angle.
- I can use a calculator in degree mode to evaluate the trigonometric ratios of an obtuse angle.
- I can calculate the measure of an obtuse angle given its sine, cosine or tangent.
- I can find the area of a triangle given two sides and the included angle.
- I can solve problems applying the formula of the area of a triangle.
- I can draw a diagram and state the sine rule.
- I can prove the sine rule.
- I can apply the sine rule to find the lengths of the sides of a triangle given two angles.

Check your knowledge questions

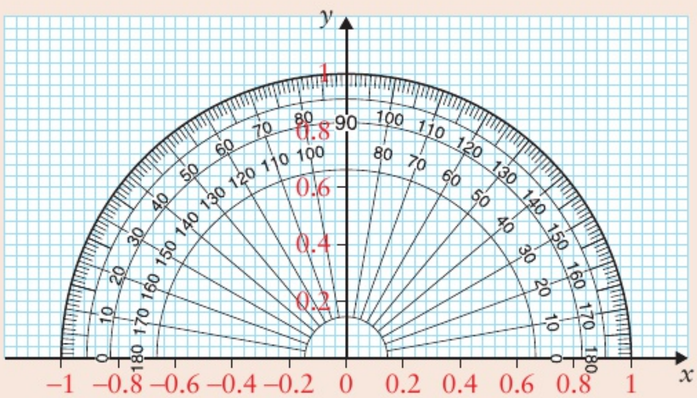
1 Use your calculator to find the coordinates of A and B.



2 Use the protractor and graph paper to find the values, rounded to two decimal places, of:

a $\sin 140°$ **b** $\cos 110°$ **c** $\sin 125°$ **d** $\cos 160°$

Check your results with a calculator.

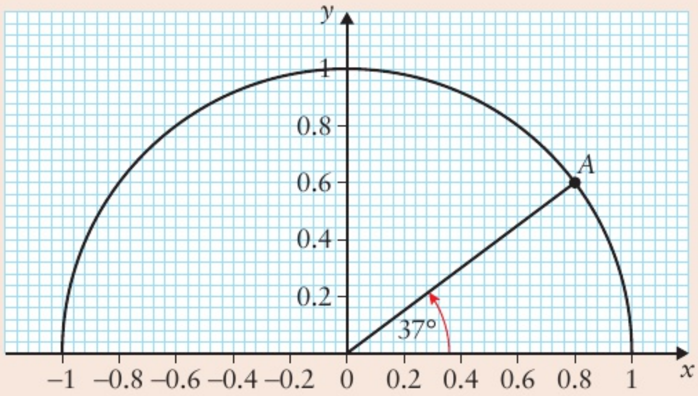


3 **a** Copy the diagram on the next page. On your diagram, draw the angle θ, $90° \leqslant \theta \leqslant 180°$, such that $\sin\theta = \sin 37°$. Find the value of θ. Then use a calculator to find the value of $\sin\theta$ rounded to two decimal places.

b Use the angle found in part a and verify with a calculator that $\cos\theta = -\cos 37°$

c Use the results of parts a and b to find $\tan\theta$. Check your result with a calculator.

d Use the results of parts a and b to verify that the values you found satisfy the equation $\cos^2\theta + \sin^2\theta = 1$



4 Find the obtuse angle that has the same sine ratio as:

a 47° **b** 80° **c** 39°

5 Use your calculator and the unit circle to find two angles θ, with $0 \leqslant \theta \leqslant 180°$, such that $\sin\theta = 0.43$

6 Use your calculator and the unit circle to find two angles θ, with $0 \leqslant \theta \leqslant 360°$, such that $\cos\theta = -0.81$

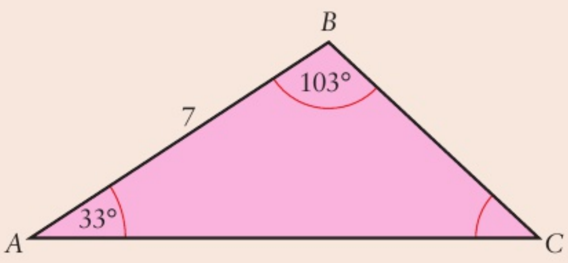
7 Use your calculator and the unit circle to find two angles θ, with $0 \leqslant \theta \leqslant 360°$, such that $\tan\theta = 0.66$

8 Find θ, with $90° \leqslant \theta \leqslant 180°$, if:

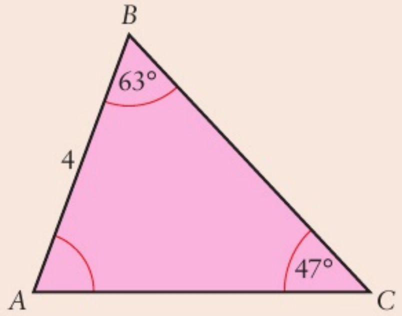
a $\sin\theta = 0.56$ **b** $\cos\theta = -0.21$

9 Calculate all the missing angles and sides of each triangle.

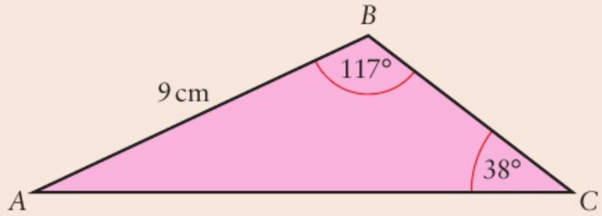
a



b



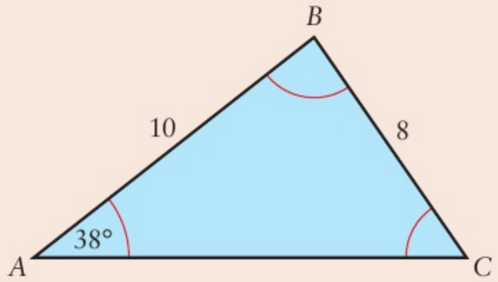
c



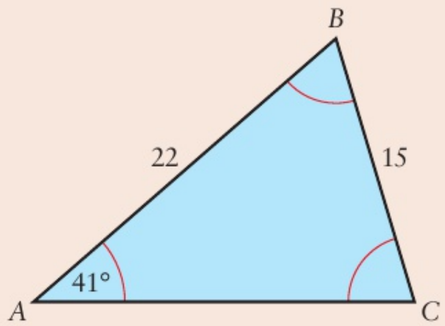
Challenge Q10

10 Calculate all the missing angles and sides of each triangle. All lengths are given in cm.

a

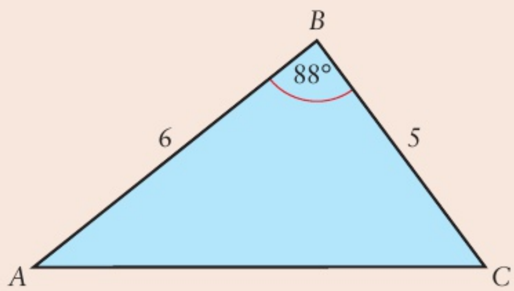


b

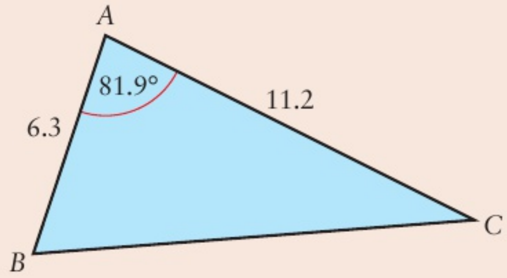


11 Find the area of each triangle. All lengths are given in cm.

a



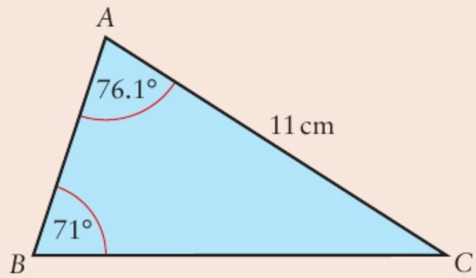
b



12 Calculate:

a the length of BC

b the area of triangle ABC.



13 The leaning tower of Pisa in Italy leans because it was built on unstable soil made of water, sand and clay. The tower is approximately 58.4 m tall from the base. The top of the tower is 4.04 m off the centre.

Challenge Q13

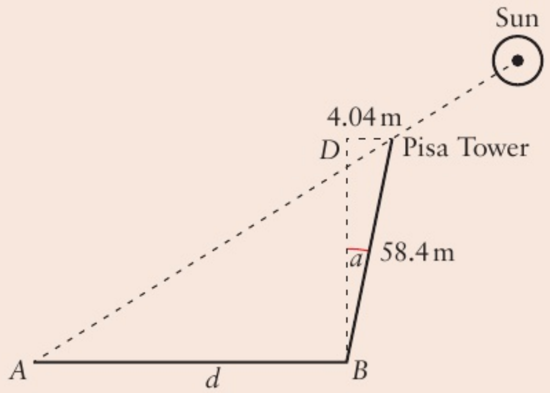
a Find the angle of lean, α, of the tower.

If the angle of depression of the sun is 40° to the top of the tower:

b write down the measure of angle A

c calculate the length, d, of the shadow cast by the tower.

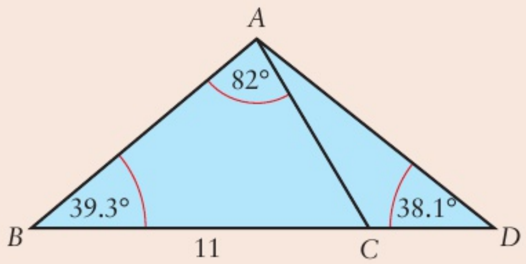
The diagram is a stylisation of the problem (the tower is represented as a line segment and the sun by a circle). It is not drawn to scale.

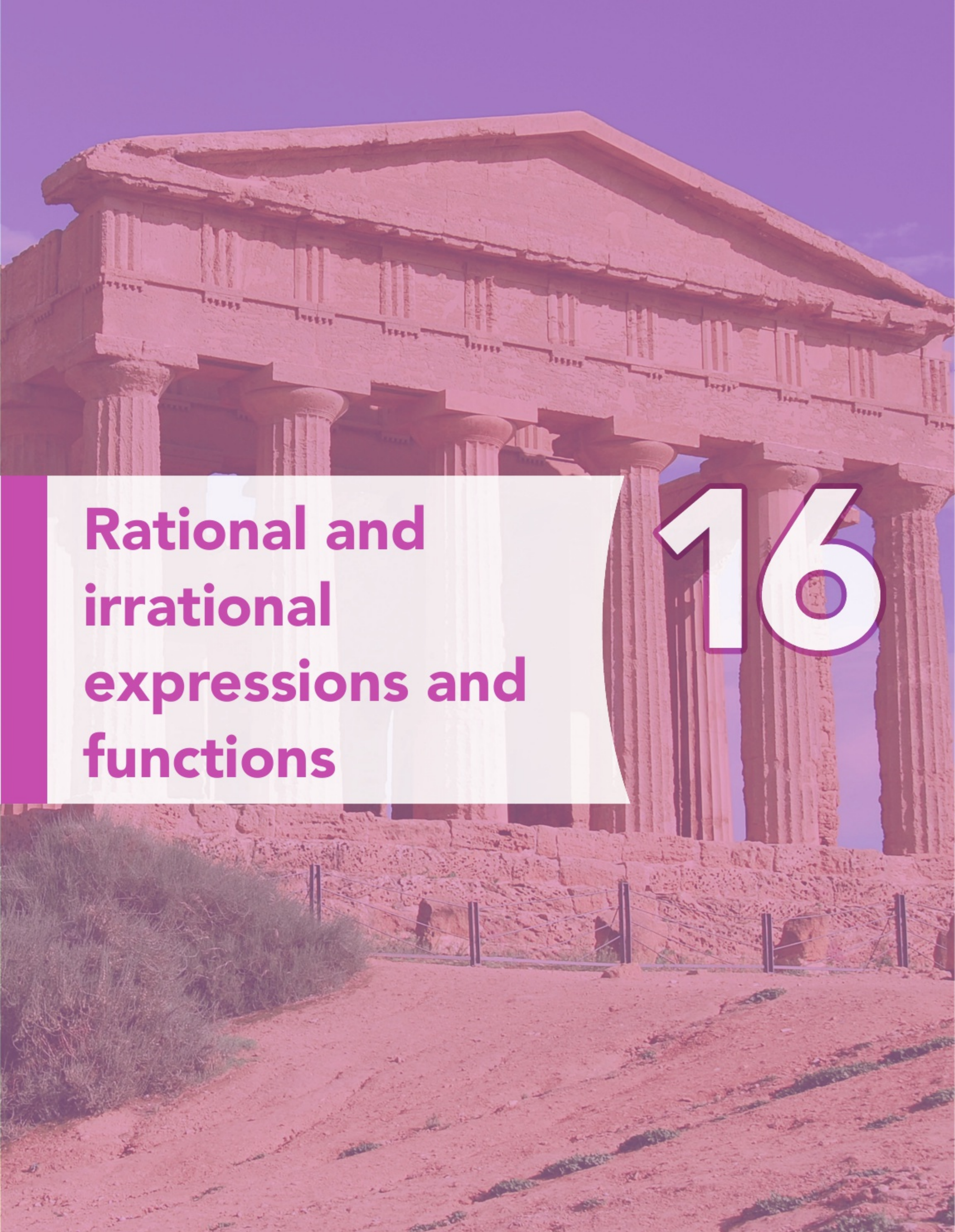


14 A 7 m telephone pole has a very bad lean and creates an angle of 97° with the ground. A guide wire is attached to the top of the pole for support so the pole will not fall down.

The angle that the ground makes with the guide wire is 34°. Calculate the length of the guide wire.

15 Calculate the length of side *AD*. All the measurements are in centimetres.





16 Rational and irrational expressions and functions

16 Rational and irrational expressions and functions

KEY CONCEPT

Logic

RELATED CONCEPTS

Equivalence, Representation, Systems

GLOBAL CONTEXT

Identities and relationships

Statement of inquiry

Understanding the limitations of natural numbers when measuring leads us to develop logical methods to represent relationships between all types of numbers.

Factual

- What types of numbers can we find on the real number line?

Conceptual

- Are the integers the only numbers we need? What about fractions; would we then have enough to describe all numbers?
- How has our understanding of number developed?
- What do denominators containing variables and square roots add to our understanding of numbers, functions and graphs?

Debatable

- Does the analysis or the graph of a function tell you more about it?

Do you recall?

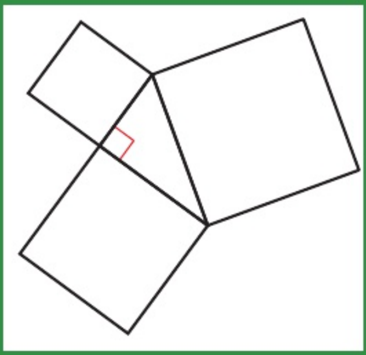
1 Place $\frac{1}{2}$, $\frac{2}{3}$ and the average of $\frac{1}{2}$ and $\frac{2}{3}$ on the real number line.

2 Give an example of a number that you know cannot be written as a fraction.

3 For which of the following calculations might you use common denominators?

$\frac{2}{3} \times \frac{3}{4}$ $\quad$ $\frac{2}{3} + \frac{3}{4}$ $\quad$ $\frac{2}{3} \div \frac{3}{4}$

4 What does Pythagoras' theorem say about this picture?



5 What is the square of $\sqrt{n}$?

6 Expand $(a + b)^2$ and $(a - b)^2$

7 Factorise $a^2 - b^2$

8 Which of these numbers is the odd one out?

1.4 $\quad$ $\frac{7}{5}$ $\quad$ $1\frac{2}{5}$ $\quad$ $\sqrt{2}$ $\quad$ 140% $\quad$ $\sqrt{\frac{49}{25}}$

16.1 Rational numbers and functions

16.1.1 Measuring numbers

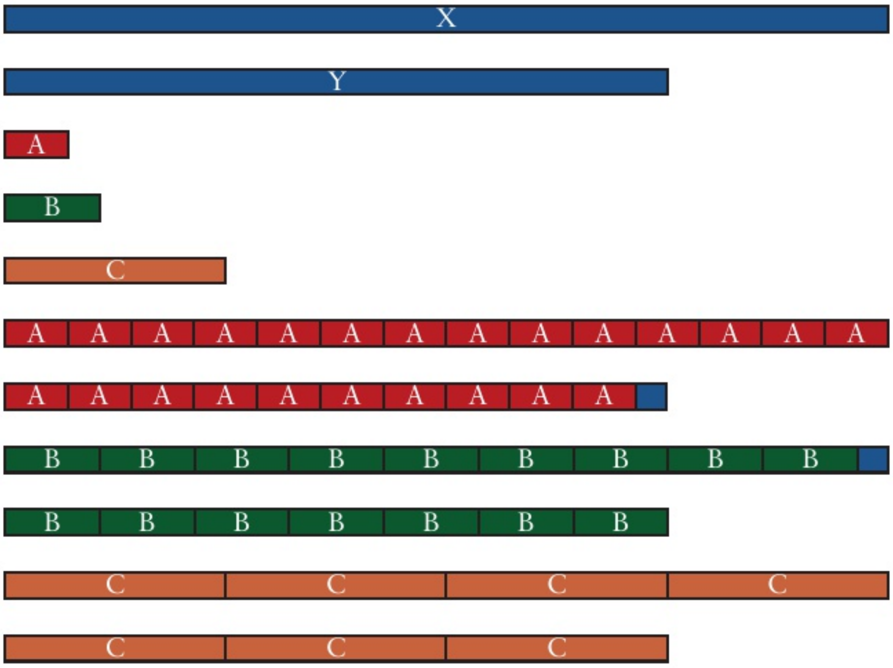
Explore 16.1

Sticks of wood are placed on top of square grids. Some of the wood is painted red. In each case, is the grid good enough to measure exactly the lengths of painted and unpainted wood?

Common measure

Sticks X and Y have different lengths. We will measure them using sticks A, B and C.

Multiple copies of stick A can be used to exactly measure stick X, but not stick Y. And multiple copies of stick B can be used to exactly measure stick Y, but not stick X. Multiple copies of stick C be be used to exactly measure both stick X and stick Y, so we say C is a **common measure**.



We can rephrase this as a question about **common factors**. Stick Y has a length of 21 cm and stick X has a length of 28 cm. What are the lengths of sticks A, B and C? What is the highest common factor of 21 and 28?

But the lengths could also be given as 0.21 m and 0.28 m. The common measure would be 0.07 m. As we are no longer in the realm of whole numbers, it is not possible to call the measure a common factor. For the Pythagoreans, however, 0.07 m would simply become 1 unit of measure and the lengths of X and Y would be 3 and 4 units respectively, in the ratio 3 : 4

Sticks of length 4.5 cm and 6 cm could be measured by a stick of length 1.5 cm. To the Pythagoreans, the 1.5 cm was the unit to use in that case, so the 4.5 cm stick would be 3 units long and the 6 cm stick would be 4 units long.

Connections

2500 years ago, Pythagoras ran a school. The students were adults and they were sworn to secrecy about the mathematical theorems that they discovered, as they believed they were learning to speak the language of the gods. One of their fundamental beliefs was that you could always find a unit length that would measure exactly two different-length sticks. They described this by saying the lengths were **commensurate** (had the same unit of measure) and it is the ratio of those measures that has given us the name **rational numbers**.

Worked example 16.1

Find the highest common measures of:

a 7 and 6 **b** 15 and 25

c $3\frac{1}{2}$ and $2\frac{1}{3}$ **d** a^2bc and abc^2

Solution

Understand the problem

In many cases finding the highest common measure is the same as finding the highest common factor (HCF). The HCF of several whole numbers is well defined. The HCF of several algebraic terms is the part common to all the terms. That is a slightly different definition, as the letters in the terms could represent whole numbers or rational or even irrational numbers.

> **Reminder**
>
> The HCF is also called the GCD (greatest common divisor).

Make a plan

We need to find a unit that divides each of the numbers in the pair a whole number of times. The unit does not have to be a whole number, or even a rational number.

For algebraic expressions, it is the same process as finding common algebraic factors.

Carry out the plan

a $7 = 7 \times 1, 6 = 6 \times 1$

There are no higher measures that would work, so 1 is the highest common measure.

b Recognising multiples of 5, $15 = 3 \times 5, 25 = 5 \times 5$

There are no higher measures that would work, so 5 is the highest common measure.

c First convert to simple fractions with a common denominator:

$$3\frac{1}{2} = \frac{7}{2} = \frac{21}{6}, \qquad 2\frac{1}{3} = \frac{7}{3} = \frac{14}{6}$$

$$\frac{21}{6} = 3 \times \frac{7}{6}, \qquad \frac{14}{6} = 2 \times \frac{7}{6}$$

There are no higher measures that would work, so $\frac{7}{6}$ is the highest common measure.

d Looking for common factors, $a^2bc = a \times abc$, $abc^2 = c \times abc$, so abc is the highest common measure.

Look back

Apart from part c, we have already seen the techniques used in this question. Remember that, in a way, fractions as measures are artificial, as they depend on the unit of measure that is chosen.

Fact

Tenths of centimetres are whole millimetres. Millionths of kilograms are whole milligrams. Older non-metric units contained some unusual possibilities, for example a 'finger' was $\frac{7}{8}$ inch, so you could measure $4\frac{3}{8}$ inches and $6\frac{1}{8}$ inches as 5 fingers and 7 fingers respectively.

Practice questions 16.1.1

1 Find the highest common measure of each pair.

a 8 and 12 b 5 and 10 c 14 and 21

d 101 and 103 e 51 and 66 f 74 and 111

g $\frac{1}{2}$ and $\frac{1}{3}$ h $\frac{2}{3}$ and $\frac{7}{8}$ i $\frac{3}{4}$ and $\frac{9}{16}$

j $2\frac{1}{2}$ and $3\frac{3}{4}$ k $4\frac{1}{8}$ and $13\frac{3}{4}$ l ax and ay

m $\frac{6x^2y}{5}$ and $\frac{9xy^2}{20}$

2 Use your answers to question 1 to simplify, where possible, the following fractions.

a $\frac{8}{12}$ b $\frac{5}{10}$ c $\frac{14}{21}$ d $\frac{101}{103}$

e $\frac{51}{66}$ f $\frac{74}{111}$ g $\frac{\frac{1}{2}}{\frac{1}{3}}$ h $\frac{\frac{2}{3}}{\frac{7}{8}}$

i $\frac{\frac{3}{4}}{\frac{9}{16}}$ j $\frac{2\frac{1}{2}}{3\frac{3}{4}}$ k $\frac{4\frac{1}{8}}{13\frac{3}{4}}$ l $\frac{ax}{ay}$

m $\frac{\frac{6x^2y}{5}}{\frac{9xy^2}{20}}$

3 Two gears rotate on fixed axes with their teeth engaged as shown. The smaller gear has 28 teeth and the larger one has 36 teeth.

a How many times must each gear fully rotate before the marks are aligned again?

b What other sizes of gears could be used to give the same number of rotations?

a What ratio do all possible answers to part b have in common?

4 Spanners, and the nuts and bolts that they fit, have been made to different standards throughout the world. As a result, a garage mechanic would need different sets of spanners to cope with cars manufactured in different countries. To help standardise sizes, some spanners that are very close in size have become interchangeable. AF spanners are measured in fractions of an inch, always using a power of 2 as the denominator, down to $\frac{1}{32}$ of an inch. Metric spanners are measured in whole numbers of mm.

a A 16 mm spanner is equivalent to $\frac{5}{8}$ AF. What AF measure would you expect to be equivalent to 8 mm and 24 mm?

b A standard wheel nut takes a 19 mm spanner. Using the measures in part a, what AF measurement would you expect to use on a wheel nut?

c What metric spanner would you use for $\frac{17}{16}$ AF?

16.1.2 The definition of a rational number

Explore 16.2

The tray shown here is designed to guide cutting a *galette des rois* (or cake or pizza) into equal parts, depending on the number of people sharing it, between 4 and 9.

How would you adapt it for other numbers of people?

Could a large full-circle protractor be used instead?

Do you think the 0 has a similar meaning to the other numbers?

The highest common measure of 52 and 65 is 13 because:

$52 = 4 \times 13$ and $65 = 5 \times 13$

Therefore the ratio $52:65 = 4:5$. We can say that 52 is $\frac{4}{5}$ of 65.

It is this ratio of whole-number multipliers of a common measure that leads us to the idea of rational numbers, most easily recognised as simple fractions.

A simple fraction consists of two integers, one called the numerator and the other the denominator. If we call the integers p and q then **rational numbers** can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ (division by zero is impossible).

This can be stated formally as:

$$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$$

You read this as the set of rational numbers ($\mathbb{Q}$) is the set of all numbers of the form $\frac{p}{q}$, where p and q are in the set of integers ($\mathbb{Z}$) and q is not zero.'

Connections

See Section 16.1.5 for more about dividing by zero.

Rational numbers can be written in different ways, and it is not always easy to see how to rewrite them in the form of a simple fraction $\frac{p}{q}$. For example:

$$3\frac{1}{7}=\frac{22}{7},\quad 0.4=\frac{2}{5},\quad -100=-\frac{100}{1},\quad 0.\dot{3}=\frac{3}{9}=\frac{1}{3},$$

$$0.45=\frac{45}{100}=\frac{9}{20},\quad 0.\dot{4}\dot{5}=\frac{45}{99}=\frac{5}{11},\quad \sqrt{\frac{120409}{84681}}=\frac{347}{291}$$

Reflect

Is it possible for the numerator and the denominator of a simple fraction to have a common factor that cannot be removed?

Investigation 16.1

It can be proved that a rational number can be written in decimal form with either a finite or recurring decimal expansion.

Communication skills

Connections

You can get a feel for this using a paper and pencil method for dividing 1 by 7, to change $\frac{1}{7}$ to decimal form.

The number of figures present in the recurring part of a decimal is called the **period**, and has some interesting properties related to the denominator of the equivalent fraction.

Fraction	Decimal	Recurring digits	Period
$\frac{1}{1}$	1		0
$\frac{1}{2}$	0.5		0
$\frac{1}{3}$	0.333333333333333	3	1
$\frac{1}{4}$	0.25		0
$\frac{1}{5}$	0.2		0
$\frac{1}{6}$	0.166666666666667	6	1
$\frac{1}{7}$	0.142857142857143	142857	6
$\frac{1}{8}$	0.125		0
$\frac{1}{9}$	0.111111111111111	1	1
$\frac{1}{10}$	0.1		0
$\frac{1}{11}$	0.090909090909091	90	2

The table, generated by a spreadsheet, shows the start of the decimal expansion for the reciprocals of the first 11 positive whole numbers. In some cases the decimals clearly recur. The final digit might not fit the pattern because of rounding so it is best to ignore it in the following analysis.

Use your calculator or a spreadsheet to continue the table up to the reciprocal of 20, completing as much as possible.

Now look at the expansions for the first few fractions with 29 as the denominator. Some patterns have been highlighted. Using those patterns and the digits around them we can reconstruct the full expansion of $\frac{1}{29}$, which has 28 recurring digits: 0.0344827586206896551724137931…

Fraction	Decimal
$\frac{1}{29}$	0.03448275862068967
$\frac{2}{29}$	0.06896551724137931
$\frac{3}{29}$	0.1034482758620690
$\frac{4}{29}$	0.1379310344827590
$\frac{5}{29}$	0.1724137931034480
$\frac{6}{29}$	0.2068965517241380
$\frac{7}{29}$	0.2413793103448280
$\frac{8}{29}$	0.2758620689655170
$\frac{9}{29}$	0.3103448275862070
$\frac{10}{29}$	0.3448275862068970
$\frac{11}{29}$	0.3793103448275860

Use the same approach to find the full expansion of $\frac{1}{17}$ and $\frac{1}{19}$

Now try $\frac{1}{23}$ and $\frac{1}{31}$. Do they both follow the same method?

What is the period of each?

You might find it interesting to take this further and study the expansion of $\frac{1}{98}$

Challenge

Practice questions 16.1.2

1 Where possible, write each of the following in the form $\frac{p}{q}$, where p and q are integers. Use a calculator to help when necessary.

a $4\frac{3}{4}$ **b** $\sqrt{961}$ **c** 2.25

d $\sqrt{2.25}$ **e** $\sqrt{\frac{36}{25}}$ **f** 3.14

g 1.414 **h** 0.1 **i** $\sqrt{2}$

j $-\sqrt{4}$

2 Use the pattern spotting method from Investigation 16.1 to find all the recurring digits of the decimal expansion of:

a $\frac{1}{43}$ **b** $\frac{2}{43}$

3 How many different sets of recurring digits can you find for the decimal expansion of $\frac{n}{41}$?

Reflect

Were there any numbers in question 1 that you could not put in the form $\frac{p}{q}$?

Can you be sure that there is a number with an infinite decimal expansion that does not recur?

Do you agree with the early Pythagoreans that all numbers are rational?

16.1.3 Working with rational expressions

Explore 16.3

The ancient Egyptians had some interesting rules for fractions: they had to be written as sums of fractions with different denominators and 1 as the numerator.

An easy example: $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$

A hard example: $\frac{23}{38} = \frac{1}{2} + \frac{1}{10} + \frac{1}{190}$ (Why not $\frac{1}{2} + \frac{1}{19} + \frac{1}{19}$?)

Choose some fractions to explore writing in this way.

Do you think every fraction can be written this way?

Finding patterns in algebraic fractions

$\frac{2}{3} + \frac{5}{3} = \frac{7}{3}$ can be read as saying that 2 of something added to 5 of the same thing gives 7 of it.

What would you expect to get from $\frac{a}{3} + \frac{b}{3}$?

What about $\frac{3}{x} + \frac{4}{x}$?

Sometimes we look for common denominators because the original denominators are not the same:

$$\frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12}$$

Similarly $\frac{x}{3} + \frac{y}{5} = \frac{5x + 3y}{15}$

Connections

The name 'rational' number comes from the ratios of commensurate numbers. Written as simple fractions they are also described as having a **numerator** and a **denominator**. The meaning of the words is revealing: numerator refers to number or quantity, and denominator to the name (think of the French *nom*) of the fraction.

Explore 16.4

The answer to $\frac{1}{2} + \frac{1}{3}$ is $\frac{5}{6}$, written as a single simple fraction.

Do you see a relationship between the numerator and denominator in the answer and the denominators in the question? Does the same relationship hold for the following?

$\frac{1}{9} + \frac{1}{10}$ $\qquad$ $\frac{1}{a} + \frac{1}{b}$ $\qquad$ $\frac{1}{3} + \frac{1}{n}$

You should now have a rule. Test it on $\frac{1}{4} + \frac{1}{6}$. Can you remove any factors common to the numerator and the denominator in the answer? Do you need to review your rule? Now write $\frac{1}{2p} + \frac{1}{2q}$ as a single simple fraction.

Hint

When there are different denominators in an addition, subtraction or comparison of fractions, look for the lowest common denominator (the lowest common multiple of both denominators).

Worked example 16.2

Simplify:

a $\frac{1}{x+1} + \frac{2}{x+1}$ $\qquad$ **b** $\frac{1}{x+1} + \frac{1}{x-1}$

c $\frac{3}{x(x-2)} - \frac{2}{x-2}$ $\qquad$ **d** $\frac{5}{(x-3)^2} - \frac{5}{x^2-9}$

Solution

a The denominators are the same, so add the numerators.

$$\frac{1}{x+1} + \frac{2}{x+1} = \frac{3}{x+1}$$

b The denominators are different and have no common factor, so the common denominator will be their product.

$$\frac{1}{x+1}+\frac{1}{x-1}=\frac{x-1}{(x+1)(x-1)}+\frac{x+1}{(x+1)(x-1)}=\frac{2x}{x^2-1}$$

c $x-2$ is a common factor of the denominators so it only appears once in the denominator of the product.

$$\frac{3}{x(x-2)}-\frac{2}{x-2}=\frac{3}{x(x-2)}-\frac{2x}{x(x-2)}=\frac{3-2x}{x(x-2)}$$

d The denominators share a common factor but it is not immediately obvious.

$x^2-9=(x-3)(x+3)$, so the common factor is $(x-3)$

However, because it occurs twice in a single denominator in $(x-3)^2$, it must appear twice in the common denominator.

$$\frac{5}{(x-3)^2}-\frac{5}{x^2-9}=\frac{5(x+3)}{(x-3)(x-3)(x+3)}-\frac{5(x-3)}{(x-3)(x-3)(x+3)}$$

$$=\frac{5x+15-(5x-15)}{(x-3)^2(x+3)}$$

$$=\frac{30}{(x-3)^2(x+3)}$$

Hint

Remember to use the lowest common denominator.

Compare this with $\frac{7}{8}-\frac{3}{4}=\frac{7}{8}-\frac{6}{8}=\frac{1}{8}$

There is no need to use $4\times 8=32$ as the common denominator because 4 is already a factor of both 4 and 8.

Hint

Take care when subtracting algebraic fractions. Use brackets if you need to when expanding the numerator of the second fraction.

Practice questions 16.1.3

1 a Simplify:

i $\frac{1}{2}-\frac{1}{4}$ ii $\frac{1}{5}-\frac{1}{25}$ iii $\frac{1}{11}-\frac{1}{121}$

b If the difference of two fraction is $\frac{x-1}{x^2}$, what might the two fractions look like?

2 Write as a single fraction:

a $1-\frac{1}{x}$

b $\frac{2}{3x}-\frac{1}{2x}$

c $x+\frac{3}{x}$

d $\frac{x}{4}+\frac{4}{x}$

e $\frac{2}{1-x}-\frac{1}{x}$

f $\frac{x}{x-2}+\frac{x}{x+2}$

g $\frac{1}{x-1}-\frac{x}{x^2-1}$

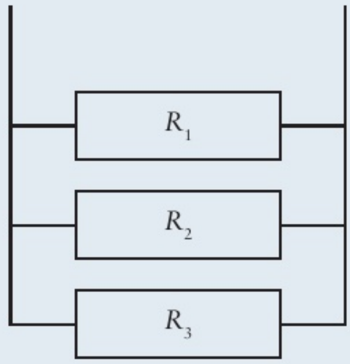
h $\frac{1}{x-1}-\frac{2}{x}+\frac{1}{x+1}$

i $\frac{1}{x^2+2x}-\frac{1}{x^2-4}$

Challenge Q3

3 When three resistors with resistances R_1, R_2 and R_3 are combined in parallel in a circuit their total resistance R is given by:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



Given that R_2 is 1 ohm less than R_1 and R_3 is 1 ohm more than R_1, find as a single fraction an expression for R in terms of R_1.

16.1.4 Graphs of rational functions

Explore 16.5

Think of a non-zero whole number. Take its reciprocal. Is the answer close to zero? Can you get closer? How close can you get?

Functions with variables in the denominator

Investigation 16.2

A phone company will sell you a phone for €250, provided you sign up to a contract for 24 months costing €30 per month. Although the contract is quite expensive, it will cover all your monthly needs in terms of calls, texts and data, so you know you will not have to pay any more than the fixed rate.

1 How much will the phone (initial price and contract) have cost you after 10 months? What is the average cost per month for those ten months?

2 At the end of two years, what will the total cost be and how much would that be on average per month?

3 If you have had the phone for x months what would be the total cost?

a Write a function $a(x)$ which represents the average cost per month of having the phone for x months.

b Assuming that you can extend the contract at the same monthly rate, how long would it take to get the average cost per month:

i below €36   ii below €31   iii below €30?

c Use technology to draw a graph $y = a(x)$. Are there parts of the graph that have no meaning in terms of the original description of buying a phone?

Add the line $y = 30$ to your graph. What is the significance of this line?

d The function was defined on a monthly basis, but what if the company allowed you to pay initially €1 per day in a month with 30 days?
What would the phone have cost you at the end of the first day?

e Evaluate $a\left(\frac{1}{30}\right)$ to find the monthly average cost after the first day.

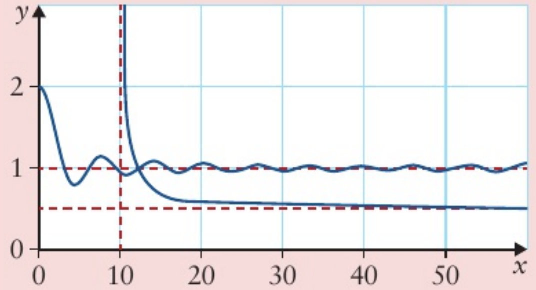
f What fraction would you use for x to find the monthly rate for just the first hour? What is the value of a in this case? What happens to the monthly rate as you choose shorter time periods?

Fact

The function $a(x)$ is an example of a **rational function**.

Fact

The graphs shown here approach straight lines (the dotted lines) without completely merging with them. Such straight lines are called **asymptotes**.



Practice questions 16.1.4

1 Consider the function $f(x) = \frac{12}{x}$

a Without using technology, complete a copy of the table (as far as possible) and then plot the graph of $y = f(x)$

x	-12	-6	-4	-3	-2	-1	0	1	2	3	4	6	12
$f(x)$													

b Which value of x had no corresponding $f(x)$? If necessary, check your table with a calculator.

c If you extended the axes far enough, do you think you would find a solution to $f(x) = 0$ (in other words, a value of 0 in the $f(x)$ row)?

2 The shortest distance from the UK to France is about 32 km.

a How long would it take a swimmer to cross at 2 km/h?

b How long would it take a rower at 5 km/h?

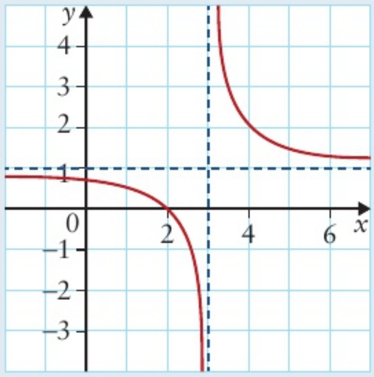
c How long would it take a ferry at 40 km/h?

d How long would it take a hovercraft at 80 km/h?

e Plot a graph of time taken against speed for the answers above and trace a smooth curve through the values.

f Check that a catamaran travelling at 60 km/h has the correct time on your graph.

3 The graph shown is of $y = f(x)$, when $f(x) = \dfrac{x-2}{x-3}$



a What are the equations of the dotted lines (asymptotes)?

b How do the asymptotes correspond to the definition of $f(x)$?

c Use technology to discover whether the graph of the function $g(x) = \dfrac{x}{x-3}$ has the same asymptotes.

d What about the graph of $h(x) = \dfrac{2x}{x-3}$ or $j(x) = \dfrac{2}{x-3}$?

16.1.5 Domain and range of rational functions

Explore 16.6

Use technology to investigate how close the function $g(x) = \dfrac{x+1}{x-1}$ can get to the lines $x = 1$ and $y = 1$

Connections

The link to rational numbers is clear.

Rational functions are formed with expressions $\dfrac{p(x)}{q(x)}$, where p and q are polynomials.

You need to understand and be able to use the following vocabulary when dealing with rational functions. These terms, already introduced, apply to many functions and their graphs, but rational functions exemplify their meaning clearly.

Domain: the set of values of x for which $f(x)$ has a value. When working out the domain, consider if there are any values of x for which $f(x)$ cannot be found. You may have noticed that division by zero indicates a value of x for which a function does not have a value. For example, if $f(x) = \dfrac{1}{x}$ then $f(0)$ has no value, so the domain of f would be all real numbers apart from zero.

Range: the set of possible values of $f(x)$. When working out the range, consider if there are any values that $f(x)$ cannot reach. For example, $a(x)$ in Investigation 16.2 has no solution for $a(x) = 30$. Note too that although there are values of x that can give $a(x) < 30$, they have no meaning in the real-life problem and so are excluded. So the range of a contains all real numbers greater than 30.

An **asymptote** is a straight line that the graph of $y = f(x)$ approaches as the absolute value of either x or y (or both) becomes very large (tends to positive or negative infinity). $y = 1$ on the graph in question 3 is an asymptote.

We are dealing with **real functions of real variables**. In other words both the domain and the range are restricted to sets of real numbers.

The problem with zero

Without using a calculator, work out quickly in your head 'twenty divided by a half'.

If you were asked to work out $20 \div 4$, you would have no problem giving the answer 5, because you know that $5 \times 4 = 20$, or, put another way, that you need four fives to make twenty.

So, how many halves do you need to make twenty? Now check your earlier answer to twenty divided by a half.

What about $20 \div 0$? This is asking how many zeros are needed to make twenty. Would a hundred be enough? A million? There seems to be a problem with dividing by zero. Try $20 \div 0$ on your calculator. How does the calculator respond?

A domain or range is often restricted because of difficulties involving division and the number zero.

- It is impossible to divide by zero, so for the fraction $\frac{1}{n}$, $n \neq 0$. If n gets close to zero then $\frac{1}{n}$ has a big absolute value.
- There is no value of n for which the fraction $\frac{1}{n}$ equals zero.

  Consider the following: $\frac{1}{n} = 0 \Rightarrow 1 = n \times 0 \Rightarrow 1 = 0$. This is an impossible result, which demonstrates that the initial equation $\frac{1}{n} = 0$ is false.

  For $\frac{1}{n}$ to approach zero, n has to have a big absolute value.

Worked example 16.3

a Use technology to plot the graph of $y = f(x)$, where $f(x) = \dfrac{1}{x-3}$

b Which number should be excluded from the domain of f ?

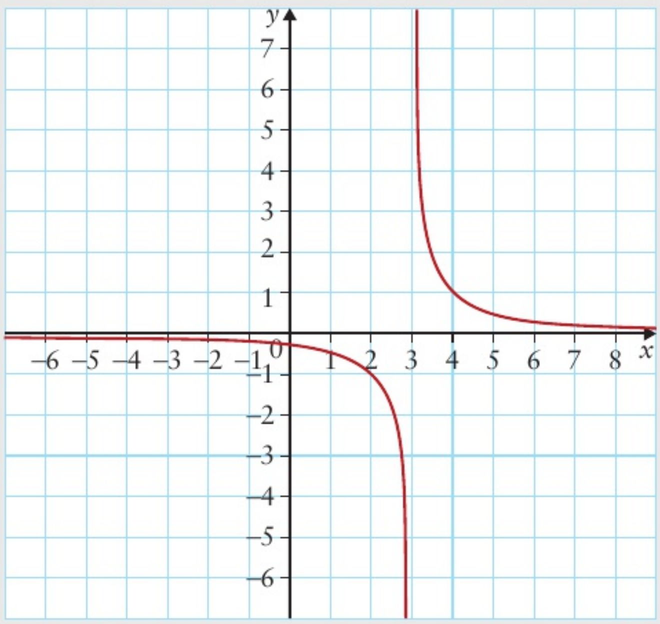
c Which number cannot be found in the range of f ?

d What are the equations of the asymptotes of the graph?

e If x is limited to be an integer, what is the largest possible value of $f(x)$?

Solution

a



b The function contains a denominator of $x-3$, so to avoid division by 0, x cannot be 3. 3 should be excluded from the domain of f.

c The graph suggests that the function cannot give a value of 0. Attempting to solve $f(x) = 0$ gives:

$$\frac{1}{x-3} = 0 \Rightarrow 1 = 0(x-3) \Rightarrow 1 = 0$$

So the assumption $f(x) = 0$ must be false and 0 is not in the range of f.

d $x = 3$ and $y = 0$

e The largest values of $f(x)$ occur in the interval $3 < x \leq 4$, and $x = 4$ is the only integer in that range. So the maximum possible value of $f(x)$ for an integer value of x is $f(4) = 1$

Connections Qc

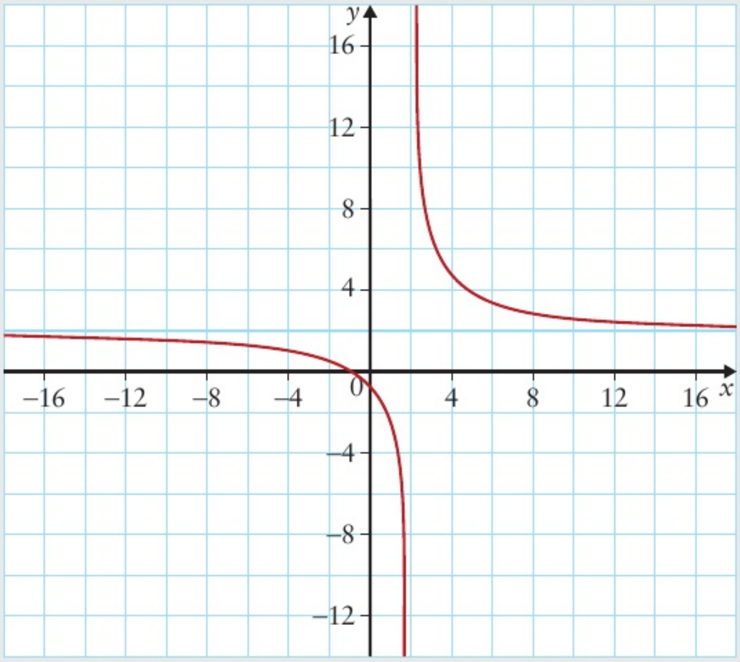
The assumption that $f(x) = 0$ leads logically to the conclusion that $1 = 0$, which is clearly impossible, so the assumption must be false. This is a simple example of **proof by contradiction**.

Worked example 16.4

Repeat steps a to e from Worked example 16.3 for $f(x) = \dfrac{2x+1}{x-2}$

Solution

a



b The function contains a denominator of $x - 2$, so to avoid division by 0, x cannot be 2.

2 should be excluded from the domain of f

c The graph suggests that the function cannot give a value of 2. Attempting to solve $f(x) = 2$ gives:

$$\frac{2x+1}{x-2} = 2 \Rightarrow 2x + 1 = 2(x - 2) \Rightarrow 1 = -4$$

So the assumption $f(x) = 2$ must be false and so 2 is not in the range of f.

d $x = 2$ and $y = 2$

e The largest values of $f(x)$ occur in the interval $2 < x \leq 3$, and $x = 3$ is the only integer in that range. So the maximum possible value of $f(x)$ for an integer value of x is $f(3) = 7$

Reflect

The vertical asymptote is clearly associated with the value of x that causes division by zero, but the horizontal asymptote is less obvious without the graph.

Use technology to experiment with graphs of the form $y = \frac{ax + b}{cx + d}$ for various values of a, b, c and d. Change one at time in whole number steps. Can you find a simple rule for the horizontal asymptote?

Practice questions 16.1.5

For each of the functions in questions 1–6:

a State the value of x that should be excluded from the domain.

b State the value of $f(x)$ that cannot be included in the range.

c Write down the equation of the two asymptotes for the graph of $y = f(x)$

d Sketch the graph and compare your result with one given by technology, such as a GDC.

1 $f(x) = \frac{x - 2}{x - 3}$

2 $f(x) = \frac{2x - 1}{x + 1}$

3 $f(x) = \frac{x}{x + 3}$

4 $f(x) = \frac{x + 1}{2x - 4}$

5 $f(x) = \frac{3}{x - 2}$

6 $f(x) = \frac{3}{2 - x}$

Challenge Q7

Fact

Fluoride in water helps protect teeth against decay. It occurs naturally, but usually not in sufficient concentration to be effective, so many countries and localities add extra fluoride to their water supplies. However, there is a limit above which there may be some detrimental health effects. The World Health Organisation have set a very safe limit of 1.5 mg/litre.

7 A water tower can contain up to 500 000 litres. It is 60% full and the water is found to have 2 mg/litre of fluoride, but the recommended maximum level is 1.5 mg/litre.

Water with a concentration of fluoride of 0.8 mg/litre is added to the tank at a rate of 2000 litres per minute. During this period no water is drained from the tank.

a How much fluoride (in mg) is in the tank at the start?

b How much fluoride is added each minute?

c How much fluoride is in the tank after an hour?

d How much water is in the tank after an hour?

e What is the concentration (in mg/litre) of fluoride after an hour?

f Repeat parts c, d and e for a period of x minutes to find the concentration of fluoride after x minutes.

g Draw a graph of the concentration of fluoride against time.

h Use your graph to help calculate whether the concentration will fall below 1.5 mg/litre before the tank is full.

i What recommendation would you make for the concentration of fluoride in the water that is being added to the tank?

Thinking skills

Communication skills

16.2 Irrational numbers

16.2.1 What are irrational numbers?

Explore 16.7

$3.317^2 = 11.002\,489$

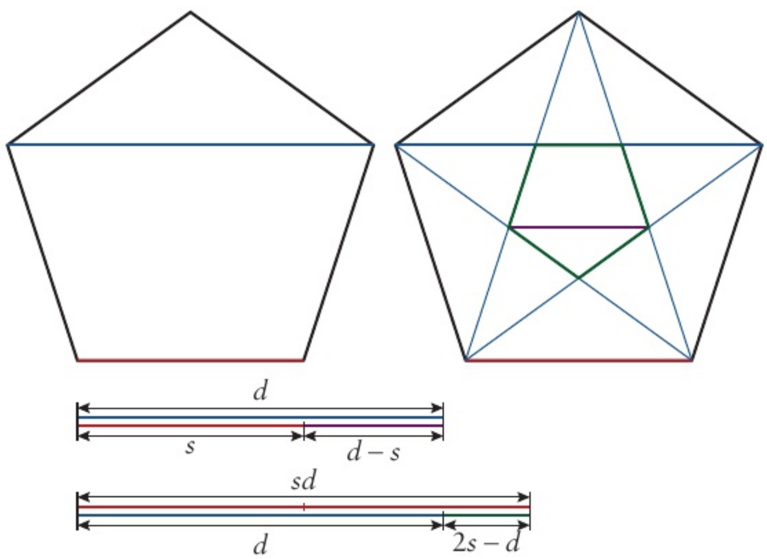
Investigate the possible final digit of a decimal number that has been squared.

What final digit would you get if you squared by hand the decimal number that your calculator displays for $\sqrt{11}$?

What general conclusion could you draw?

An **irrational** number is a real number that is not rational.

Although the Pythagoreans believed that all numbers are rational, their own method of common measures applied to a regular pentagon shows the existence of irrational numbers, using a proof by contradiction.



Connections

2500 years ago, the Pythagorean Hippasus first proved that irrational numbers exist. There is no written record of his proof, but he worked on the properties of regular pentagons and so the outline presented here follows his likely method.

Hippasus' fellow Pythagoreans understood that it undermined their fundamental belief that all numbers are rational. Legend has it that they were so threatened by his findings that they took him out to the middle of the Mediterranean and threw him overboard.

Assume that a side, s, and a diagonal, d, of a regular pentagon have a common measure, m. Draw all the diagonals, which then create a smaller regular pentagon. It can be proved that the diagonal of the new pentagon has length $d - s$ and the side has length $2s - d$. Since a common measure of two lengths also measures multiples and differences of the lengths, m must measure the diagonal and side of the smaller pentagon.

Repeat the process inside the smaller pentagon to create one smaller still. And then keep repeating. The full proof would involve scaling the original picture so that m is a positive integer and hence arguing that m cannot have a value smaller than 1, but as the process of constructing the pentagons goes on forever (called 'infinite descent') the sides and diagonals must eventually depend on a common measure smaller than 1.

Square roots

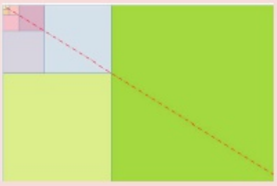
The golden ratio involves square roots. But not all irrational numbers are square roots: π is also an irrational number (proved by Johann Lambert in the 1760s). Eventually mathematicians realised that the irrational numbers are not the exception, but the rule. It is difficult to imagine more than infinity, but there are an infinite number of rational numbers and infinitely more irrational numbers.

Among the many types of irrational number, this chapter deals mainly with square roots and their combinations with rational numbers. A square root that cannot be simplified is referred to as a **surd**.

Summarising number sets

Natural numbers and, more generally, integers can always be written in rational form and so are subsets of the rational numbers. Any real number that is not rational is irrational. In the Venn diagram on the facing page, the examples of irrational numbers are placed outside the set of rational numbers. Beyond the real numbers are imaginary and complex numbers that you may encounter in the future. The square root of minus one, which cannot lie on a real number line, is placed outside the real numbers as a symbol of that greater definition of number.

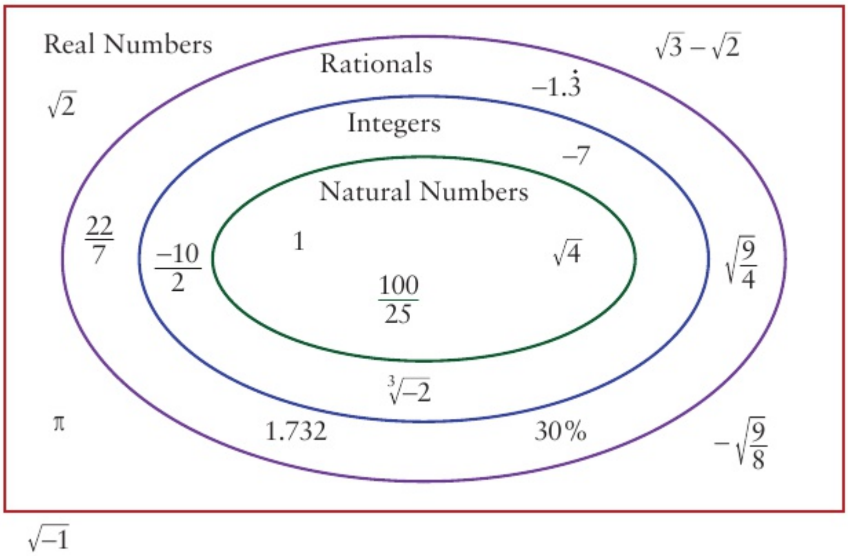
Fact



The length and width of a rectangle exemplify the golden ratio if, when a square of the same width is removed, the length and width of the remaining rectangle have the same ratio as the original.

Hint

The symbol $\sqrt{\ }$ is called a **radical** or **root**.



Reflect

Why is the square root of -1 outside the box? Could there be a number on the real number line whose square is -1?

Not all square roots are irrational. Can you spot the rational numbers in this list?

$\sqrt{7}$ $\quad\sqrt{8}$ $\quad\sqrt{9}$ $\quad\sqrt{0}$ $\quad\sqrt{\frac{2}{3}}$ $\quad\sqrt{\frac{16}{9}}$ $\quad\sqrt{3.6}$ $\quad\sqrt{0.36}$

Challenge

Combining rational numbers and irrational numbers

What happens when you combine rational numbers and irrational numbers?

The sum of a rational number and an irrational number is always irrational. For example, $5 + \sqrt{3}$ is irrational.

Worked example 16.5

Given that $a = \sqrt{2} + 1$ and $b = \sqrt{2} - 1$ are both irrational, determine whether $a - b$ and ab are rational or irrational.

Solution

$a - b = (\sqrt{2} + 1) - (\sqrt{2} - 1) = 2$

$ab = (\sqrt{2} + 1)(\sqrt{2} - 1) = 2 - 1 = 1$

Both $a - b$ and ab are rational.

Connections

The method used here can also be applied to complex numbers that you may encounter later in school. The difference of two squares formula is frequently applied in many areas of mathematics.

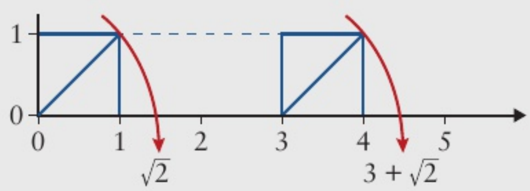
Worked example 16.6

Given that $\sqrt{2}$ is irrational, show that $3 + \sqrt{2}$ is also irrational.

Solution

We can use Pythagoras' theorem to help represent square roots on the real number line.

Study the diagram below. Pythagoras' theorem establishes the length of the diagonal of each square as $\sqrt{2}$. This is then transferred by compass construction (rotation) to the number line as shown.



We know $\sqrt{2}$ does not coincide with a rational number. Therefore $3 + \sqrt{2}$ is not rational either, as its position is a positive translation of 3 from $\sqrt{2}$.

A similar argument can be used for $2\sqrt{2}$ as an enlargement of $\sqrt{2}$.

Can you demonstrate this with a sketch?

Practice questions 16.2.1

Hint Q1

Use a calculator to check your answers.

1 Which of the following are rational numbers?

a $\sqrt{3}$ **b** $\sqrt{49}$ **c** $\sqrt{3} + 1$

d $\sqrt{2} \times \sqrt{8}$ **e** $(\sqrt{2})^2$ **f** $(\sqrt{2})^3$

g $(\sqrt{3} + 1)(\sqrt{3} - 1)$

2 a and b are irrational numbers. Which one of the following must be an irrational number? Give examples of values of a and b that could give rational answers for the other parts.

a $a + b$ **b** $\frac{a}{b}$ **c** a^2 **d** $\sqrt{a}$

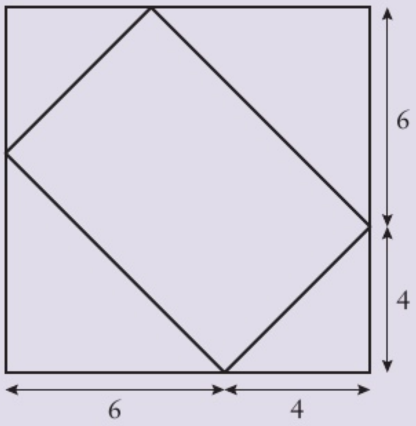
3 c is rational and d is irrational. Which, if any, of the following could be rational? Give examples.

a cd **b** $c + d$ **c** $(c + d)^2$ **d** cd^2

16.2.2 Products and quotients of surds

Explore 16.8

The picture shows a rectangle drawn inside a square.



Find the area of the square and of the four triangles, and hence the area of the rectangle.

Use Pythagoras' theorem to find the dimensions (length and width) of the rectangle as exact surds.

How are the answers to parts a and b connected?

Repeat the calculations for a rectangle drawn in the same way but dividing the edge of the square in the ratio 3 : 7

Repeat for other divisions and draw a general conclusion.

Rules for products and quotients of surds

For rational numbers a and b, there are two general rules:

$\sqrt{a}\sqrt{b} = \sqrt{ab}$ and $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

These rules allow us to simplify surds, with the aim of reducing the size of the numbers involved. For example:

$$\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$$

$$\frac{\sqrt{63}}{\sqrt{7}} = \sqrt{\frac{63}{7}} = \sqrt{9} = 3$$

Reflect

Can you see where the commutative and associative laws were used in the proof in the Fact box?

Fact

To prove general results like this, we can use the commutative law:
$ab = ba$ and the associative la:w
$a(bc) = (ab)c$

Using the two laws:
$x^2y^2 = xxyy = xyxy = (xy)^2$

Now consider $\sqrt{a}\sqrt{b}$, where $a \geqslant 0$ and $b \geqslant 0$

Let $\sqrt{a} = x$ and $\sqrt{b} = y$

Then $a = x^2$ and $b = y^2$

So $ab = x^2y^2 = (xy)^2$
$= (\sqrt{a}\sqrt{b})^2$

Taking square roots:
$\sqrt{ab} = \sqrt{a}\sqrt{b}$

Worked example 16.7

Simplify $\sqrt{200}$

Solution

Understand the problem

Simplifying means reducing the problem to something more manageable. As we are taking a square root, we look for square factors.

Make a plan

200 has a number of square factors, but to be efficient we should look for the largest one.

Carry out the plan

$200 = 100 \times 2$

So $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$

Look back

Can you now do a similar problem in your head?

Worked example 16.8

Simplify $\dfrac{\sqrt{5}}{\sqrt{20}}$

Solution

Understand the problem

There is a factor of 5 that would be cancelled in a rational fraction, but the surds must be dealt with first.

Make a plan

Applying the rule $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ will create a simple fraction inside the square root to allow cancellation.

Carry out the plan

$$\frac{\sqrt{5}}{\sqrt{20}} = \sqrt{\frac{5}{20}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Look back

The last step uses $\dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{4}$

Worked example 16.9

Write $3\sqrt{5} + \sqrt{125}$ in the form $a\sqrt{b}$

Solution

There are two square roots, but 5 and 125 have something in common as powers of 5.

$\sqrt{125} = \sqrt{25 \times 5} = 5\sqrt{5}$

So $3\sqrt{5} + \sqrt{125} = 3\sqrt{5} + 5\sqrt{5} = 8\sqrt{5}$

Practice questions 16.2.2

1 Simplify:

a $\sqrt{27}$ b $\sqrt{72}$ c $\dfrac{\sqrt{90}}{\sqrt{10}}$

d $\dfrac{\sqrt{2}}{\sqrt{8}}$ e $\dfrac{\sqrt{37}}{\sqrt{333}}$ f $\dfrac{\sqrt{88}}{\sqrt{11}}$

g $\sqrt{a^2b^2}$ h $\dfrac{\sqrt{n^3}}{\sqrt{4n}}$ i $(\sqrt{2})^2$

j $(\sqrt{ab})^2$ k $(\sqrt{3} + 1)(\sqrt{3} - 1)$ l $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

m $\sqrt{a^2b}$ n $\sqrt{m^3}$ o $\dfrac{\sqrt{4x^5}}{2x^2}$

2 Write in the form $a\sqrt{b}$, where a and b are integers and b is prime.

a $\sqrt{32}$ b $\sqrt{3} + \sqrt{27}$ c $\sqrt{80} + \sqrt{125} - \sqrt{180}$

3 Without a calculator, simplify:

a $\sqrt{12}$ b $\sqrt{20}$

c $\sqrt{128}$ d $\sqrt{63}$

e $\sqrt{98}$ f $\sqrt{405}$

g $\sqrt{45} - 2\sqrt{20} + \sqrt{125}$ h $\sqrt{48} + \sqrt{12} - 2\sqrt{27}$

i $\sqrt{490} - \sqrt{250} + 2\sqrt{40}$ j $3\sqrt{8} + 2\sqrt{18} - \sqrt{50}$

4 Expand the following sets of brackets.

a $(\sqrt{5} + 1)^2$ b $(1 + \sqrt{2})(1 + \sqrt{3})$

c $(2\sqrt{2} - \sqrt{3})^2$ d $(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})$

e $(1 + \sqrt{n})^2$ f $(\sqrt{n} + \sqrt{2})(\sqrt{n} - \sqrt{2})$

16.2.3 Rationalising the denominator

Explore 16.9

Without a calculator, but knowing that $\sqrt{2} \approx 1.414$, find an approximation for each of $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$. Then try with a calculator.

Consider further examples based on:
$\sqrt{3} \approx 1.732$ $\sqrt{5} \approx 2.236$ $\sqrt{11} \approx 3.317$

The process of removing surds from the denominator is called **rationalising the denominator** (in other words, making the denominator a rational number). Rationalising the denominator frequently uses two facts:

$(\sqrt{x})^2 = x$

and

$(a + b)(a - b) = a^2 - b^2$, where a or b or both could be surds. This method uses the difference of two squares.

Worked example 16.10

Show that $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Solution

First consider the easier problem of writing $\frac{3}{5}$ as a decimal. One way of doing this is to change to tenths, since the first place of a decimal is tenths:

$$\frac{3}{5} = \frac{3}{5} \times \frac{2}{2} = \frac{6}{10} = 0.6$$

Multiplying by $\frac{2}{2}$ is equivalent to multiplying by 1, so the fraction does not change size, but is rewritten in a form that is easier to change into a decimal number.

We can rationalise $\frac{1}{\sqrt{3}}$ in a similar way, multiplying by a fraction that is equal to 1, but which changes the denominator into a rational number.

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{(\sqrt{3})^2} = \frac{\sqrt{3}}{3}$$

Looking back, we can see that multiplying by $\frac{\sqrt{3}}{\sqrt{3}}$ is equivalent to multiplying by 1.

In general, $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$

Worked example 16.11

Rationalise the denominator in $\dfrac{1}{\sqrt{3}-\sqrt{2}}$

Solution

There are two different surds in the denominator. A different method is needed to eliminate both surds.

Because $(\sqrt{3}-\sqrt{2})$ is of the form $(a-b)$ where a or b or both could be a surd, we can use the difference of two squares to rationalise it.

$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 3-2 = 1$

So, $\dfrac{1}{\sqrt{3}-\sqrt{2}} = \dfrac{1}{\sqrt{3}-\sqrt{2}} \times \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \dfrac{\sqrt{3}+\sqrt{2}}{1} = \sqrt{3}+\sqrt{2}$

Looking back, we can use a calculator to confirm that $\dfrac{1}{\sqrt{3}-\sqrt{2}}$ and $\sqrt{3}+\sqrt{2}$ have the same value. Remember that simpler calculators may require the brackets around the denominator.

Worked example 16.12

Rationalise the denominators in:

a $\dfrac{1}{\sqrt{a}-\sqrt{b}}$ **b** $\dfrac{1}{\sqrt{a}+b}$

Solution

These examples require both $\sqrt{a}\sqrt{a} = a$ and the difference of two squares method.

a $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$

So $\dfrac{1}{\sqrt{a}-\sqrt{b}} = \dfrac{1}{\sqrt{a}-\sqrt{b}} \times \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \dfrac{\sqrt{a}+\sqrt{b}}{a-b}$

b $(\sqrt{a}+b)(\sqrt{a}-b) = a-b^2$

So $\dfrac{1}{\sqrt{a}+b} = \dfrac{1}{\sqrt{a}+b} \times \dfrac{\sqrt{a}-b}{\sqrt{a}-b} = \dfrac{\sqrt{a}-b}{a-b^2}$

Practice questions 16.2.3

1 Reflecting on the examples you have seen so far, which statement do you think is true in each of the following? Illustrate your answers by providing an example.

a The sum of two rational numbers is:
 i always rational ii sometimes irrational.

b The sum of two irrational numbers is:
 i always irrational ii sometimes rational.

c The product of two irrational numbers is:
 i always rational ii always irrational.
 iii sometimes irrational

d The sum of a rational number and an irrational numbers is:
 i always irrational ii sometimes rational.

2 Rationalise the denominators in the following, checking your answers by obtaining decimal approximations on your calculator.

a $\dfrac{1}{\sqrt{2}}$ b $\dfrac{5}{\sqrt{5}}$ c $\sqrt{\dfrac{4}{3}}$

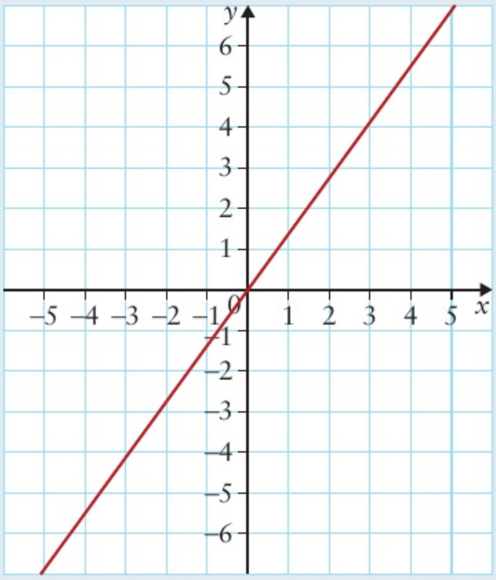
d $\dfrac{1}{2-\sqrt{3}}$ e $\dfrac{1}{3-\sqrt{2}}$ f $\dfrac{2}{\sqrt{5}+\sqrt{3}}$

3 Rationalise the denominators in each of these expressions.

a $\dfrac{1}{\sqrt{a}}$ b $\dfrac{n\sqrt{m}}{\sqrt{nm}}$ c $\dfrac{1}{\sqrt{a}-1}$

d $\dfrac{1}{\sqrt{x}+\sqrt{y}}$ e $\dfrac{\sqrt{p}+\sqrt{q}}{\sqrt{p}-\sqrt{q}}$

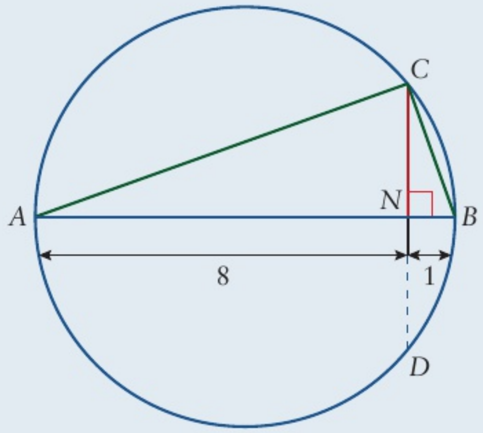
4 The diagram shows the graph of $y = \sqrt{2}\,x$
It passes through (0, 0).
If the graph is extended indefinitely, will it pass through any other intersections of grid lines? Justify your answer.



5 The two shorter sides of a right-angled triangle are $\sqrt{n} + \sqrt{3}$ and $\sqrt{n} - \sqrt{3}$

Given that both the hypotenuse and n are integers, what possible values of n are less than 50?

6 AB is the diameter of a circle.



a Calculate the exact lengths of NC, AC and BC.

b Simplify any answers involving surds.

Hint Q6

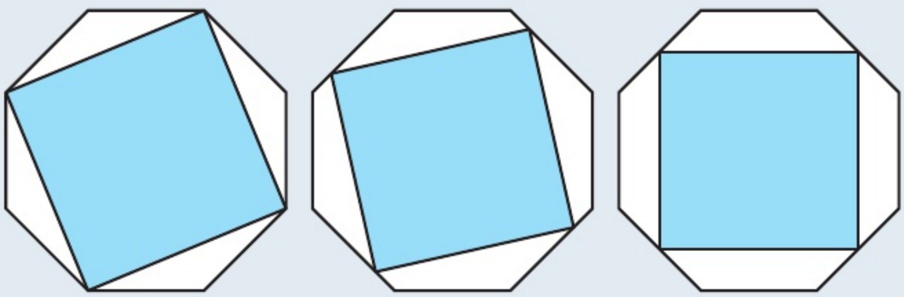
You could use similar triangles or the intersecting chord theorem to find NC from the chords AB and CD.

Reminder

Look back at Chapters 8 and 11 for facts and theorems about similar triangles and circle geometry.

7 This question leads you to establish the area of the largest square that can be drawn inside a regular octagon of side 1 unit. Your answer should be exact. To have an exact answer you need to keep surds in your calculations and not approximate them with decimal fractions.

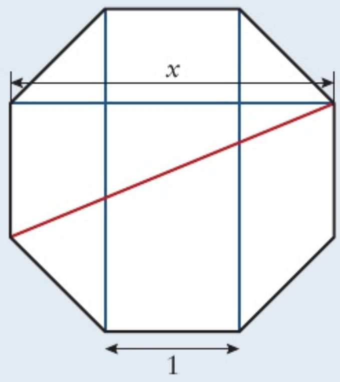
Which of these three pictures has the largest square? Drawing the diagonals of the squares may help you to provide a convincing argument.



Challenge Q7

Thinking skills

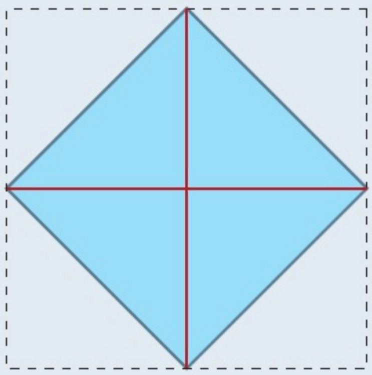
Use the picture of the octagon below to calculate the length x and hence the length of the longest diagonal of the octagon.



Look at the picture of the square below to consider whether there is a quick way to find the area of a square from the length of its diagonals. Hence, find the area of the largest square that can be drawn inside a regular octagon of side 1 unit.

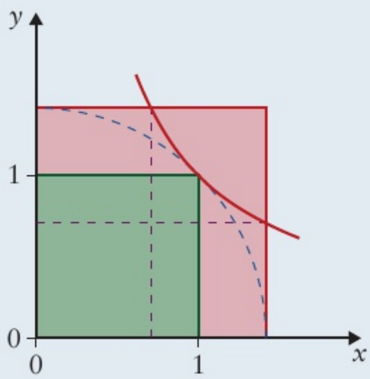
Hint Q7

The same formula works for a kite and a rhombus.



Challenge Q8

8 The dashed curve is a quarter of a circle and the dashed lines divide the large square into four equal parts. What do you think is the equation of the solid curve? Why?



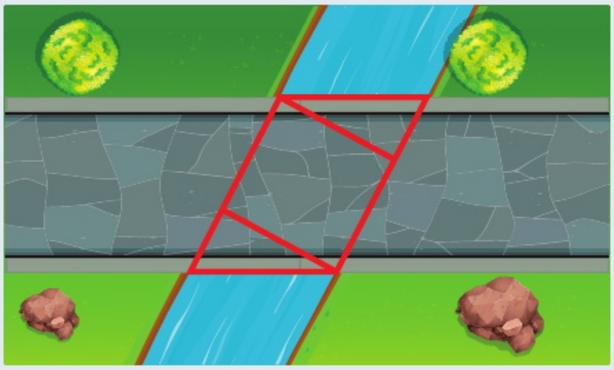
Challenge Q9

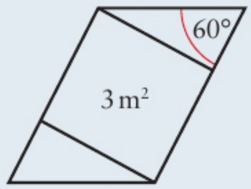
9 A small tunnel is needed to allow a stream to pass under a footpath at an angle of 60° as shown.

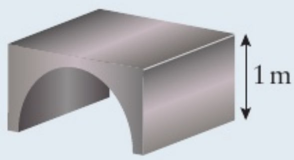
The main block of the tunnel is a prism of concrete with a square top surface of area $3\,\text{m}^2$ and a rectangular cross section of height 1 m, out of which is cut a semi-circular section of area $1\,\text{m}^2$ to allow enough water to flow.

To complete the tunnel so that the water flows straight through, the main block is extended with two pieces taken from a second identical block, cut to give triangular top surfaces as shown. The concrete has a density of 2.4 tonnes per m^3.

Find the exact weight of the total concrete structure.







16 Rational and irrational expressions and functions

Self assessment

- I can express numbers as whole number multiples of a common measure.
- I know that the ratio of such whole number multiples is called a rational number and therefore rational numbers can be expressed as simple fractions.
- I can recognise algebraic rational expressions from the expressions that make up their numerator and denominator.
- I can find common denominators of rational expressions and use them to add, subtract and compare the expressions.
- I know that division by zero is impossible and can use this to limit the domain of a function. I know that near the value that would cause division by zero, a rational function can reach positive and negative numbers with a very large absolute value (approaching infinity).
- I know that for simple rational expressions there may be values that the function approaches but does not reach for large absolute values of the variable.
- I can recognise graphs of simple rational functions of the form $f(x) = \dfrac{ax+b}{cx+d}$ and can identify the equations of the vertical and horizontal asymptotes.
- I understand that surds are examples of irrational numbers, meaning that they cannot be written as simple fractions.
- I know that the sum of a rational number and an irrational number is always irrational.
- I know that the sum or product of irrational numbers may be rational, particularly for the pair $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$
- I can simplify surds that contain square factors and quotients of surds with common factors.
- I know the two techniques for rationalising denominators involving surds.
- I can apply the techniques for surds to both numerical and algebraic expressions.

Check your knowledge questions

1 Find the common measure of:

a $\dfrac{3}{5}$ and $\dfrac{4}{5}$ b $3\dfrac{1}{3}$ and $4\dfrac{1}{6}$ c $\sqrt{8}$ and $\sqrt{18}$

2 Write as a single fraction:

a $\dfrac{1}{x}+\dfrac{1}{y}$ b $2-\dfrac{3}{n}$ c $p+\dfrac{1}{p}$

d $\dfrac{1}{x-1}-\dfrac{1}{x}$ e $\dfrac{2x}{x^2-1}-\dfrac{1}{x-1}$

3 For each of the following functions, state which numbers are excluded from:

i the domain ii the range.

a $f(x) = \dfrac{1}{x}$ b $f(x) = \dfrac{x-5}{x-4}$

c $f(x) = \dfrac{3-x}{x}$ d $f(x) = \dfrac{2x}{x-2}$

4 Find, without technological aid, equations for the asymptotes of the graphs of $y = f(x)$ for the following functions.

a $f(x) = \dfrac{2}{x - 1}$ b $f(x) = \dfrac{2x}{x - 1}$ c $f(x) = \dfrac{x + 1}{x - 1}$

5 Use technology to graph the function $f(x) = \dfrac{1}{x^2 + 1}$ and state the range of the function.

6 You are organising a reception with food for a large group. For €1000 you can rent a room that will take up to 150 people. Catering will cost €30 per person. You have no other costs.

a If 100 people are attending, how much will it cost per person?

b Write down a formula for $C(x)$, where $C(x)$ is the cost per person for x people.

c What is the domain of C?

d Plot a graph of $y = C(x)$

e If you do not want the reception to cost more than €35 per person, what is the minimum number of people that should attend?

f Write down the range of values of C and explain the significance of the lower limit of the range.

7 A decorating company need 400 litres of paint for the interior of a new hotel. They would like a custom mix for a pale blue colour.

The paint manufacturer pours paint into a large container, controlling the flow to allow mixing. So far 80 litres have been mixed using 25% blue paint and 75% white paint.

The decorator inspects the mixture and decides it is too dark. They would like just 15% concentration of blue in the mix.

a The manufacturer considers correcting the mixture by now adding white paint at 9 litres per minute and blue at 1 litre per minute.

i What would be the concentration of blue paint 5 minutes later?

ii How long would it take to reach the 15% concentration required?

iii How much paint would have been produced by then?

b The manufacturer decides instead to correct the mixture by adjusting the flow of white paint once, and then making no further changes. Once the 400 litres are complete, the concentration will be 15%.

  i If blue paint is still added at 1 litre per minute, what rate would be required for the white paint?

  ii Plot a graph of concentration of blue paint against time for this rate, up to the point where all the paint is prepared.

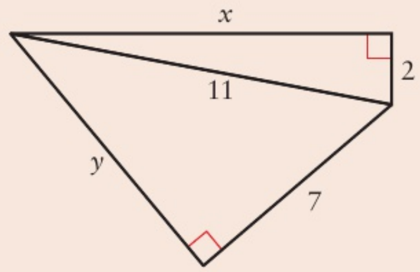
8 Which of the following are irrational?

a $\sqrt{196}$ **b** $\sqrt{1000}$

c $\dfrac{\sqrt{7}}{\sqrt{28}}$ **d** $(\sqrt{3}-\sqrt{2})^2$

e $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$ **f** $\dfrac{(2\sqrt{2}+3)(\sqrt{2}-1)}{\sqrt{2}+1}$

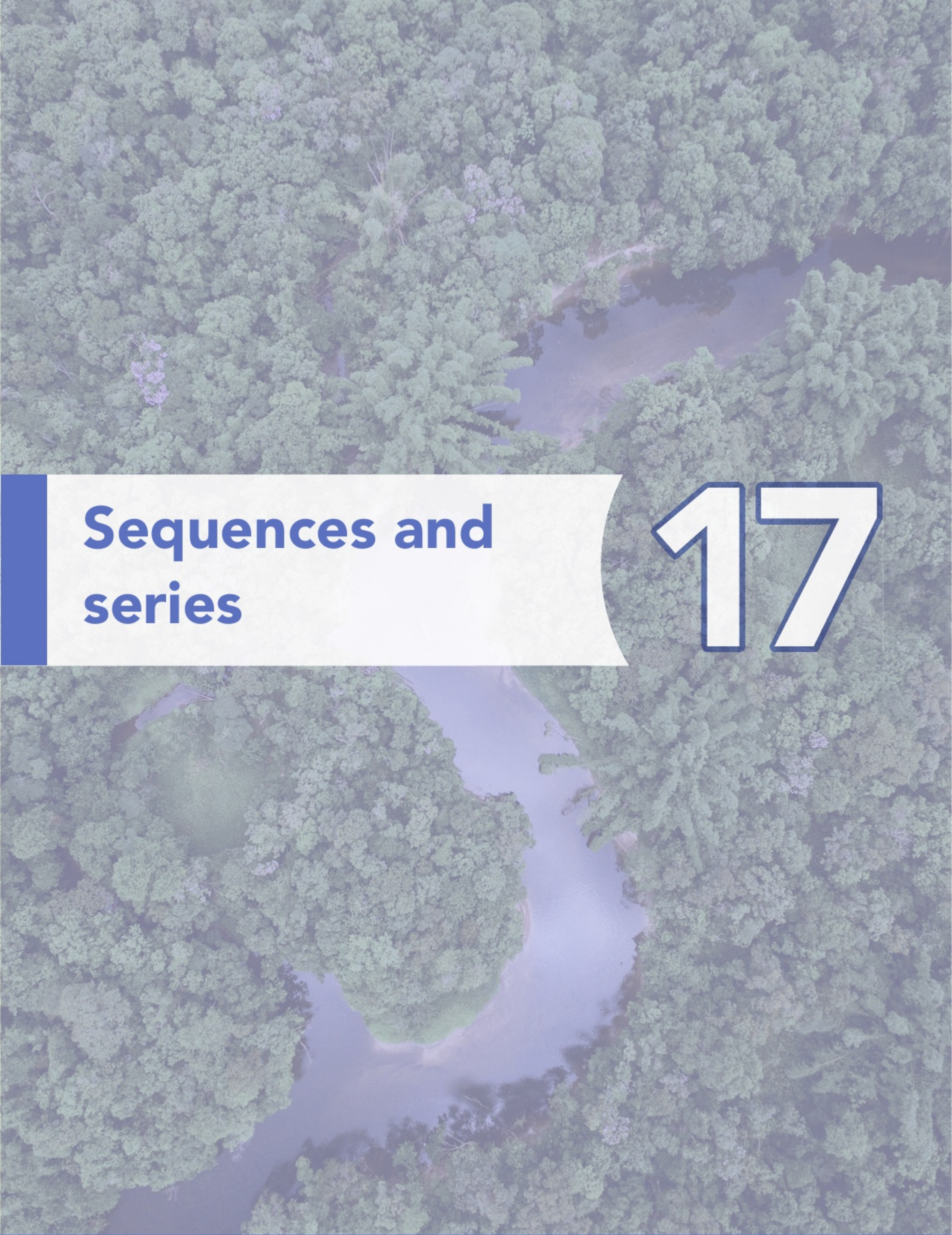
9 Find the exact values of x and y. Give your answers in the form $a\sqrt{b}$, where a and b are integers.



10 Rationalise the denominators in each of the following.

a $\dfrac{3}{\sqrt{6}}$ **b** $\dfrac{\sqrt{2}}{\sqrt{3}}$ **c** $\dfrac{\sqrt{a}}{\sqrt{b}}$

d $\dfrac{2}{3-\sqrt{7}}$ **e** $\dfrac{\sqrt{13}+\sqrt{11}}{\sqrt{13}-\sqrt{11}}$ **f** $\dfrac{\sqrt{a}-1}{\sqrt{a}+1}$



17 Sequences and series

17 Sequences and series

KEY CONCEPT

Relationships

RELATED CONCEPTS

Generalisation, Patterns

GLOBAL CONTEXT

Globalisation and Sustainability

Statement of inquiry

Understanding relationships by identifying and generalising patterns can help us to predict possible scenarios and preserve fragile natural resources.

Factual

- What is a sequence? What is an arithmetic sequence? What is a geometric sequence?
- What is the *n*th partial sum of a sequence?
- What real-life problems can be solved using sequences and series?

Conceptual

- How do you recognise if a sequence is arithmetic or geometric?
- What is the difference between recursive and explicit formulae?

Debatable

- Is it always possible to find an explicit formula for any sequence?
- Are there any differences between the terms ‘pattern’ and ‘sequence’?

Do you recall?

1 Use the laws of indices to evaluate:

a $5^2 \times 5^4$

b $3^2 \times 4^2$

c $7^8 \div 7^2$

d $(2^3)^4$

2 Solve this system of simultaneous equations.

$2x + 3y = 15$

$x - 3y = 3$

3 Evaluate:

a $2x^2 - 3y + 4x$, when $x = -2$ and $y = 3$

b $3n + 4$, when $n = 7$

c 4×5^n, when $n = 3$

4 Solve:

a $6n + 14 = 2$

b $n^2 - 7n = -10$

17.1 From patterns to generalisations

The way humans learn is built on patterns. We learn through repetition and recognition of regularity. Many mathematicians and philosophers say that mathematics is the study of patterns. Studying patterns gives opportunities to observe, make conjectures, predict and then generalise.

Explore 17.1

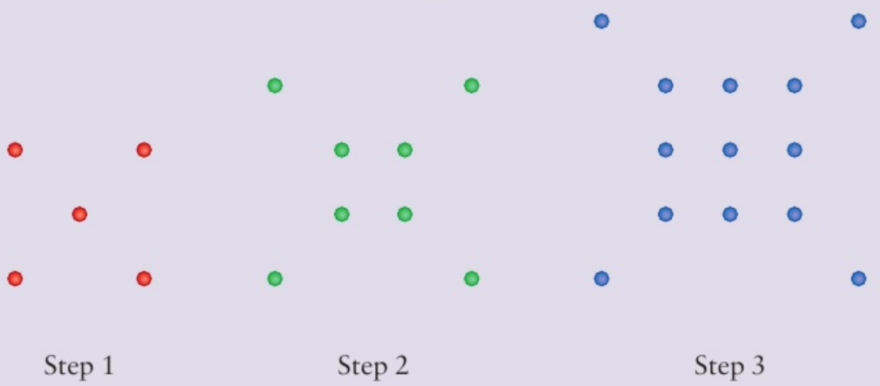
What comes next? Describe three different ways of continuing the following list of numbers by giving the next three numbers: 1, 2, 4, …

Explore 17.2

How do you see the pattern in this sequence growing?

Draw the next two steps of the sequence and complete the table.

How many dots will there be in the 100th step?



Step	Number of dots
1	5
2	
3	
4	
5	
100	

A **sequence** is a list of items whose order (position in the sequence) follows a pattern or rule. The items can be numbers, objects or pictures. Each item in the list is called a **term**.

Sequences are usually written using subscript notation to denote the first term, second term and so on.

For example, for the numerical sequence 1, 4, 9, 16

The first term is $u_1 = 1$

The second term is $u_2 = 4$

The third term is $u_3 = 9$

The fourth term is $u_4 = 16$

The whole of a sequence $u_1, u_2, u_3, \ldots$ is usually written as $\{u_n\}$, for $n \in \mathbb{Z}^+$

Connections

For computer programmers it can be convenient to start numbering a sequence with 0 instead of 1, so that the terms are $u_0, u_1, u_2, \ldots$

Reminder

$\mathbb{Z}^+$ stands for the set of all positive integers.

$\mathbb{N}$ stands for the set of all the natural numbers (including zero).

Reflect

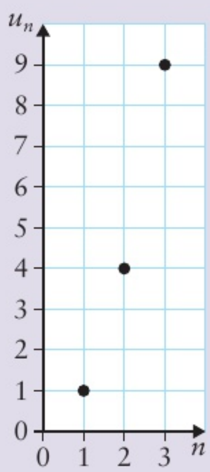
Why is it convenient to denote a sequence using subscripts? Can you think of another convenient notation? When is subscript notation used in other areas of mathematics?

Explore 17.3

Communication skills

In the grid below the first three terms of a sequence $\{u_n\}$ are graphed. What are the values of u_1, u_2 and u_3?

Can you describe how functions and sequences are related?



In this chapter we will look at sequences generated by well-defined mathematical rules or formulas. There are two main types: explicit formulae and recursive definitions.

An **explicit formula** (or ***n*th term rule**) links the value of the term to the term's position n in the sequence.

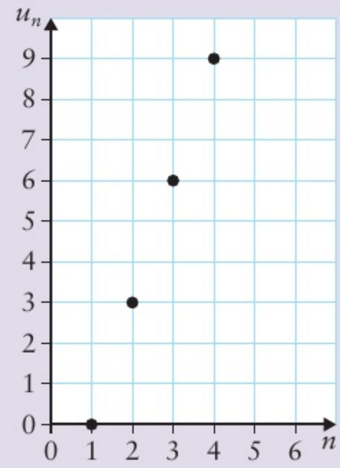
For example, the formula $u_n = n^2$, for $n \in \mathbb{Z}^+$, generates the following sequence by substituting $n = 1, 2, 3, 4, \ldots$ into the formula:

$u_1 = 1^2 = 1$, $u_2 = 2^2 = 4$, $u_3 = 3^2 = 9$, $u_4 = 4^2 = 16$ and so on.

Thinking skills

Explore 17.4

Can you find the explicit formula of the sequence represented in the graph?



Reminder

You can use a graphing utility in sequence mode to insert a sequence and then use the table feature to check the terms of the sequence and their position.

Worked example 17.1

A sequence is defined by $u_n = 3n - 4$, for $n \in \mathbb{Z}^+$

a List the first, second and eighth terms of the sequence.

b Determine whether or not 218 and 220 are terms of the sequence.

Solution

a We need to find u_1, u_2 and u_8

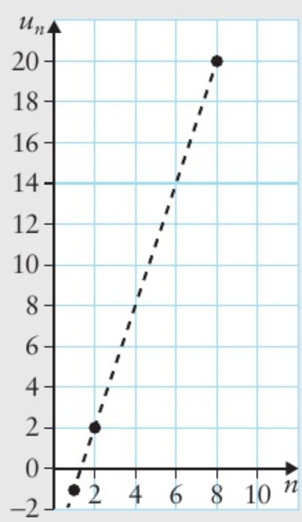
To do this, substitute $n = 1$, 2 and 8 into the formula.

$u_1 = 3 \times 1 - 4 = -1$

$u_2 = 3 \times 2 - 4 = 2$

$u_8 = 3 \times 8 - 4 = 20$

The points are on a straight line so the sequence generated is linear. Note that the terms of the sequence increase by 3 as n increases by 1 (look at the slope of the line).



b The first question can be rephrased as: can you find a positive integer n such that $u_n = 218$?

We solve the equation $3n - 4 = 218$ for n:

$3n - 4 = 218$

$3n = 222$

$n = 74$

So 218 is the 74th term of this sequence.

Similarly, to find if 220 is a term of the sequence we solve the equation $3n - 4 = 220$ for n:

$3n - 4 = 220$

$3n = 224$

$n = 74.67$

Since n is not an integer, 220 is not a term of the sequence.

For $n = 74$, $u_{74} = 218$, and for $n = 75$, $u_{75} = 221$

A **recursive definition** (or **recursive formula**) of a sequence relates new terms to previous terms in the sequence. To define a sequence recursively, you need to be given one or more of the first terms, so that you can work out the next ones.

For example, the second term of the sequence $\{u_n\}$ defined recursively by $u_n = 2 \times u_{n-1} - 1$ and $u_1 = 2$ can be found by replacing u_{n-1} in the recursive formula with $u_1 = 2$:

$u_2 = 2u_1 - 1 = 2 \times 2 - 1 = 3$

The third term can be found by replacing u_{n-1} with $u_2 = 3$:

$u_3 = 2u_2 - 1 = 2 \times 3 - 1 = 5$

And so on.

Fact

Recursive definitions can also be called term-to term rules.

Explore 17.5

The sequence 2, 5, 11, 23, 47, 95, … is defined recursively. Can you find the recursive definition?

Worked example 17.2

a A sequence is defined as $u_n = 3 \times u_{n-1} + 1$, where $u_1 = 4$
Find the second, third, and fourth terms of the sequence.

b A sequence is defined as $u_n = u_{n-1} + u_{n-2}$, where $u_1 = 1$ and $u_2 = 1$
Find the third, fourth and fifth terms of the sequence.

Solution

a The sequence is defined recursively: each term (from u_2 on) is 3 times the previous term plus 1.

Substitute the previous term in the recursive formula to get the next term.

$u_2 = 3u_1 + 1 = 3 \times 4 + 1 = 13$

$u_3 = 3u_2 + 1 = 3 \times 13 + 1 = 40$

$u_4 = 3u_3 + 1 = 3 \times 40 + 1 = 121$

The sequence is 4, 13, 40, 121, …

b The sequence is defined recursively: each term is the sum of the previous two terms, from u_3 on.

Substitute the previous terms in the recursive formula to get the next one.

$u_3 = u_1 + u_2 = 1 + 1 = 2$

$u_4 = u_3 + u_2 = 2 + 1 = 3$

$u_5 = u_4 + u_3 = 3 + 2 = 5$

The sequence is 1, 1, 2, 3, 5, …

Reflect

In Worked example 17.2, why was only u_1 given in part a but u_1 and u_2 given in part b?

Investigation 17.1

Research skills

The recursive sequence in part b of Worked example 17.2 (1, 1, 2, 3, 5, …) is the famous **Fibonacci sequence**. Research the many applications of it in nature, such as the role it plays in the arrangement of leaves, branches, flowers or seeds in plants, in evolution or in abstract mathematics or in links to the golden ratio.

Practice questions 17.1

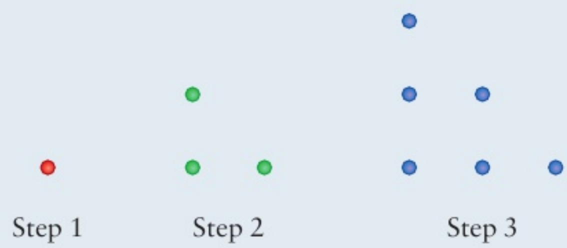
In questions 1–4, the first three terms are shown for each sequence.

a Draw the next three terms.

b How many dots will there be in the 100th term?

c How many dots will there be in the *n*th term? Can you find an explicit formula that describes how the pattern grows?

1



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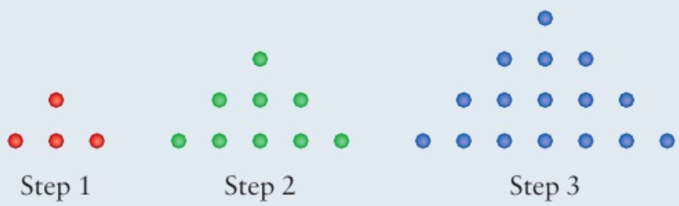

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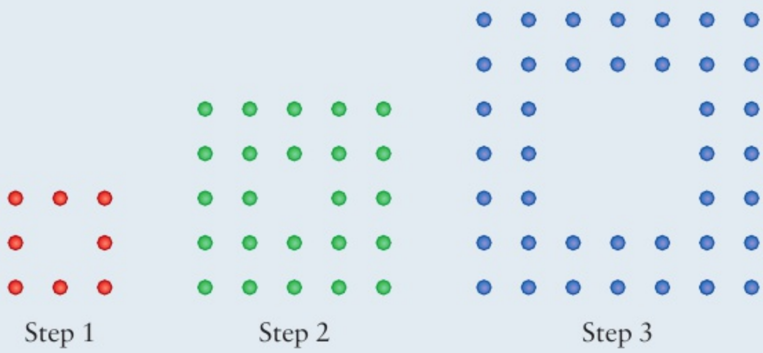

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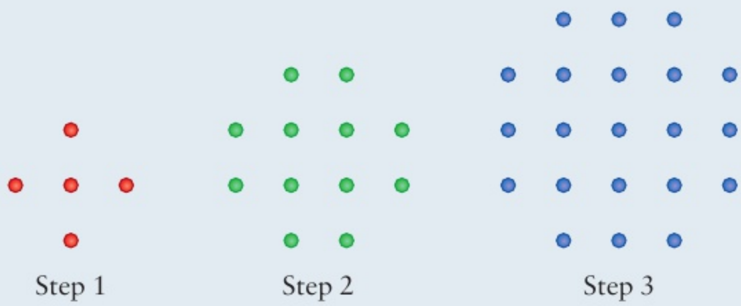


5 Continue each sequence in at least two different ways.

a $1, 2, 3, \dots$ b $2, 4, 6, \dots$ c $1, 4, 8, \dots$

6 Find the first four terms of the following sequences.

a $u_n = 4n - 2$ b $u_n = -6n + 4$ c $u_n = n^2 + n - 1$

d $u_n = n^3$ e $u_n = \sqrt{n} + 3$ f $u_n = n^n$

g $u_n = (1 + n^{-1})^n$ h $u_n = (-1)^n\left(\frac{3}{4}\right)^n$ i $u_n = \cos(180° \times n)$

7 Find the first four terms of the following sequences.

a $u_n = 3u_{n-1} - 4, u_1 = -7$

b $u_n = 2u_{n-1} + 1, u_1 = 0.5$

c $u_n = 4u_{n-1} - 2, u_1 = 3$

d $u_n = u_{n-1}^2, u_1 = 1$

e $u_n = u_{n-1}^2, u_1 = 0$

f $u_n = -u_{n-1} + 2u_{n-2}, u_1 = 1$ and $u_2 = 3$

g $u_n = u_{n-1} \times u_{n-2}, u_1 = 1$ and $u_2 = 2$

h $u_n = u_{n-1} \div u_{n-2}, u_1 = 2$ and $u_2 = 4$

8 Find a recursive definition for each sequence.

a $4, 7, 10, 13, \dots$ b $3, 5, 7, 9, \dots$

c $4, 8, 16, 32, \dots$ d $1, 5, 25, 125, \dots$

e $1, 2, 3, 5, \dots$ f $2, \sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2}, \dots$

9 Find an explicit formula for each sequence.

a $4, 7, 10, 13, \dots$ b $3, 5, 7, 9, \dots$

c $-1, -5, -9, -13, \dots$ d $2, 5, 10, 17, \dots$

e $-2, 2, -2, 2, \dots$ f $-\frac{2}{5}, \frac{2}{25}, -\frac{2}{125}, \frac{2}{625}, \dots$

10 The third term of a sequence is 152. The sequence is defined recursively by $u_n = 3u_{n-1} - 1$, for $n = 2, 3, \dots$ Find the first and second terms.

11 Consider the sequence 2, 7, 12, 17, …

a Find an explicit formula for the sequence.

b Find the 23rd term.

c Is 352 a term of the sequence? If so, what is its position in the sequence?

12 Consider the sequence $u_n = 2^{n-1}$

a Find the 4th, 6th and 8th terms.

b Is 2048 a term of the sequence? Explain how you found your answer.

13 Consider the sequence $u_n = n^2 + n$

a Find the 7th term.

b Is 764 a term of the sequence? Explain how you found your answer.

17.2 Series and sigma notation

Sometimes we need to calculate the sum of the terms of a sequence.

For example, if we have a sequence, then we can retrieve a series from this sequence by considering the following partial sums:

Fact

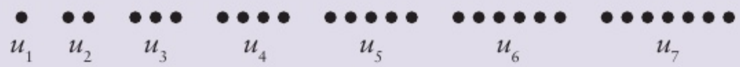
The sum of a finite sequence is called a finite series. The notation S_n is often used to indicate the sum of the first n terms (S_n is also called the nth partial sum). If the sequence is infinite, then the resulting series is called an infinite series. In this context, finite means a specific number of terms, such as 5, and infinite means that it goes on forever.

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
.
.
$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

A series is a type of sequence, the sequence of **partial sums**.

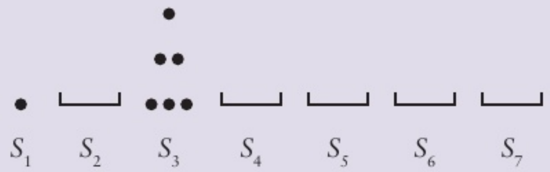
Explore 17.6

Consider the following sequence.



Can you identify this sequence?

Now, here is a part of a series:



Can you complete this series? Can you describe the partial sums?

Communication skills

Worked example 17.3

Work out the partial sums S_1, S_2, S_3, S_4 for the sequence $u_n = 2n - 2$

Solution

The problem asks for the first term (S_1), the sum of the first two terms (S_2), the sum of the first three terms (S_3) and the sum of the first four terms (S_4).

First we will calculate u_1, u_2, u_3, u_4. Then we will add them together: the first two terms, then the first three terms, then the first four terms.

$u_1 = 2 - 2 = 0$

$u_2 = 2 \times 2 - 2 = 2$

$u_3 = 2 \times 3 - 2 = 4$

$u_4 = 2 \times 4 - 2 = 6$

Thus,

$S_1 = u_1 = 0$

$S_2 = u_1 + u_2 = 0 + 2 = 2$

$S_3 = u_1 + u_2 + u_3 = 2 + 4 = 6$

$S_4 = u_1 + u_2 + u_3 + u_4 = 2 + 4 + 6 = 12$

Hint

Note that $S_3 = S_2 + u_3$ and $S_4 = S_3 + u_4$

In general, $S_n = S_{n-1} + u_n$

Note that $S_1 = u_1$ for any sequence. Also note that the sequence $u_n = 2n - 2$ generates the sequence of even numbers.

Can you find an explicit formula for the sequence of its partial sums?

The notation S_n is used to denote the sum of the first n terms of a sequence. Suppose you want to find the sum of the terms of a sequence from a given start point (not necessarily the first term) to a given end point. In this case, it is not convenient to use S_n as notation. We, therefore, introduce a new, more versatile notation, called the **sigma notation**.

Fact

The name comes from the uppercase Greek letter sigma, Σ.

For example, we could write the sum from term 6 to term 10 as:

$$\sum_{r=6}^{10} u_r = u_6 + u_7 + u_8 + u_9 + u_{10}$$

The index r is called the index of summation, 10 is the upper limit of summation and 6 is the lower limit of summation.

Fact

In general, sigma notation gives instructions on which terms of the sequence, and therefore how many terms, are to be added.

$$\sum_{r=i}^{k} u_r = u_i + u_{i+1} + u_{i+2} + \ldots + u_k, \text{ for } i \leqslant k$$

The lower limit of summation, i, indicates which term to start with, and the upper limit of summation, k, indicates which term to end with. The index of summation can take any variable (such as r, n) and the lower limit of summation can start from any natural number.

Worked example 17.4

Work out $\sum_{r=3}^{5}(2 + 3r^2)$

Solution

The problem asks for the sum from the third term to the fifth term, inclusive, of the sequence $u_r = 2 + 3r^2$

We will calculate u_3, u_4, u_5 using the explicit formula of the sequence and then add them up.

$u_3 = 2 + 3 \times 3^2 = 29$

$u_4 = 2 + 3 \times 4^2 = 50$

$u_5 = 2 + 3 \times 5^2 = 77$

$$\sum_{r=3}^{5} u_r = u_3 + u_4 + u_5 = 29 + 50 + 77 = 156$$

Note that the lower limit of summation does not have to be 1. It can be any natural number.

Many graphic display calculators (GDCs) have a 'sum sequence' feature that can be used to calculate finite series.

Worked example 17.5

Evaluate $\displaystyle\sum_{n=0}^{5000} \frac{(-1)^n}{(2n+1)}$

Solution

The problem asks for the sum from the 0th term to the 5000th term, inclusive, of the sequence $u_n = \dfrac{(-1)^n}{(2n+1)}$

To find the value of this expression, we would need to calculate each term $u_0, u_1, u_2 \ldots, u_{5000}$

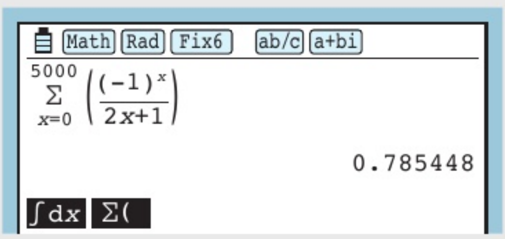
$$u_0 = \frac{(-1)^0}{(0+1)} = 1,\ u_1 = \frac{(-1)^1}{(2+1)} = -\frac{1}{3},\ u_2 = \frac{(-1)^2}{(4+1)} = \frac{1}{5}, \text{ and so on}$$

Such calculation is possible but is very time consuming. Using a GDC is much quicker.

$$\sum_{n=0}^{5000} u_n = (u_0 + u_1 + u_2 + u_3 + u_4 + \ldots + u_{5000})$$

$$= \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} + \ldots + \frac{1}{10\,001}\right) \approx 0.7854$$

Here is an example of a GDC output:



Fact

The series in Worked example 17.5 is called the Leibniz formula for π (from the mathematician Gottfried Leibniz). The more terms we add, the better it approximates $\frac{\pi}{4}$

It is a slow formula, in the sense that to have an approximation that agrees with 4 significant figures of π (after multiplying by 4), we need to add about 5000 terms.

Reflect

Can you find the sum of each of these infinite series without using a calculator? Explain your answers.

a $\sum_{n=1}^{\infty} \frac{3}{10^n}$ **b** $\sum_{n=1}^{\infty} n$

Practice questions 17.2

1 Evaluate the partial sums S_1, S_2, S_5, S_8 for each sequence.

a $u_n = 2n + 1$ **b** $u_n = n^2 - n$ **c** $u_n = (-1)^n(2n)$

d $u_n = 2^n$ **e** $u_n = \frac{n-1}{n+1}$

2 Evaluate the following expressions.

a $\sum_{r=3}^{6}(1 + 2r^2)$ **b** $\sum_{i=1}^{3}(2 - 6i)$ **c** $\sum_{r=1}^{3}(r^r)$

d $\sum_{r=0}^{4}(1.5^r)$ **e** $2\sum_{i=1}^{3} i$ **f** $\sum_{i=3}^{3}\frac{1}{i+2}$

3 Write the following sums using sigma notation. There is more than one correct answer.

a $\frac{1}{3(1)} + \frac{1}{3(2)} + \frac{1}{3(3)} + \cdots + \frac{1}{3(7)}$

b $4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28$

c $-\frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \frac{1}{2^6}$

d $\frac{2}{1^3} - \frac{2}{2^3} + \frac{2}{3^3} - \frac{2}{4^3} + \frac{2}{5^3}$

e $3 + 5 + 7 + 9 + 11 + 13 + 15 + \cdots + 27$

f $\frac{1}{7} + \frac{3}{7} + \frac{9}{7} + \frac{27}{7} + \frac{81}{7}$

4 Determine whether each statement is true or false.

a $\sum_{i=2}^{7} 3i^2 = \sum_{i=2}^{7} 3i \times \sum_{i=2}^{7} i$ **b** $\sum_{k=3}^{6} k^2 = \sum_{k=1}^{4}(k + 2)^2$

5 The market of industries that offer eco-friendly solutions to sustainable development challenges is expanding. It has been calculated that from the beginning of 2016 to the beginning of 2021 the nine 'Green Giants' (a group of global major companies that sell products or services that are sustainable) have generated collectively an annual revenue of $u_n = 100 + 11n$ (in billions of dollars), where n is the year and $n = 1$ corresponds to 2016, from their green business lines alone. Estimate the total revenue earned by the nine Green Giants from the beginning of 2016 to beginning of 2021.

Challenge Q6

6 Consider the sequence of rows of pascal's triangle: 0, 1, 2, 3, 4, 5, 6….

row													
0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1

We can create a sequence where a_i is the sum of the elements of the ith row in the triangle shown above, where a_i is the sum of the elements of the ith row in the triangle shown above, for example, $a_0 = 1$, $a_1 = 1 + 1$ etc.

a Can you identify this sequence?

b Can you find the general formula for this sequence?

17.3 Arithmetic sequences

Explore 17.7

Consider the following three sequences:

2, 5, 8, 11, 14, 17, …

2, 9, 16, 23, 30, 37, …

2, 13, 24, 35, 46, 57, 68, …

Can you list the next three terms for each?

What common characteristics do you notice?

Can you find a recursive formula for each of the three sequences and describe the differences and similarities between the formulae you found?

In this section we will explore in greater depth sequences whose consecutive terms have a common difference. These sequences are called **arithmetic sequences**.

A sequence $\{u_n\}$ is **arithmetic** if the difference between any two consecutive terms is constant. That is:

$u_2 - u_1 = u_3 - u_2 = u_4 - u_3 = \ldots = u_n - u_{n-1} = d$

The constant d is the **common difference** of the sequence.

For example, the sequence

$$\underbrace{2, \quad 7,}_{5}\underbrace{\quad}_{5}\underbrace{12, \quad 17,}_{5} \quad \ldots$$

is an arithmetic sequence with common difference 5.

Worked example 17.6

Find the missing terms in the following arithmetic sequence:

2, $\square$, $\square$, $\square$ 38, 47

Solution

Since this is an arithmetic sequence, we need to know one term and the common difference. Then we use the fact that $u_n = u_{n-1} + d$ to find the missing terms.

The first term is 2, and the common difference is the difference between any two consecutive terms.

The fifth and sixth terms are given: 38 and 47

Therefore the common difference is $47 - 38 = 9$

So the missing terms are: $u_2 = 2 + 9 = 11$, $u_3 = 11 + 9 = 20$ and $u_4 = 20 + 9 = 29$

Looking back, we can check our answer: the next term after 29 would be $29 + 9 = 38$, which is given as the fifth term. So our answer fits the data given.

Explore 17.8

Consider the following six terms of an arithmetic sequence.

4, 9, 14, 19, 24, 29, …

To find the tenth term, you can repeatedly add 5 until you reach the tenth term.

However, if we were asked for the 100th term this process would not be efficient!

Consider the following:

$u_1 = 4$, $u_2 = 4 + 5 = 9$, $u_3 = 9 + 5 = 4 + 5 + 5 = 4 + 2 \times 5 = 14$,

$u_4 = 14 + 5 = 4 + 2 \times 5 + 5 = 4 + 3 \times 5, \ldots$

Can you work out a similar form for u_5, u_6 and u_{100}?

Can you generalise your findings such that you can find the nth term if you are given the first term u_1 and the common difference d?

Worked example 17.7

Find the missing terms and the 100th term in the following arithmetic sequence.

5, $\square$, $\square$, $\square$, 41, 50

Solution

Fact

For the nth term of an arithmetic sequence:
$u_n = u_1 + (n - 1)d$

Since this is an arithmetic sequence, we need to know one term and the common difference. Then we use the fact that $u_n = u_{n-1} + d$ to find the missing terms. To find the 100th term we use the relationship: $u_n = u_1 + (n - 1)d$

Finding the common difference and the missing terms can be done as in Worked example 17.6.

The common difference is $50 - 41 = 9$, and the missing terms are:

$u_2 = 5 + 9 = 14$, $u_3 = 14 + 9 = 23$, and $u_4 = 23 + 9 = 32$

For the 100th term we have $u_{100} = 5 + (100 - 1) \times 9 = 896$

Looking back, one way to check our answer would be to work backwards from the answer. That is, consider 896 to be the first term with a common difference of -9

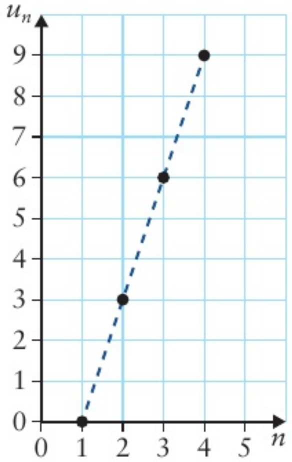
The 100th term in this sequence is $u_{100} = 896$

Therefore, the first term, $u_1 = 896 + (100 - 1) \times (-9) = 5$ which confirms given first term.

Reflect

Are arithmetic sequences and linear functions related? If so, how?

Here is a graph of the first few terms of an arithmetic sequence.



Note that $u_1 = 0$, $u_2 = 3$, $u_3 = 6$, and $u_4 = 9$

Also notice that if we sketch a straight line using any two points it will contain all the points. This fact, as well as the explicit formula you found earlier in Explore 17.4, imply that any arithmetic sequence is in fact a linear function from the set of natural numbers to the set of real numbers.

Which elements of the explicit formula of the arithmetic sequence correspond to the slope, and to the dependent and independent variables of a linear function?

To answer these questions, first simplify the explicit formula:

$$u_n = u_1 + (n - 1)d = u_1 - d + dn$$

We have two variable quantities: u_n and n. u_1 and d are constants. We can make the following analogy:

- u_n can be matched with the dependent variable y.
- n can be matched with the independent variable x.
- d can be matched with the gradient (or slope) of the function.

Reflect

What effect does a negative common difference have on an arithmetic sequence?

Worked example 17.8

The 10th term of an arithmetic sequence is 62 and the 13th term is 80. Find the 20th term.

Solution

We know neither the first term nor the common difference, but we know the 10th and the 13th terms. In order to find the 20th term we need to find u_1 and d.

We can write a system of equations using the explicit formula of arithmetic sequences and solve for u_1 and d. Once we have u_1 and d, we can find the 20th term.

We use the formula $u_n = u_1 + (n - 1)d$ for u_{10} and u_{13} to write a system of equations with u_1 and d as unknown variables:

$62 = u_1 + (10 - 1)d$

$80 = u_1 + (13 - 1)d$

We solve the system:

$18 = 3d$

$d = 6$

$u_1 = 8$

Therefore, $u_{20} = 8 + (20 - 1)6 = 122$

We could have used a different method. Considering the part of the sequence from the 10th to the 13th as an arithmetic sequence of four terms enables us to find d.

The 10th term is the first term in the new sequence and the 13th term is the 4th term in the new sequence, so we have:

$a_1 \ldots a_4$

$62, \boxed{}, \boxed{}, 80$

Thus

$a_4 = a_1 + (4 - 1)d \Rightarrow 80 = 62 + 3d \Rightarrow d = 6$

Now the part of the sequence between the 10th and the 20th terms is an arithmetic sequence $\{b_1, \ldots, b_{11}\}$ with 11 terms, thus the 20th term in the old sequence is the 11th term in the new one:

$u_{20} = b_{11} = 62 + (11 - 1) \times 6 = 62 + 60 = 122$

Reminder

This is a system of simultaneous equations and can be solved by elimination, substitution or graphically.

Hint

Note that starting the count at 10 and ending at 13 gives four terms. Going from term 10 to term 20 gives 11 terms.

Practice questions 17.3

1 Determine whether or not each sequence is arithmetic.

a 13, 10, 7, 4, …

b −4, −2, 0, 2, 4, …

c 3, 9, 27, 81, …

d 2, 5, 10, 17, 26, …

e $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \ldots$

f $u_n = 2 + 3n$

g $u_n = 1 + (n - 1)6$

h $u_n = 3^n \times n$

i $u_{n+1} = u_n + 7, u_1 = 4$

2 Find the (general) explicit formula for each arithmetic sequence given the following information.

a $u_1 = 43$ and $d = -3$

b $u_1 = -7$ and $d = 0.5$

c $u_1 = 12$ and $u_2 = 22$

d $u_1 = 5$ and $u_2 = 0$

e $u_1 = 1$ and $u_2 = 1 + \pi$

f $u_4 = 52$ and $u_7 = 58$

g $u_6 = -64$ and $u_{13} = -85$

3 The third term of an arithmetic sequence is 22 and the eighth term is 87. Which term has the value 243?

4 How many terms are in the arithmetic sequence: 3, 9, 15, … , 111?

5 An object in free fall experiences a constant acceleration of 32 ft/s^2 due to gravity. This means that the object's speed increases by 32 ft/s. Starting with an initial speed of 0 ft/s, find the sequence of speeds of the object in free fall after 1, 2, 3, 4, …, n seconds.

6 When a company first started it had 35 employees. It was decided to increase the number of employees by 4 at the beginning of each year.

a Find the total number of employees during the second and third years.

b After how many years will the company have 71 employees?

Challenge Q7

7 The first four terms of an arithmetic sequence are 3, $a + b$, $2a + b + 1$, $3b$, where a and b are constant. Find a and b and the common difference of the arithmetic sequence.

8 Carbon dioxide (CO_2) is a gas that traps heat. It is released into the atmosphere by human activities such as deforestation and burning fossil fuels, or natural events such as respiration and volcanic eruptions. Increasing levels of CO_2 cause global temperatures to rise, with dangerous results.

In the years between 2010 and 2019, the growth of atmospheric levels of CO_2 followed an arithmetic sequence: in 2010 there were 388 ppm (parts per million), in 2013 there were 395 ppm, in 2016 there were 402 ppm.

If levels of the carbon dioxide continue to follow the same pattern, how many parts per million of CO_2 will there be in 2024?

17.4 Geometric sequences

The previous section explored in depth a particular type of sequence: sequences whose consecutive terms differ by a constant. This section explores a different type of sequence: sequences whose consecutive terms have a common ratio. Such sequences are called **geometric**.

Explore 17.9

Give the next three terms for each of the following sequences and describe a rule for finding the next terms.

2, 6, 18, 54, 162, 486, …

$6, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots$

5, −10, 20, −40, 80, …

What common characteristics do you notice?

Can you find a recursive formula for each of the three sequences and describe the differences and similarities between the formulas you found?

A sequence $u_1, u_2, u_3, \ldots, u_i, u_{i+1}, \ldots$ is a **geometric sequence** of **common ratio r** if there exists a non-zero constant r such that:

$$\frac{u_2}{u_1} = \frac{u_3}{u_2} = \ldots = \frac{u_n}{u_{n-1}} = r$$

For example, the sequence

$$\underbrace{3, \quad 6,}_{\times 2} \; \underbrace{\quad 12,}_{\times 2} \; \underbrace{\quad 24,}_{\times 2} \quad \ldots$$

is a geometric sequence with common ratio 2.

Reflect

Why can none of the terms of a geometric sequence be zero?

Explore 17.10

In the square below, each row is a geometric sequence of ratio 2 and each column is a geometric sequence of ratio 5. Complete a copy of the square. What sequence do you obtain in the diagonal?

2	4	8		
10				
50				

Worked example 17.9

Find the missing terms in the following geometric sequence:

2, $\square$, $\square$, $\square$, 20 000, 200 000

Solution

This is a geometric sequence so we need to know one term and the common ratio. Then we use the fact that $u_n = u_{n-1} \times r$ to find the missing terms.

The first term is 2, and the common ratio is the ratio between any two consecutive terms.

The fifth and sixth terms are given: 20 000 and 200 000, so the common ratio is $\frac{200\,000}{20\,000} = 10$

The missing terms are: $u_2 = 2 \times 10 = 20$, $u_3 = 20 \times 10 = 200$ and $u_4 = 200 \times 10 = 2000$

Looking back, this answer fits the given data since the next term after 2000 is 20 000, which is a given term.

Hint

$u_n = u_{n-1} \times r$ is the recursive formula for any geometric sequence.

Explore 17.11

Consider the following six terms of a geometric sequence:

4, 12, 36, 108, 324, 972, …

To find the tenth term you can repeatedly multiply by 3 until you reach the tenth term.

However, if you are asked for the 100th term, this process will not be efficient!

Consider the following arrangement:

$u_1 = 4$, $u_2 = 4 \times 3 = 12$, $u_3 = 12 \times 3 = 4 \times 3 \times 3 = 4 \times 3^2 = 36$,

$u_4 = 36 \times 3 = 4 \times 3^2 \times 3 = 4 \times 3^3$, ...

Can you guess a similar form for u_5, u_6 and u_{100}?

Can you generalise this to find the nth term if you are given the first term u_1 and the common ratio r?

Worked example 17.10

The sixth term of a geometric sequence is 2916 and the eighth term is 26 244. Find the possible values that the common ratio and u_1 can take.

Solution

We are given two terms of a geometric sequence and we have to find the common ratio. The question says 'the possible values', so there might be more than one solution.

We write expressions for the sixth and eighth terms using the explicit formula. This will give a system of two equations in u_1 and r that we can solve for r.

$2916 = u_1 r^{6-1}$

$26\,244 = u_1 r^{8-1}$

Solve the two equations by dividing the second equation by the first:

$9 = r^2$

Then $r = \pm 3$

For $r = 3$, $2916 = u_1 \times 3^{6-1} \Rightarrow u_1 = \dfrac{2916}{3^5} = 12$

For $r = -3$, $2916 = u_1 \times (-3)^{6-1} \Rightarrow u_1 = \dfrac{2916}{-3^5} = -12$

We have two possible geometric sequences that satisfy the given conditions:

12, 36, 108, 324, 972, 2916, 8748, 26 244, …

and

−12, 36, −108, 324, −972, 2916, −8748, 26 244, …

Note that when the common ratio is negative, the terms in the sequence alternate between positive and negative.

Looking back, we could have used a different method:

Consider the terms from term 6 to term 8 to be a new sequence $\{g_n\}$ with three terms.

Thus, $g_3 = g_1 \times r^{3-1} \Rightarrow 26\,244 = 2916 \times r^2 \Rightarrow r^2 = 9$

The remainder of this method follows the same logic as the first method.

Fact

For the nth term of a geometric sequence: $u_n = u_1 \times r^{n-1}$

Worked example 17.11

A geometric sequence has first term 1 and common ratio 2.

a Write an explicit formula for the sequence.

b List the first four terms. Then graph the sequence on a coordinate plane.

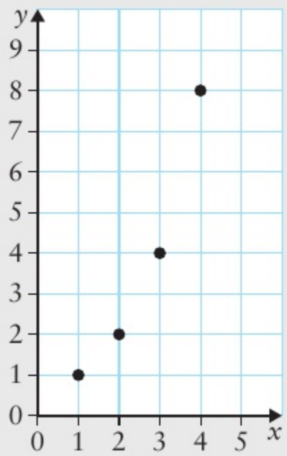
Solution

a $u_1 = 1$ and $r = 2$

The explicit formula is $u_n = u_1 r^{n-1} = 2^{n-1}$

b $u_1 = 1, u_2 = 2, u_3 = 4, u_4 = 8.$

To plot the points on a coordinate plane, the x-coordinate is the subscript of the term (or the position of the term in the sequence), and the y-coordinate is the value of term. So the points are (1, 1), (2, 2), (3, 4), (4, 8)



Reflect

What is the equation of the curve on which the points lie?

Worked example 17.12

Challenge

The Amazon rainforest is the largest tropical rainforest in the world. At the beginning of 2021, it covered an area of about 660 million hectares (ha) with a concentration of animals and plants higher than any other place on Earth. Unfortunately, every year around 2 million ha of Amazon rainforest are destroyed.

There are calls to stop the rainforest destruction. A law has been suggested to stop destruction altogether and to start annual reforestation on the basis that throughout the coming year, an area equivalent to about 1% of the forested area at the start of the year will be repopulated with trees.

Assuming that this law is successful:

a work out the forested area at the beginning of 2022, 2023 and 2024.

b when will they achieve a forested area of 700 million ha?

Solution

a At the beginning of 2021 the forested area is 660 million ha.

The forested area grows at an annual rate of 1.01 of its present size. So this appears to be a geometric sequence with first term $u_1 = 660$ and common ratio $r = 1.01$. During 2021 the reforestation is repopulating 1% of the existing area, so at the beginning of 2022 the forested area will be:

$u_2 = 660 \times 1.01 = 666.6$ million ha

During 2022 the forested area increases by 1% again, so the area at the beginning of 2023 will be:

$u_3 = 666.6 \times 1.01 = 673.3$ million ha

Similarly, the forested area at the beginning of 2024 will be:

$u_4 = 673.3 \times 1.01 = 680.0$ million ha

b To find out when the forested area reaches 700 million ha, we use the first term, the common ratio, and the explicit formula $u_n = u_1 r^{n-1}$

$700 = 660 \times 1.01^{n-1} \Rightarrow 1.01^{n-1} = 1.06$

To find n we use logarithms:

$$(n-1)\ln 1.01 = \ln 1.06 \Rightarrow n - 1 = \frac{\ln 1.06}{\ln 1.01} = 5.85$$

This means that $n = 6.85$. Therefore, we can say that they will achieve this goal in approximately 7 years, that is in the year 2028.

Practice questions 17.4

1 Determine whether or not each sequence is geometric. If it is, find the common ratio.

a $7, 21, 63, 189, \ldots$

b $1.5, -3, 6, -12, \ldots$

c $1, 2, 6, 24, 120, \ldots$

d $-1, 2, 5, 8, \ldots$

e $2, 0.6, 0.18, 0.054, \ldots$

f $1, 4, 9, 16, 25, \ldots$

g $3, 5, 9, 17, \ldots$

h $0, 4, 8, 12, \ldots$

i $7, 2.8, 1.12, 0.448, \ldots$

2 Write the first four terms of each geometric sequence.

a $u_1 = 5, r = 3$

b $u_1 = 3, r = \frac{1}{3}$

c $u_1 = 0.2, r = -\frac{1}{5}$

d $u_1 = 2, r = \sqrt{2}$

e $u_1 = 1, r = \pi$

f $u_1 = 1, r = -1$

g $u_1 = 2, r = 3x$

h $u_1 = a, r = x$

3 How many terms are there in each of the following geometric sequences?

a $-4, -12, -36, \ldots, -19\,131\,876$

b $0.5, -2.5, 12.5, \ldots, 7812.5$

c $10, 30, 90, \ldots, 5\,314\,410$

d $5, 2, \frac{4}{5}, \ldots, \frac{1024}{1\,953\,125}$

e $1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{531\,441}$

4 The 3rd term of a geometric sequence is 98 and the 5th term is 4802. Find the two possible values for the 8th term.

5 The 2nd term of a geometric sequence is -208 and the 5th term is 13 312. Find the 10th term.

Challenge Q6

6 A geometric sequence has first term 10 and common ratio $\frac{1}{3}$. What is the first term that is less than 10^{-7}?

7 Jennifer has received a job offer with an annual starting salary of £35,600 and a guaranteed annual salary increase of 1% on the salary of the previous year. Find Jennifer's salary at the beginning of her 5th year of employment.

8 In 2021, the number of students in a small school is 360. It is estimated that the student population will increase by 2% each year.

a Write a formula for the student population.

b Estimate the student population in 2026. Round your answer to the nearest integer.

9 A clothing factory opens a new line of sustainable clothes, which it advertises on a new website. In the first week the number of hits is 323. The communication manager estimates that the number of hits will increase by 1.6% per week. Estimate the number of weekly hits in 7 weeks' time.

Self assessment

- I can predict what comes next in a sequence.
- I can write the terms of a sequence using subscript notation.
- I can graph a sequence on a coordinate grid.
- Given the explicit formula of a sequence, I can list the terms of the sequence.
- Given a sequence, I can determine if a value is a term of the sequence.
- I can write the recursive definition of a sequence.
- I can list the terms of a sequence given its recursive definition.
- I can explain what the *n*th partial sum of a sequence is.
- I can define what a series is.
- Given a sequence, I can calculate the *n*th partial sum.
- I can explain what sigma notation of a sequence is.
- I can use sigma notation to calculate the sum/partial sum of a sequence.
- Given the sum of a sequence, I can write it in sigma notation.
- I can explain what an arithmetic sequence is.
- I can explain what a geometric sequence is.
- I can determine whether or not a sequence is arithmetic or geometric.
- I can explain what the common difference of an arithmetic sequence is.
- I can explain what the common ratio of a geometric sequence is.
- I can solve problems about arithmetic sequences using the appropriate strategy.
- I can solve problems about geometric sequences using the appropriate strategy.
- I can model a real-life problem using sequences.
- I can apply my knowledge of sequences to solve real-life problems.

Check your knowledge questions

1 Continue the pattern 1, 3, 5, … in at least three different ways. Explain how each pattern grows.

2 Consider the sequence with explicit formula $u_n = n^2 - 3$. Find:

a the first six terms

b the sum of the first five terms.

3 Write down the first five terms of the sequence:

a starting at 6 and increasing by 4 each time

b starting at 1 and decreasing by 3 each time.

4 Find the first five terms of the sequence with recursive formula:

a $u_1 = 3$ and $u_n = 4u_{n-1} + 6$ for $n \geqslant 2$

b $u_1 = 1$, $u_2 = 2$ and $u_n = 2u_{n-1} \times u_{n-2}$ for $n \geqslant 3$

5 Evaluate the partial sums S_1, S_2, S_4, S_6 for each sequence.

a $u_n = 3n + 4$ b $u_n = -n^2 - n + 1$ c $u_n = (-1)^n(3n + 1)$

d $u_n = 3^n$ e $u_n = \dfrac{2n-1}{2n+1}$

6 Evaluate the following expressions.

a $\displaystyle\sum_{r=0}^{4}(3r^2)$ b $\displaystyle\sum_{i=1}^{2}(7 + 5i)$ c $\displaystyle\sum_{r=2}^{4}(r^{-r})$

d $\displaystyle\sum_{r=0}^{7}(0.5^r)$ e $\displaystyle 3\sum_{i=1}^{6} i$

7 Find the unknowns, given that the sequences are arithmetic.

a $3, 7, \boxed{}, 15, \ldots$ b $-2, \boxed{}, -14, \ldots$

8 Find k given that $2k + 1$, $5k - 1$, $3k + 4$ are consecutive terms of an arithmetic sequence.

9 Find the explicit formula of the arithmetic sequence such that $u_4 = 10$ and $u_{23} = 67$

10 Consider the sequence $-3, -7, -11, -15, \ldots$

a Show that this sequence is arithmetic.

b Find an explicit formula for the sequence.

c Find the 101st term.

d Find the smallest value of m for which the mth term of the sequence is less than -800.

11 An arithmetic sequence has 20 terms. The first term of the sequence is 10 and the last term is 67. Find the 10th term.

12 Write down the common ratio for these geometric sequences.

a $10, 30, 90, \ldots$ **b** $4, -8, 16, \ldots$ **c** $5, \frac{5}{\sqrt{2}}, \frac{5}{2}, \ldots$

13 $k - 3$, $k - 1$ and $k + 3$ are three consecutive terms of a geometric sequence.

a Find the value of k.

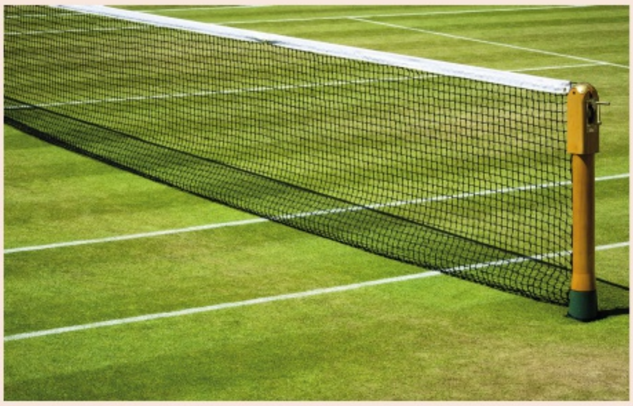
b Find the common ratio of the sequence.

14 The men's Wimbledon tennis tournament is held annually in Wimbledon, UK. In the first round of the tournament, 64 matches are played. In each successive round, the number of matches played decreases by one half.

a Find an explicit formula for the number of matches in the nth round.

b For what values of n does your formula make sense?

c Find the total number of matches played in the men's Wimbledon tournament.



15 The first row of a concert hall has 26 seats and each row after the first row has one more seat than the row before it. There are 33 rows of seats.

a Find an explicit formula for the number of seats in the nth row.

b 36 students want to sit in the same row. How close to the front can they sit?

18

Probability

18 Probability

KEY CONCEPT

Logic

RELATED CONCEPTS

Change, Models, Representation, Systems

GLOBAL CONTEXTS

Fairness and development

Statement of inquiry

By logically evaluating patterns, models can be developed to represent measures of chance so that we can develop systems to evaluate fairness.

Factual

- What is the difference between experimental and theoretical probability?

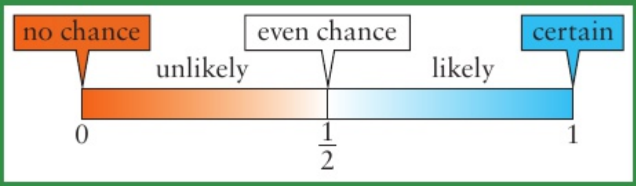
Conceptual

- What is the difference between simple and compound events?
- How is the probability of the union of two events found?

Debatable

- How certain can predictions be?

Do you recall?



1 **a** Describe the chance of each of these events happening, using the terms: certain, likely, unlikely, impossible (no chance) and even chance.

A A new-born baby is a girl.

B The average height of all MYP students this year is 100 cm.

C There will be a holiday in the UK on 25th December.

D My house is 2 km away from school and I can walk non-stop from home to school.

E I draw a card from a deck of 52 playing cards and I get a King or a Queen.

F I will have a mathematics assessment task during the last three months of the school year.

b Put the events A to F in order from least likely to most likely.

2 A card is chosen at random from a group of ten cards numbered 0 to 9. What is the probability:

a of choosing an odd number

b of not choosing a 9

c of choosing an odd or even number

d of choosing a number larger than 10?

3 A coin was flipped 100 times and landed on heads (H) 47 times. Find the experimental probability of this coin landing on heads.

4 Find the value of:

a 9!

b 1!

c 0!

Hint Q1

The probability of an event is its chance of happening. This will be a number from 0 to 1.

Hint Q2

The theoretical probability P of an event E is found by dividing the number ways the event can happen $n(E)$ by the total number of possible outcomes $n(U)$, where U is the sample space of the experiment that E is part of.

$$P(E) = \frac{n(E)}{n(U)}$$

18.1 Probability review

Explore 18.1

Place 20 identical thumbtacks in a clear plastic cup. Cover the cup with plastic wrap and a rubber band. Shake the cup, remove the plastic wrap and turn it upside down on the desk.

Before counting the number of tacks pointing up or down, what can you say about each of the following statements?

a 'Point up' is equally likely as 'point down'.

b If one tack is flipped, the probability that it will land point up is about 50%.

c If a cup containing 200 tacks was emptied onto a desk, I expect 100 tacks to land point down.

Now, count the number of tacks pointing up and the number pointing down. What can you say about the differences between the results of your experiment and the statements a-c above?

18.1.1 Theoretical and experimental probability

Theoretical probability

An **outcome** is a result of a random experiment.

An **event** is a set of outcomes of an experiment.

Fact

Before flipping a coin you do not know the result. This is an example of a **random experiment**. In particular, a random experiment is a process by which we observe something uncertain.

Rolling a die and observing the number on the top surface, counting cars at a traffic light when it turns green, measuring daily rainfall in a certain area, and so on, are a few experiments in this sense of the word.

Fact

Experimental (or empirical) probability is an estimate of the likelihood that an event will occur based on gathered data from an **experiment**. **Theoretical** probability is an 'expected' probability based on knowledge of the situation. *Theoretical* signifies what should happen, while *experimental* signifies what actually happened in similar situations.

The set of all the possible outcomes of a random experiment is called the **sample space**.

For example, when you roll a fair 6-sided die, the sample space is $\{1, 2, 3, 4, 5, 6\}$

Each outcome is equally likely to happen, so the probability of rolling a 4 is

$$P(4) = \frac{1}{6}$$

Rolling an even number is an event made up of three outcomes, $\{2, 4, 6\}$. Outcomes themselves are also called simple events because they can happen in one way. To roll a 4, for example, is simple because you can roll a 4 only in one way. Rolling an even number, an event, can be achieved in three different ways: 2, 4, or 6.

When you flip a fair coin the sample space is {heads, tails}. Each outcome is equally likely, so

$$\text{P(heads)} = \text{P(tails)} = \frac{1}{2}$$

A deck of cards contains 52 cards, 13 cards in each of the 4 suits: Hearts, Clubs, Diamonds and Spades. If you randomly pick a card from the deck, the probability that it is a Club is:

$\text{P(Club)} = \frac{13}{52} = \frac{1}{4}$. All of these are theoretical probabilities.

Experimental probability

Although each outcome when you roll a die has a theoretical probability of $\frac{1}{6}$, it does not mean that if you roll a die six times it will land on each of the faces once. You could roll six 4s in a row, although that is very unlikely.

Reflect

Can you work out how unlikely it is that you will roll six 4s in a row?

Connections

Meteorologists use experimental probability to predict the weather based on what has happened in similar conditions in the past.

In an experiment, a die is rolled 200 times and the following results are obtained.

Score	1	2	3	4	5	6
Frequency	26	34	32	24	48	36

The experimental probability of each score can be found by dividing the corresponding frequency by 200. This is also called the **relative frequency**.

$$\text{experimental probability} = \text{relative frequency} = \frac{\text{frequency of chosen outcome}}{\text{total frequency}}$$

This table gives the relative frequencies for each score in the previous table.

Score	1	2	3	4	5	6
Relative frequency	$\frac{26}{200} = 0.13$	$\frac{34}{200} = 0.17$	$\frac{32}{200} = 0.16$	$\frac{24}{200} = 0.12$	$\frac{48}{200} = 0.24$	$\frac{36}{200} = 0.18$

The theorical probability of each outcome is $\frac{1}{6} = 0.16666...$, so the experimental and theoretical probabilities are not the same. Does that mean that the die is biased (not fair)? How can we improve the results to be more certain of our conclusion?

For the theoretical and experimental probabilities to become close we need to repeat an experiment a large number of times.

Worked example 18.1

In an experiment, two dice are rolled 36 times and the **sum** of the faces are recorded as shown below.

Sums when rolling two dice 36 times								
4	6	4	8	10	8	6	11	9
2	3	6	12	5	6	8	4	8
7	11	5	2	8	3	6	4	10
3	5	3	7	8	3	11	6	4

a What is the experimental probability of rolling a sum of 6?

b What is the theoretical probability of rolling a sum of 6?

c How do the empirical and theoretical probabilities compare?

Solution

a Experimental probability is the probability observed in the chart above. A sum of 6 was rolled 6 times out of 36 rolls.

The experimental probability $= \frac{6}{36} = \frac{1}{6} \approx 0.167 = 16.7\%$ (3 s.f.)

b Theoretical probability is based upon what is expected when rolling two dice. We can list the possible outcomes of the experiment resulting in a sum of 6:

$\{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}$

Moreover, since there are 6 faces on each die, the total number of possible outcomes is $6 \times 6 = 36$

The theoretical probability of rolling a sum of 6 is:

5 times out of 36 rolls $= \frac{5}{36} \approx 0.139 = 13.9\%$ (3 s.f.)

c Obviously, the two probabilities are not the same. The experimental probability may approach the theoretical probability when the number of trials is extremely large.

Practice questions 18.1

1 What is the probability that a number chosen at random from the numbers 1 to 100 is:

a divisible by 3

b not divisible by 3

c divisible by 12

d not divisible by 12?

2 A card is chosen at random from a standard deck of 52 cards. What is the probability that the card is:

a black

b not black

c green

d a seven

e not a seven

f a picture card

g a Spade

h not a Spade

i a King or a Queen?

3 For a school raffle, 100 green tickets numbered 1 to 100, 50 red tickets numbered 1 to 50 and 20 blue tickets numbered 1 to 20 are put in a barrel. When a ticket is chosen at random from the barrel, what is the probability that the ticket is:

a blue

b red or green

c an even number

d less than 10

e more than 60

f more than 30 and less than 50

g not green

h either a 28 or a 68?

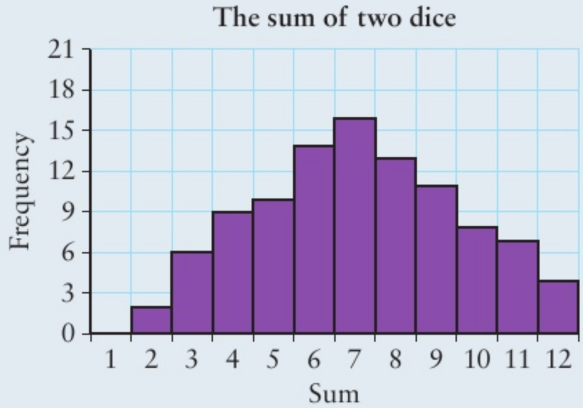
4 Use the GeoGebra link on the right to find out how many times you need to roll a die so that the difference between the theoretical and experimental probability of each score is less than 1% (this may take a while!).

5 A factory that manufactures batteries sampled 100 of its batteries and found three of them to be faulty.

a Use this result to estimate the probability of a battery being faulty.

b Do you think that your answer to part a is a good estimate? Explain your answer.

6 Two dice were rolled 100 times and their scores were added. The results are shown in the frequency diagram.



Use the diagram to determine the probability of getting a sum of:

a exactly 7 **b** less than 5 **c** more than 9.

7 Siva will either play football or video games when he comes home from school. He noticed that in a regular week he will play football twice and video games on the other days.

a What is the probability of Siva playing video games?

b What do you notice about the sum of the probabilities of playing football or video games? Does that make sense?

8 Three coins were flipped 50 times and the number of heads was recorded each time. The results are given below.

1	1	2	1	2	2	3	1	1	1
2	1	2	2	0	1	2	2	2	0
3	3	1	1	2	0	3	2	3	2
0	2	2	1	3	1	2	1	1	0
2	1	1	1	2	0	3	2	2	1

a List the possible outcomes.

b Work out the experimental probabilities of each outcome.

c What is the probability of getting one or two heads?

d Compare your answers to part b with the theoretical probability of each outcome.

9 Navid recorded the type of fuel used by the cars taking students to school one morning. He obtained the results shown in the table.

Electric	Hybrid	Petrol	Diesel
24	38	119	238

What is the probability that a student arrives in school in a car that is more environmentally friendly? Do you think the results would be the same at your school? Give a reason for your answer.

10 Robin and Tico find a coin. They decide to flip it 100 times to find out whether it is fair. They record 57 heads and 43 tails. Robin says that the coin is not fair, but Tico is not convinced. Who do you think is correct? Investigate further to explain your answer.

11 A social media platform has 2.60 billion monthly active users (MAU). 1.73 billion of those are daily active users (DAU). What is the probability that a MAU is also a DAU?

12 A bag contains red, blue and green discs. The probability of drawing a red disc is $\frac{1}{3}$ and the probability of drawing a blue disc is $\frac{2}{5}$

a Calculate the probability of drawing a green disc.

b What can you deduce about the number of discs in the bag?

c Assuming that there are 30 discs, how many of each colour are there?

18.2 Combined events

In this section, you will use sample spaces to find the theoretical probability of combined events. A combined event could be rolling a die and flipping a coin at the same time.

Initially, you will consider events that do not influence each other. For example, the score on the die does not influence the outcome on the coin. These events are called **independent**.

18.2.1 Lists and tables

We need to use efficient ways of organising the complete sample space of compound events, where more than one activity is involved. We could use a list, a table or a tree diagram. Sometimes other types of diagrams are used.

To illustrate various concepts in this section, we shall invent a fictitious club consisting of a set of members U of two girls: Sophia and Cathy; and three boys: Marc, Rob, and Luca.

Explore 18.2

Consider forming a committee of two members, containing a girl and a boy from this club. Can you list all possible committees? Hence, find the probability that Sophia is on this committee.

Worked example 18.2

A fair die is rolled and a fair coin is flipped.

a List all the possible outcomes (that is, find the sample space).

b Find the probability of rolling an even number and flipping a head.

Solution

a A sample space is the set of all the possible outcomes. We can list these systematically.

The die can show scores of 1, 2, 3, 4, 5 or 6. The coin can either show heads (H) or tails (T). So, each score on the die can be followed by either of these faces. The sample space is

{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T}

Alternatively, we can show this in a table.

Die	H	T
1	1H	1T
2	2H	2T
3	3H	3T
4	4H	4T
5	5H	5T
6	6H	6T

Both methods show that there are $2 \times 6 = 12$ possible outcomes, all of which are equally likely.

b We can use the sample space or table to find the probability of a particular outcome.

To find the probability of rolling an even number and flipping a head we can highlight the outcomes in the sample space list that fit this description.

{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T}

We could also use the table:

Die	H	T
1	1H	1T
2	2H	2T
3	3H	3T
4	4H	4T
5	5H	5T
6	6H	6T

Both ways give a probability of $\frac{3}{12} = \frac{1}{4}$

Complements

Events either happen or do not happen. If the event E represents the event 'rolling a 6 on a die', then its **complement** E' denotes this event not happening, that is, 'not rolling a 6 on a die'.

Since it is certain that one or the other happens,

$P(E) + P(E') = 1$ so $P(E') = 1 - P(E)$

Sometimes it is easier to calculate the probability of a complementary event that that of the original event.

Fact

The sum of the probabilities of all the outcomes of an experiment is always equal to 1.

Worked example 18.3

Days missed	Number of students
0	31
1	47
2	25
3	18
4	11
5	8
6	2

The absentee records for a year group of MYP3 students are displayed in this frequency table. If a student is selected at random, find the probability that the student has missed:

a 4 days of school

b fewer than 3 days

c more than 2 days

d 8 days of school.

Solution

We need to find the total number of students first. The sum of the second column is 142 students.

a 11 students out of 142 have missed four days of school.

Therefore, P(4 days missed) $= \frac{11}{142}$

b Fewer than 3 days corresponds to 0, 1 or 2 days.

Number of outcomes $= 31 + 47 + 25 = 103$

P(fewer than 3 days) $= \frac{103}{142}$

c More than 2 days corresponds to 3 days or more. We either add all entries from 3 days or above or use the complement, which is less than the 3 days, calculated earlier.

P(more than 2 days) $= 1 - \frac{103}{142} = \frac{39}{142}$

d No one has been absent for 8 days.

Therefore, P(8 days missed) = 0

Reflect

When do you think it might be easier to calculate a probability using a complementary event?

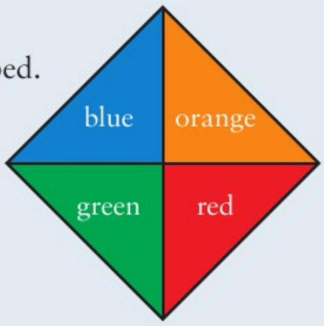
Practice questions 18.2.1

1 A card is drawn at random from a standard pack of 52 playing cards and a fair die is rolled.

- **a** List the sample space.
- **b** Determine the probability of drawing a red card and rolling at least a 3.

2 This fair spinner is spun and a fair coin is flipped.

By making a table of possible outcomes, find the probability of the spinner landing on red and getting a head on the coin.



3 The spinner from question 2 is spun twice. List the possible outcomes and find the probability that:

a it lands on green and red (in any order)

b it lands on the same colour twice.

4 Tiago draws a playing card at random from a standard pack of 52 cards. He records the suit and then puts the card back in the pack. He draws a second card. Make a table of the possible outcomes and find the probability that:

a both cards are black

b the cards are of different suits.

Hint Q4

Use the complement rule.

5 A fair game is one in which each player is equally likely to win or to lose. Ricky and Jess play a game with three coins. When they flip the coins, Ricky wins if there are more heads than tails and Jess wins otherwise. Is this a fair game? Explain your reasoning.

6 Pablo rolls two dice and notes how many 6s he rolls. What is the probability of Pablo rolling:

a two 6s

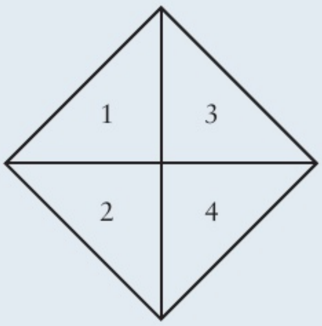
b one 6

c no 6s?

Hint Q6

Use the complementary event.

7 This spinner is spun and a die is rolled at the same time. Make a table to show the possible outcomes.



Find the probability that:

a the spinner and the die show the same score

b the score on the spinner is bigger than the score on the die.

8 If you repeated the experiment in question 7 sixty times, how many times would you expect the score on the spinner to be bigger than the score on the die?

9 How many times do you need to flip a coin to be more than 99% sure that you will have flipped at least one head?

Challenge Q9

18.2.2 Venn diagrams

Venn diagrams are useful for finding probabilities when the data can be sorted into sets.

Explore 18.3

Some students are asked whether they compost food waste (C) or recycle plastics (P) at home. It was found that 33 students composted at home. Of those 33, 10 did both. 45 students recycled plastics and 32 did neither. Can you show how to find the probability of choosing a student that recycles plastic only?

Worked example 18.4

Of 120 employees in a company outside a large city, 35 use the company's bus service and of these, 5 are among the 47 employees working in management.

If an employee is chosen at random, what is the probability that the employee:

a uses the bus service but is not in management

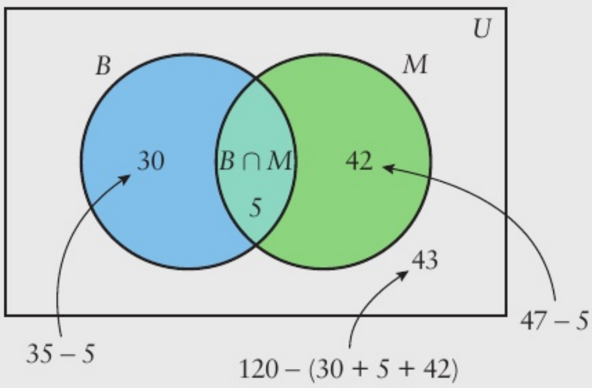
b neither uses the bus service nor works in management?

Solution

Let B be the set of employees using the bus service, M the set of employees working in management, and, U, the set of all employees.

$n(U) = 120$ (ie number in U is 120); $n(B) = 35$, $n(M) = 47$, and $n(B \text{ and } M) = 5$

We use the information given to place numbers in each part of the Venn diagram.

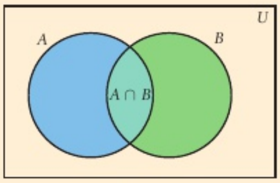


Connections

You have seen Venn diagrams before. You will use the same notation here and apply it to probability questions.

Reminder

The intersection of two sets is where they overlap. It contains the members that have the characteristics of both sets. In symbolic form we write it as $A \cap B$, which we read as 'A intersection B'.



The union of two sets contains the elements of both sets combined. It contains the members that have the characteristics of one of the sets or both. In the diagram, the union is made up of the shaded regions. In symbolic form we write it as $A \cup B$, which we read as 'A union B'.
Note that $n(A)$ includes $n(A \cap B)$ and $n(B)$ includes it too.
Thus, if you write $n(A \cup B) = n(A) + n(B)$ then $n(A \cap B)$ is counted twice. This leads to the General Addition Rule for sets:
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
which leads to the General Addition Rule for event probabilities:
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

a 30 employees who use the bus service do not work in management.

Therefore, P(B and **not** M) = $\frac{30}{120} = \frac{1}{4}$

b 43 employees are not in either of the two categories.

Therefore, P(**not** B and **not** M) = $\frac{43}{120}$

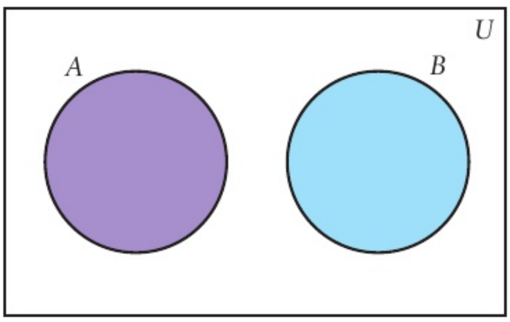
Reflect

Why can you not use a list, table or tree diagram to solve the question in Worked example 18.7?

Mutually exclusive events

Mutually exclusive events are events that cannot happen at the same time. Examples include: looking up and looking down, a flipped coin landing on both heads and tails, and one card taken from an ordinary pack of cards being both a spade and a heart.

This can be illustrated using a Venn diagram. Mutually exclusive events have nothing in common. Therefore, their sets do not overlap, and the intersection is empty.



This means, if A and B are mutually exclusive events then the following must be true:

$\mathrm{P}(A \cap B) = 0$

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

Worked example 18.5

A class has 35 students. 16 students chose to study French as a foreign language. 21 students chose Spanish as a foreign language, and 5 chose no foreign language.

a Use the information to draw a Venn diagram.

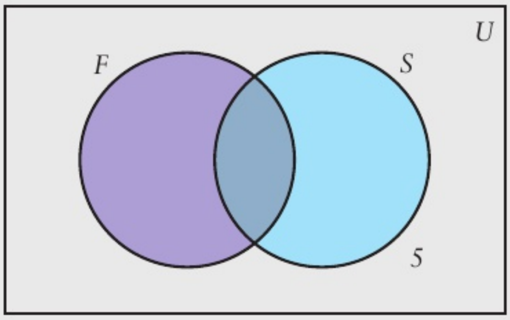
b A student is chosen at random from the group. Find the probability that this student studies:

i either French or Spanish **ii** both French and Spanish

iii French only **iv** only one language.

Solution

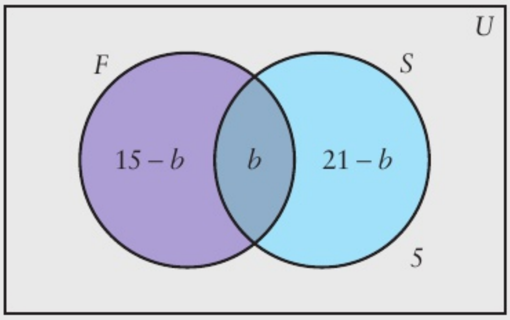
a Let the set of French learners be F, and Spanish learners S. Here is a Venn diagram showing the information supplied. Note that since 5 out of the 35 chose neither language, we place 5 in the space outside both sets.



Let the number of those studying both languages be b. Note that since 5 students do not study any language, then 30 will be studying either language, that is:

$n(F \cup S) = 30$

Here is an updated diagram.



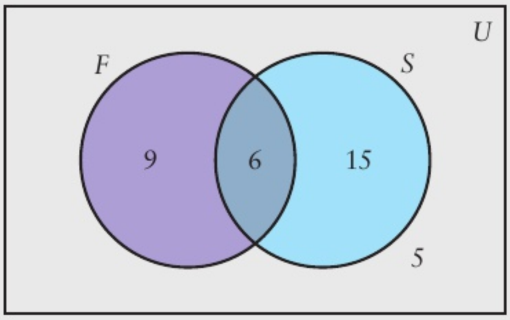
To study both is the set $F \cap S$ represented by b.

Since the union $F \cup S$ contains 30 students, then:

$15 - b + b + 21 - b = 30$

So, $b = 36 - 30 = 6$

We can now finalise the Venn diagram:



b **i** To study either French or Spanish is represented by the union:

$$P(F \cup S) = \frac{30}{35} = \frac{6}{7}$$

ii $P(F \cap S) = \frac{6}{35}$

iii P(French only) $= \frac{9}{35}$

iv Only one language means either French only, which has 9 students, or Spanish only, which has 15 students. They are also mutually exclusive:

$$\text{P(one language only)} = \frac{9}{35} + \frac{15}{35} = \frac{24}{35}$$

Reflect

- Can you verify the general addition rule using the data in Worked example 18.5.
- Can you approach the calculations involved differently?

Practice questions 18.2.2

1 A group of 35 people were asked whether they liked fishing (F) or bowling (B). The results were: 10 liked both fishing and bowling, 20 liked bowling and 7 liked neither.

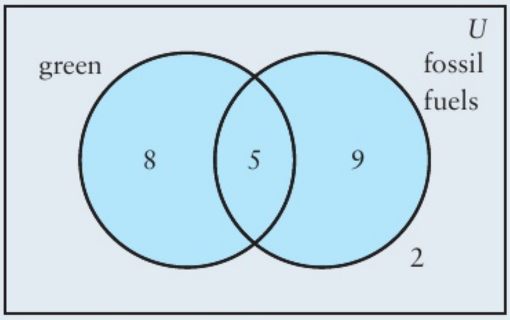
a Put this information in a Venn diagram.

b Find the following probabilities.

i $P(F)$ ii $P(F \cap B')$ iii $P(F')$

c Describe in words each of the groups in part b.

2 A class does a survey to find out how students get to school. Students are categorised according to the type of fuel used. The results are shown in the Venn diagram.



a Explain how the two students outside both sets might travel to school.

b A student is chosen at random from the class. Copy and complete the following table using the information in the diagram.

	P(G)	P(F)	P($G \cap F$)	P($G \cup F$)
Explain what is meant in words	The probability that the fuel used is green.			
Find the probability	$\frac{13}{24}$			

3 In a group of 16 students, 9 have been to Thailand and 4 have been to Peru. 2 students have visited both countries.

a Use this information to draw a Venn diagram.

A student is randomly chosen from this group. Find the probability that this student:

b has visited Thailand, but not Peru

c has visited neither Thailand nor Peru.

4 A factory has 150 employees. 58 of them have a beard. 6 of the bearded employees are among the 37 working with heavy machinery.

a Use this information to draw a Venn diagram.

One employee is chosen at random. Find the probability that:

b they have a beard, but do not work with heavy machinery

c they neither have a beard, nor work with heavy machinery.

5 In the same factory as question 4, 2 employees are selected at random to represent the team. Find the probability that:

a both have beards

b neither of them works with heavy machinery.

6 There are 24 students in Alberto's class. 11 students wear glasses. 6 of the students who wear glasses have blue eyes. There are 10 students with blue eyes altogether. Find the probability that a randomly chosen student does not have blue eyes and does not wear glasses.

7 There are 30 students in a class. 23 students play at least one sport, 11 play a musical instrument and 1 student does neither. Find the probability that a randomly chosen student in this class both plays sport and plays an instrument.

8 In a small town there are three supermarkets: X, Y and Z.

55% of the population shop at X, 42% shop at Y and 35% shop at Z

2% of the population shop at all three.

21% shop at X and Y, 17% shop at X and Z and 8% shop at Y and Z

Determine the probability that a randomly chosen person shops at:

a none of the supermarkets

b exactly one of the supermarkets

c at least one of the supermarkets.

Challenge Q8

Hint Q8

Change the percentages into decimals. Remember that people shopping at X and Y might also shop at Z.

Investigation 18.1

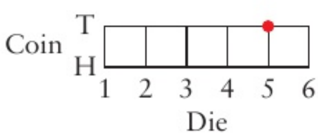
Questions 2 and 7 in Practice questions 18.2.2 describe surveys carried out on a class of students. Carry out each survey on your own class and answer the questions using your own results.

18.2.3 Dot and tree diagrams

Dot diagrams are also called **lattice diagrams** or **sample space diagrams**. They help you to quickly identify the different outcomes.

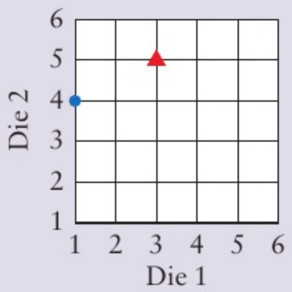
Consider the example of rolling a die and flipping a coin from the previous section.

This is what the dot diagram for the outcomes may look like. The circle corresponds to the outcome 5T.



Dot diagrams can be used to show all the possible outcomes of two events.

Explore 18.3



The dot diagram shows the possible outcomes when you roll two dice. The score marked by the triangle corresponds to a 3 on the first die and a 5 on the second die. The circle corresponds to a 1 on the first die and a 4 on the second die.

a Can you tell how many possible outcomes there are?

b Using a copy of the diagram, can you tell how many doubles there are?

c Can you find the probability of rolling a double score?

Hint Qb

A double means getting the same score on both dice.

Worked example 18.6

Two fair dice are rolled and the sum of the numbers on the top faces is noted. Find the probability of getting:

a a sum of 6

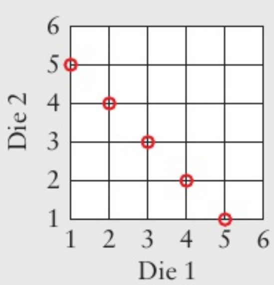
b a sum of at least 10

c a sum of less than 9 and at least one 3.

Solution

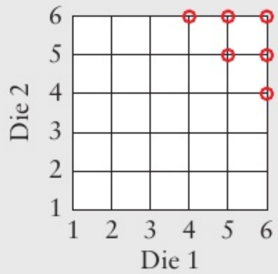
For each part of the question we can draw a dot diagram showing all the possible outcomes.

a We circle each point where the sum of the coordinates is 6 and then count the number of circles.



We have circled 5 points. There are 36 possible outcomes so the probability of a sum of 6 is $\frac{5}{36}$

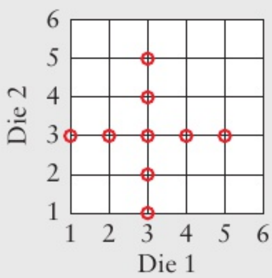
b We circle all the outcomes where the sum is least 10. This means a sum of 10, 11 or 12.



There are 6 outcomes that fit this description, so the probability is $\frac{6}{36} = \frac{1}{6}$

c To roll at least one 3, *either* die 1 *or* die 2 must land on a 3. For die 1 this is represented by all the points on the vertical line at 3 and for die 2 this is represented by all the points on the horizontal line at 3.

Then, to find the outcomes that have a sum of less than 9 (as well as at least one 3), we circle all the points that represent a sum of less than 9 from the points on the vertical and horizontal lines at 3.



9 outcomes fit this description, so the probability is $\frac{9}{36} = \frac{1}{4}$

A dot diagram is an effective technique to visualise outcomes of combined events quickly.

The simple multiplication rule

Consider the following situation. In a large school, 55% of the students are male. It is also known that the percentage of vegetarians among males in this school is 22%. What is the probability of selecting a student at random from this population and finding that the student is a male vegetarian?

Applying common sense, we can think of the problem in the following manner.

The chance of picking a male student is 55%. From those 55% of the population, we know that 22% are vegetarians, so the chance that we select a male vegetarian is 22% of the 55%. That is, $0.22 \times 0.55 = 12.1\%$

This is an example of the multiplication rule for combined events.

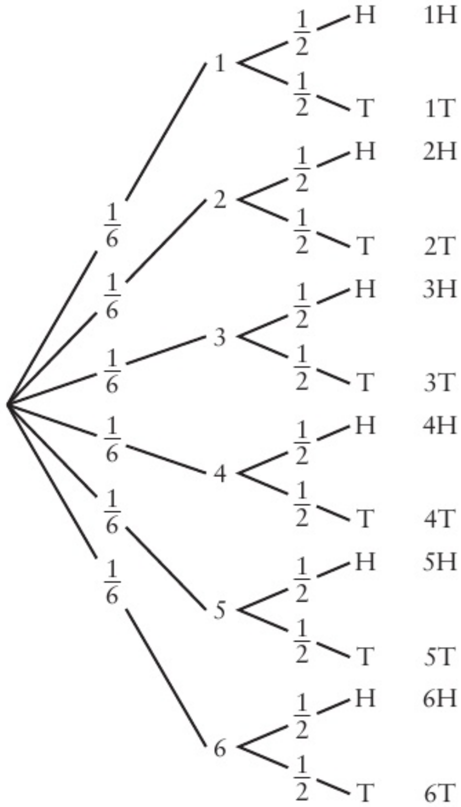
A **tree diagram** represents the multiplication rule and is an effective way of representing the outcomes of combined events. The tree diagram on the next page shows the outcomes of rolling a die and flipping a coin.

Fact

Two events A and B are independent if knowing that one of them occurs does not change the probability that the other occurs.

If two events A and B are **independent**, then

$P(A \cap B) = P(A) \times P(B)$



> **Fact**
>
> The sum of the probabilities of each set of branches is always equal to 1.
>
> $\frac{1}{6} + \frac{1}{6} + \ldots + \frac{1}{6} = 1$, or
>
> $\frac{1}{2} + \frac{1}{2} = 1$, or
>
> $\frac{1}{12} + \frac{1}{12} + \ldots + \frac{1}{12} = 1$
>
> because one of those outcomes is certain to happen.

To calculate the probability of a particular outcome, we multiply the probabilities on the branches leading to that outcome. In this example, each outcome has a probability of $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$, because each score on the die occurs with probability $\frac{1}{6}$ and each side of the coin occurs with probability $\frac{1}{2}$.

Worked example 18.7

Look at the list of outcomes for flipping a coin three times. If you list the possible outcomes, you should have:

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

There are eight possible outcomes, each of which has a probability of

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

Now answer the following questions.

a What is the probability of getting two heads?

b What is the probability of getting more than one head?

c What is the probability of getting at least one head?

Solution

a There are three outcomes that feature two heads: HHT, HTH and THH.

That means the corresponding probability is $3 \times \frac{1}{8} = \frac{3}{8}$

We write this as $P(2H) = \frac{3}{8}$

b More than one head means two or three heads. The outcomes that fit that description are HHT, HTH, THH, HHH.

Therefore $P(T > 1) = \frac{4}{8} = \frac{1}{2}$

c At least one head means one, two or three heads. This is all cases except the one where we have no heads. That means that in this case it is easier to use the complement.

$P(H \geqslant 1) = 1 - P(H = 0) = 1 - \frac{1}{8} = \frac{7}{8}$

This gives the same answer as adding the probabilities of the seven outcomes fitting the description, but is quicker and shows good insight.

Reminder

Simplify your answers when possible.

Worked example 18.8

Alfonso has a bag with 8 blue discs and 5 red discs. He pulls out a disc, records its colour and then *puts it back*. He then repeats this operation.

a What is the probability that Alfonso obtains:

i 2 blue discs

ii 2 discs of the same colour

iii 2 discs of a different colour?

b Answer parts **i–iii** above, assuming that Alfonso *does not* put the first disc back.

Solution

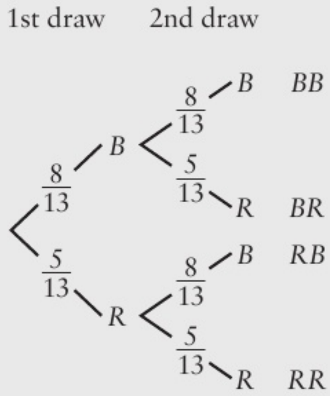
We may simply list the different possible outcomes and find the corresponding probabilities. This can be done through the construction of a probability tree.

When we construct the tree, we need to identify the possible outcomes and the probabilities that go with each of the branches. We should make sure that the sum of the probabilities on the branches at each of the splits is equal to 1. In this case they are all the same:

$$\frac{8}{13} + \frac{5}{13} = \frac{13}{13} = 1$$

These probabilities follow from the fact that we have a total of:

$8 + 5 = 13$ discs



a **i** To find the probability of getting 2 blue discs, we need to walk these branches in the probability tree. Since the outcome of the first disc does not influence the outcome of the second disc, the events are independent and we can use the multiplication rule. The probability of getting 2 blue discs is

$$\text{P}(BB) = \frac{8}{13} \times \frac{8}{13} = \frac{64}{169} = 0.379$$

ii 2 discs of the same colour means either 2 blues, which we calculated in part a, or 2 reds, which we can find in a similar way:

$$\text{P}(RR) = \frac{5}{13} \times \frac{5}{13} = \frac{25}{169} = 0.148$$

We then add the probabilities of these events, since they are mutually exclusive, the addition principle applies.

$$\text{P}(BB \cup RR) = \frac{64}{169} + \frac{25}{169} = \frac{89}{169} = 0.527$$

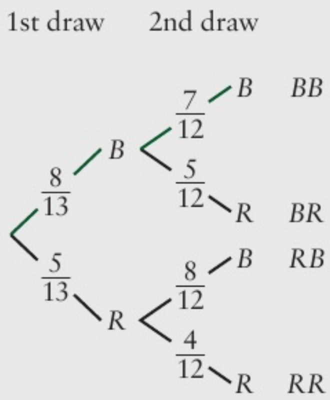
iii 2 discs of a different colour, means a blue and a red disc, in any order. These are the two events that we did not consider in part b, so we can use the complement rule.

$$\text{P}(BR \text{ or } RB) = 1 - \frac{89}{169} = \frac{80}{169} = 0.473$$

Alternatively, we could work it out using the multiplication rule as before.

$$\text{P}(BR) + \text{P}(RB) = \frac{8}{13} \times \frac{5}{13} + \frac{8}{13} \times \frac{5}{13} = \frac{80}{169} = 0.473$$

b The answers will not be the same as in part a, because, after the first draw, the sample space will change. The tree diagram below shows the new situation. The first and second draws' outcomes are not independent!



i To get two blue discs, the first and the second disk must be blue.

This is represented by the top branches as shown.

$$P(BB) = \frac{8}{13} \times \frac{7}{12} = \frac{56}{156} = 0.359$$

ii 2 discs of the same colour means either 2 blues, which we calculated in part a, or 2 reds, which we can find in a similar way:

$$P(RR) = \frac{5}{13} \times \frac{4}{12} = \frac{20}{156} = 0.128$$

We then add the probabilities of these events, since they are mutually exclusive and therefore the addition rule applies.

$$P(BB \cup RR) = \frac{56}{156} + \frac{20}{156} = \frac{76}{156} = 0.487$$

iii 2 discs of a different colour means a blue and a red disc, in any order. We did not consider these two events in part b, so we can use the complement rule.

$$P(BR \cup RB) = 1 - \frac{76}{156} = \frac{80}{156} = 0.513$$

Alternatively, we could work it out using the multiplication rule as before.

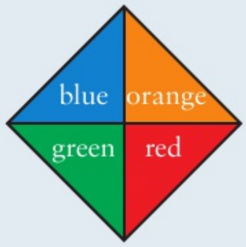
$$P(BR) + P(RB) = \frac{8}{13} \times \frac{5}{12} + \frac{5}{13} \times \frac{8}{12} = \frac{80}{156} = 0.513$$

Practice questions 18.2.3

1 A fair 6-sided die is rolled and a fair coin is flipped. Use a tree diagram to find the probability of getting an odd score on the die and tails on the coin.

2 Two fair 6-sided dice are rolled. Use a dot diagram to find the probability of getting the same score on both dice.

3 Two spinners like the one here are spun simultaneously. The scores are added. Draw a dot diagram of the possible outcomes and find the probability of getting:



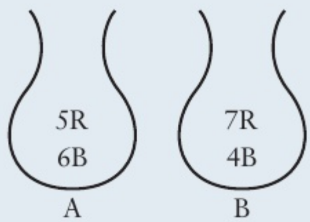
a a 3 on both spinners

b an even result for both spinners

c a sum of 6

d a sum of less than 5.

4 Beatriz has put blue and red marbles in vases A and B, as illustrated in the diagram. She takes a marble out of one of the vases and then replaces it. She then picks a second marble. Each time, it is twice as likely that she will choose vase A. What is the probability that she chooses two marbles of the same colour?



Connections Q5

The language used in probability may resemble that of fractions.

5 Noa can choose to come to school by bike or take the bus. Her choice will depend on the weather. When it does not rain, the probability that she will take the bike is 85%. When it does rain the probability is only 25%.

The probability that it rains on any given day is 20%. When Noa takes the bus, she arrives at school on time 4 out of 5 times, whereas when she takes the bike, she is on time 9 out of 10 times. Find the probability that she arrives at school on time on any given day.

6 Yichen rolls two dice and records the difference in the scores.

a Draw a table to represent the scores.

b Find the probability that the difference is:

i exactly 1 ii more than 3 iii between 0 and 4.

7 Agnes draws a card from a deck, then replaces it and draws a second card. Draw a tree diagram to answer the following questions. What is the probability that Agnes draws:

a two cards of the same suit

b two cards of a different suit

c two cards of the same colour?

Self assessment

- I can calculate the experimental probability of an event, given a table of results.
- I can calculate the theoretical probability of a single event and of combined events.
- I can find the expected number of outcomes based on both experimental and theoretical probability.
- I can organise outcomes of combined events in different ways and know how to select the best method.
- I can use lists, tables, tree diagrams, dot diagrams and Venn diagrams to help me to work out probabilities.
- I can distinguish between dependent, independent and mutually exclusive events.

Check your knowledge questions

1 The hair colour of teachers in a Swiss international school are recorded as shown in the table.

Hair colour	Number
Brown	23
Black	14
Red	7
Blond	11
Grey	5

What is the probability that the hair of a randomly chosen teacher of this school is:

a blond b not blond c grey d red?

2 If two dice are rolled, what is the probability that the total score is:

a 7 b at least 9

c no more than 3 d more than 6 and less than 10.

3 A box contains 12 apples, 6 lemons and 8 pears. What is the probability that a randomly chosen fruit is:

a a pear b a banana c not a citrus fruit?

4 Four coins are flipped. Before answering the questions below, record the possible outcomes in a list. What is the probability of obtaining:

a 4 tails b 2 heads and 2 tails

c more heads than tails?

5 Renée and Ashley both play basketball. The probability that Renée scores on a free throw is 0.8. The probability that Ashley scores on a free throw is 0.7. If they both get one shot, what is the probability that:

a both score

b both miss

c at least one of them scores?

6 Ramon either cycles or takes the bus to school. When he cycles, the probability that he is on time is 0.8. When he takes the bus, he is on time 90% of the time. He cycles 75% of the time.

a Make a tree diagram of the given situation.

b What is the probability that Ramon is on time on a randomly chosen day?

7 A group of 50 students is asked which sports they play. The students replied as follows:

38 played football, 16 played basketball and 7 played neither.

a Make a Venn diagram to represent this information.

b How many students played both basketball and football?

8 Of a group of 80 people, eye colour and the wearing of glasses are recorded. The results are in the table. Using a probability argument, state whether or not eye colour and the wearing of glasses are independent.

Eye colour	Wearing glasses	Not wearing glasses
Brown	18	25
Other	17	20

19 Statistics

19 Statistics

KEY CONCEPT

Form

RELATED CONCEPTS

Representation, Patterns, Models

GLOBAL CONTEXT

Fairness and development

Statement of inquiry

Representing data in different forms helps us to identify patterns within data and relationships between sets of data, which in turn helps us to make fair decisions.

Factual

- What are the different sampling methods?
- How can you use cumulative frequencies to estimate the median of a data set?
- What is the difference between correlation and causation?

Conceptual

- How can you be sure that you have a selected a fair sample?
- What is the difference between univariate and bivariate data?

Debatable

- How can models be used to help make decisions?
- Is all data useful data?

Do you recall?

1 For the list of numbers below:

a state the mode

b identify the median

c calculate the mean.

23 42 13 4 12 3 31 6 6

2 For the data in the table:

a identify the mode

b calculate the mean.

Level	1	2	3	4	5	6	7	8
Frequency	0	13	16	25	21	17	6	2

3 For the continuous data in the table:

a identify the modal group

b calculate an estimate for the mean by using the midpoint of each group as a representative value.

Group	$0 \leqslant x < 10$	$10 \leqslant x < 20$	$20 \leqslant x < 30$
Frequency	2	16	25

Group	$30 \leqslant x < 40$	$40 \leqslant x < 50$
Frequency	7	7

19.1 Collecting data

19.1.1 Choosing to take a census or a sample

Fact

Statistics is the science of collecting, classifying, describing, representing and analysing data to enable decision making.

When developing a statistical model to help you to make fair decisions, it is important to make sure that the data used to make the model is fair and accurate. If you have data that is biased or inaccurate, then your decisions will be affected.

The way data is collected depends on the type of data, how much time you have and the resources you have to collect it.

Communication skills

Fact

A **census** is where you collect the data from an entire population; the population can be a collection of people or items.

A **sample** is where you choose to collect data from a subset of the population to make your model.

Explore 19.1

Consider the following examples. For each scenario, state whether you would conduct a census or whether you would choose to take a sample. Give reasons for your answer.

The World Health Organization is concerned about the water quality available in the world's schools. They would like to test the water.

A washing machine manufacturer has to test an important part of the machine to see how long it will work under extreme conditions. They have to test the part until it breaks.

You would like to find the mean height of the students in your class.

Your friends are discussing their allowances. You would like to determine the mode of the values.

A botanist is writing a book about a particular tree. She needs to include the average leaf size and the average diameter of the trunk.

Many countries carry out a regular census. The data gathered is used to inform decisions on services and policies.

When considering whether to conduct a full census or to a take a sample, there are many factors that need to be considered. Here are two major considerations.

- If the population is large, conducting a census will be time-consuming and expensive, so it is better to take a sample.
- If you are performing destructive tests, you won't be able to break all of the items, so you will need to take a sample.

19.1.2 Choosing a sampling method – review

There are many different ways in which you can take a sample from a population in order to collect data. Which method you choose depends on the type of data you are dealing with. Some of the methods are random, some are not. In random sampling, every item in the population has an equal chance of being chosen.

Fact

Choosing randomly does not give a truly random sample! You must use random number tables or random number generators to guarantee statistical randomness.

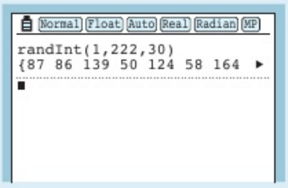
Random sampling methods

Simple random sampling

This method makes sure that you are not choosing specific items for your sample, that is, it is not biased. You can assign a number to each item in the population and then use the random number generator on your calculator or random number tables to select items for your sample.

Hint

You can use a graphical display calculator to generate a set of random numbers.

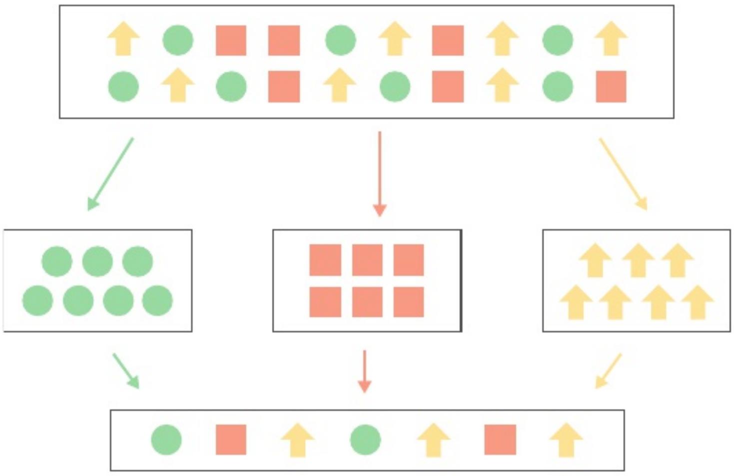


Systematic random sampling

This method forms a sample by choosing items systematically from a list. For example, for a population size of 3000, a suitable sample size would be 30 items. $\frac{3000}{30} = 100$ so you can randomly choose a starting item from the first 100 items, and then select every 100th item after that. For example, if you start with the 10th item, then you would select the 110th, 210th, 310th and so on.

Stratified sampling

This method divides the population into groups called **strata** (singular **stratum**) and selects a random sample from each group. The proportion of the sample represented by each group is the same as its proportion of the population. For example, if you are interested in finding out the music interests of students in your entire school and you suspect that the age groups differ in their preferences, then your strata will be the different age groups, and within each group you would take a random sample. For example, if one fifth of the students in the school are in Grade 9, then this year group should make up one fifth of the sample.



Non-random sampling methods

Convenience sampling

This method is used if you need to take a sample, but you cannot access the population easily. Instead, it uses items or people that are easily available. For example, if you want to find out the opinion of the school community on the school dress code but you don't have time to ask everyone for their opinion, you could send a questionnaire to the parents of our classmates.

Quota sampling

This method allows you to predetermine which type of item you would like to have in your sample. For example, if you are asking members of a community for their opinions, you could set yourself a target of asking a set number of men and women, say 100 men and 110 women. Then you stop asking members of each group once you have reached these quota.

Explore 19.2

The table below lists the percentage of each country's population with a basic level of access to clean water.

If the countries in the table are taken to be the population, then the world mean value for the percentage of people with access to the basic level of clean water is 81%.

The values in the % column are the percentage of each country's population with access to the basic level of clean water, and the numbers in the next column are the numbers assigned to each country to enable a sample to be generated.

Does the mean of a sample change when you use different sampling methods?

Is there a sampling method that gives a better estimate of the population mean?

Fact

The table is a section of the larger table of data that can be found here:

	Country	%		Country	%		Country	%		Country	%
1	Afghanistan	67	56	Djibouti	76	111	Libya	99	166	Saint Lucia	98
2	Albania	91	57	Dominican Republic	97	112	Liechtenstein	99	167	Saint Pierre Miquelon	91
3	Algeria	94	58	Ecuador	94	113	Lithuania	98	168	Saint Vincent & Grenadines	95
4	American Samoa	99	59	Egypt	99	114	Luxembourg	99	169	Samoa	97
5	Andorra	99	60	El Salvador	97	115	Madagascar	54	170	San Marino	99
6	Angola	56	61	Equatorial Guinea	65	116	Malawi	69	171	Sao Tome and Principe	84
7	Anguilla	97	62	Estonia	99	117	Malaysia	97	172	Saudi Arabia	99
8	Antigua & Barbuda	97	63	Eswatini	69	118	Maldives	99	173	Senegal	81
9	Armenia	99	64	Ethiopia	41	119	Mali	78	174	Serbia	86
10	Australia	99	65	Falkland Islands	95	120	Malta	99	175	Seychelles	96
11	Austria	99	66	Faroe Islands	99	121	Marshall Islands	88	176	Sierra Leone	61
12	Azerbaijan	91	67	Fiji	94	122	Martinique	99	177	Singapore	99
13	Bahamas	99	68	Finland	99	123	Mauritania	71	178	Sint Maarten	95
14	Bahrain	99	69	France	99	124	Mauritius	99	179	Slovakia	99

When a sample is taken correctly from a population, the mean of the sample will be a good approximation of the mean of the population.

The method of sampling that you choose can affect the validity of the process and can result in an inaccurate estimate, in particular if the sample is biased. A balance has to be made between efficiency and accuracy. Although it is more accurate to use data from the whole population, is it more costly and time-consuming than using a sample.

Worked example 19.1

Gwil is a member of his school council. He has been asked to find out what the students think about the school's recycling facilities. He decides to give a questionnaire to a sample of 60 students and thinks that different year groups will have different opinions.

There are 110 students in Grade 9.

There are 115 students in Grade 10.

There are 125 students in Grade 11.

There are 118 students in Grade 12.

a State which sampling method would be the best one for Gwil to use.

b Explain how he could use this method to select 60 students for his sample.

Solution

Understand the problem

The population is divided into groups. As the number of students in each grade varies, he must have the same proportion of students in the sample as in the population. This would be a stratified sample.

Make a plan

Gwil can determine the proportion of each grade in the population as a fraction and then find these proportions of the population to give the number of people in each grade that would be required in the sample.

Carry out the plan

a Stratified sampling.

He can determine the number of students from each grade to include in the sample by calculating the fraction of the population represented by each grade and multiplying each fraction by the sample number of 60.

b There are 468 students in total in the school.

Summarising the calculations:

Grade	Number of students in the grade	Fraction of students in the grade	Number of students in the sample	Number in the sample
9	110	$\frac{110}{468}$	$\frac{110}{468} \times 60 = 14.102\ldots$	14
10	115	$\frac{115}{468}$	$\frac{115}{468} \times 60 = 14.7435\ldots$	15
11	125	$\frac{125}{468}$	$\frac{125}{468} \times 60 = 16.0256\ldots$	16
12	118	$\frac{118}{468}$	$\frac{118}{468} \times 60 = 15.128\ldots$	15

Gwil would then choose the given number in the sample randomly from the different grades. Gwil is selecting students, so it is sensible in this case to round the calculated value to an integer; we can't ask 0.102 people.

Look back

Adding the number of students from each grade in the sample ($14 + 15 + 16 + 15$) gives 60, as required. The grade with the greatest number of students, Grade 11, has the highest number of people in the sample, while the grade with the fewest students, Grade 9, has the lowest number of people in the sample. So the grades in the sample are in the same proportion as in the population. The required number of students from each grade can then be selected randomly.

Reflect

What other sampling methods could Gwil have used?

Worked example 19.2

Communication skills

Gwil conducted his survey on the school's recycling facilities by sending a questionnaire to students selected at random from the four grades. He received the following number of responses.

Grade	Number in the sample	Number of responses
9	14	10
10	15	9
11	16	15
12	15	15

Hint

percentage response rate $= \dfrac{\text{number of responses}}{\text{number required for the sample}} \times 100$

a Calculate the response rate for each grade.

b Given that Gwil has decided to accept a response rate of 60% for each group, how could these calculations affect his survey results?

Solution

The number of students to be sampled in each grade was calculated so that each group in the sample was fairly represented. The number of responses indicate that not everyone responded to the survey. Calculating the response rate will allow us to determine whether there are enough students in the sample to ensure that it is fair and representative of the population.

a We can calculate the response rate for each group and compare the values with the accepted 60% guide.

Grade	Number in the sample	Number of responses	Response rate %
9	14	10	$\frac{10}{14} \times 100 = 71.42$
10	15	9	$\frac{9}{15} \times 100 = 60$
11	16	15	$\frac{15}{16} \times 100 = 93.75$
12	15	15	$\frac{15}{15} \times 100 = 100$

b The lowest response rate is from the Grade 10 students, which means that there is a lower proportion of the Grade 10 students in the sample than there is in the population. However, this is still at the acceptable level of 60% so Gwil's sample could still be considered representative of the population.

The response rates are different for different groups and this affects the proportion of the sample that each group represents. But there is an arbitrary cut-off value of 60%, which means that the sample is acceptable.

Reflect

Can you think of an example of when you would require a higher response rate for the sample to be a good representation of the population?

What would have happened if Gwil had chosen a 75% response rate?

Practice questions 19.1

1 For each of the examples below, state whether you would choose to carry out a census or a sample. Explain your decision.

a A company would like to determine the mean temperature at which their SIM cards start to melt.

b Michelle is going to host a party for the mathematics graduates at her college. She wants to know their preferences for a venue.

c The World Health Organization is planning to conduct a statistical analysis of the use of vaccines in different countries.

d Tina thinks that her school friends spend too long browsing social media sites. She would like to determine the mean screen-time for her and her friends.

2 Describe one advantage and one disadvantage for each of the following sampling methods.

a Random sampling
b Systematic sampling
c Stratified sampling
d Convenience sampling
e Quota sampling

3 A new industrial development is due to be based in a small town in Switzerland. The town's council are concerned about the effect it could have on the local community, so they decide to canvass opinion from the town's population. The council have the following information about the population.

Gender	Age range (years)			Total
	0–15	16–19	20+	
Male	384	811	4750	5945
Female	328	879	4944	6151
Total	**712**	**1690**	**9694**	**12 096**

Explain how you would generate a stratified random sample of 200 people, given the information in the table. Show your working.

4 Kahoru is developing a website to provide information for students who are new to his university. He has a few ideas about what to include but is not sure whether new students would like to know about recycling stations. He decides to ask a sample of current first year students and has been given a spreadsheet of names from the Student Welfare office. The list contains 4000 names. Kahoru only needs a sample of 50.

a Describe how he would generate a sample if:
   i he chose to use simple random sampling
   ii he chose to use systematic random sampling.

b Explain how Kahoru could make sure that the systematic sample has additional randomness.

5 Helena is a psychology student. She is exploring how the amount of sleep a student has affects their performance in recall tests. She decides to use convenience sampling and asks her classmates to be her test subjects.

a Describe one advantage and one disadvantage for using this method.

b As a statistician, what would your advice be?

6 Calculate the response rates for the following examples. State the decision you would make if the accepted response rate is 65%.

a Sample size of 100, number of respondents 90

b Sample size of 350, number of respondents 320

c Sample size of 45, number of respondents 27

d Sample size of 300, number of respondents 192

19.2 Organising and describing data

19.2.1 Organising data

Communication skills

Explore 19.3

The table shows the percentage scores obtained by a large school's Grade 10 students in a mid-year exam.

23	24	32	89	97	88	83	14	12	59
21	18	88	32	96	78	76	29	76	16
82	32	19	85	76	82	19	26	21	59
79	76	87	88	85	15	25	89	23	21
65	43	21	26	86	85	16	43	23	21
54	78	23	13	21	29	89	78	88	76
89	16	21	24	27	54	56	76	15	59
76	34	42	87	89	86	21	45	87	78
5	76	56	66	68	91	54	23	31	24
68	9	87	89	99	21	34	21	43	45
89	87	78	87	69	87	76	23	56	78
42	78	98	78	56	32	79	78	76	89

Try to organise the data in a way that allows you to draw meaningful conclusions. What information can you acquire from the organised data?

When we collect data, we usually have a lot of information in a format that doesn't easily allow for the identification of any patterns. In order to discover patterns, we need to organise the data and represent it in different forms that make it easier to describe trends.

19.2.2 Describing data: Measures of central tendency

A statistic is a measure that represents a data set. Statistics that represent the centre value of a set of data are called **measures of central tendency**. Examples are the mean, the median and the mode.

Worked example 19.3

The data from Explore 19.3 is organised into a frequency table.

a State the modal group.

b Estimate the mean score.

c Estimate the median score.

d Comment on the values you obtained for parts a to c.

Test score (%)	Frequency
$0 \leqslant x < 10$	2
$10 \leqslant x < 20$	16
$20 \leqslant x < 30$	25
$30 \leqslant x < 40$	7
$40 \leqslant x < 50$	7
$50 \leqslant x < 60$	8
$60 \leqslant x < 70$	7
$70 \leqslant x < 80$	16
$80 \leqslant x < 90$	27
$90 \leqslant x < 100$	5

Solution

We can identify the modal group from the table by looking for the group with the highest frequency.

We need to use the midpoint of each group as a representative value of the group. This is why we have been asked to 'estimate' the mean. The data is collected in a frequency table, so we know how many values are in each group.

Reminder

Recall that the $\sum$ symbol used in the equation for the mean is the Greek uppercase letter *sigma*. $\sum fx$ instructs us to find the sum of the values we generate by multiplying each data value by the corresponding frequency value. $\sum f$ instructs us to add up all the frequency values.

Fact

Cumulative frequency is the running total of the frequency values. To find the cumulative frequency of a group, add the sum of the previous frequency values to the frequency of the group.

We can find the midpoint of each group using:

$$\text{midpoint} = \frac{\text{upper bound} + \text{lower bound}}{2}$$

We can find the mean of the data by using the summation formula:

$$\text{mean} = \frac{\sum fx}{\sum f}$$

where x is the midpoint value and f is the frequency.

The median is the data value in the middle of the observations when they are in numerical order. Therefore we need to determine the position of the middle value.

a The modal group is the group with the highest frequency, and this can be seen from the table.

The modal group is $80 \leqslant x < 90\%$

The calculations for the mean and median can be organised in a table.

Test score (%)	Midpoint (x)	Frequency (f)	xf	Cumulative group	Cumulative frequency
$0 \leqslant x < 10$	5.5	2	11.0	$x < 10$	2
$10 \leqslant x < 20$	15.5	16	248.0	$x < 20$	18
$20 \leqslant x < 30$	25.5	25	637.5	$x < 30$	43
$30 \leqslant x < 40$	35.5	7	248.5	$x < 40$	50
$40 \leqslant x < 50$	45.5	7	318.5	$x < 50$	57
$50 \leqslant x < 60$	55.5	8	444.0	$x < 60$	65
$60 \leqslant x < 70$	65.5	7	458.5	$x < 70$	72
$70 \leqslant x < 80$	75.5	16	1208.0	$x < 80$	88
$80 \leqslant x < 90$	85.5	27	2308.5	$x < 90$	115
$90 \leqslant x < 100$	95.5	5	477.5	$x < 100$	120
		$\sum f = 120$	$\sum fx = 6360$		

b $\text{mean} = \frac{\sum fx}{\sum f} = \frac{6360\%}{120} = 53\%$

c The median is the data value that is halfway through the ordered data. In this case the total frequency is 120, which means that the middle value is the mean of the 60th and 61st data value, which lies in the data group ($50 \leqslant x < 60\%$). We can estimate where in this group the median could be by using a method called **interpolation**, but for now it is sufficient to describe the median in terms of the median group.

The median group is $50 \leqslant x < 60\%$.

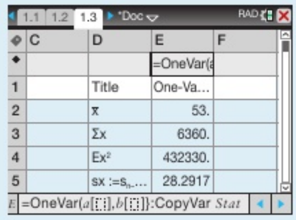
d In this example, neither the mean value nor the median group coincide with the modal group.

The modal group is $80 \leqslant x < 90\%$, the mean is estimated to be 53% and the median group is $50 \leqslant x < 60\%$. The mean is in the median group, so these values are in agreement. However, the modal group is different. This would justify further investigation for underlying patterns.

The mean and median are similar, but different from the modal group. Is there a valid reason for this? Yes, by looking at the data in the table, we can see that there are higher frequencies at the upper end and at the lower end of the scores than in the middle. This suggests that the mean and median could be in between the two groups with the highest frequencies.

Hint

You can use a GDC to perform the calculations by using the one variable statistics (1-Var Stats) option.



Practice questions 19.2

1 a Organise the data below into a grouped frequency table, with values from 0 to 19, using a group size of 5.

b Using your table:

i identify the modal group   ii estimate the mean

iii find the median group.

c Comment on the values you found in part b. What do they tell you about the data?

2	4	3	8	17	18	3	14	12	6
1	18	8	3	9	7	6	9	8	16
2	3	19	8	7	2	19	6	1	6
9	7	8	8	5	15	5	8	3	2
6	4	1	2	6	5	16	3	2	1
4	7	3	13	1	2	9	8	8	7
9	16	1	4	7	5	5	7	15	19

2 As an indicator of fairness and development, the number of students with access to university education in each country across the world is monitored. The table below shows the percentage of the population with university education in 1970. The frequency is the number of countries with a value within the given band.

% of population with university education (P)	Number of countries (frequency)
$0 \leqslant P < 1$	65
$1 \leqslant P < 2$	32
$2 \leqslant P < 3$	14
$3 \leqslant P < 4$	14
$4 \leqslant P < 5$	7
$5 \leqslant P < 6$	4
$6 \leqslant P < 7$	4
$7 \leqslant P < 8$	1
$8 \leqslant P < 9$	0
$9 \leqslant P < 10$	1
$10 \leqslant P < 11$	1
$11 \leqslant P < 12$	1

a State the modal group.

b Calculate an estimate for the mean percentage of the world's population with a university education in 1970.

c Determine the median group.

3 The table below shows the percentage of the population with a university education for the year 2010. As in question 2, the frequency is the number of countries with a value within the given band.

% of population with university education (P)	Number of countries (frequency)
$0 \leqslant P < 5$	63
$5 \leqslant P < 10$	31
$10 \leqslant P < 15$	24
$15 \leqslant P < 20$	16
$20 \leqslant P < 25$	6
$25 \leqslant P < 30$	3
$30 \leqslant P < 35$	1

a State the modal group.

b Calculate an estimate for the mean percentage of the world's population with a university education in 2010.

c Determine the median group.

d Compare your statistic values for 1970 (calculated in question 2) and 2010. Was there a change in the percentage of the world's population with a university education over the 40-year period?

4 In 2020, the UK government conducted a survey to determine the impact of a campaign to promote diversity in the workplace. One of the factors on which they focused was the age range of the workforce.

The table shows the number of employees (in thousands) in each age range.

Age range	Number of UK employees (thousands)
$16 \leqslant P < 25$	2321
$25 \leqslant P < 30$	4582
$30 \leqslant P < 35$	4703
$35 \leqslant P < 40$	5311
$40 \leqslant P < 45$	5983
$45 \leqslant P < 50$	5983
$50 \leqslant P < 55$	4914
$55 \leqslant P < 60$	3877
$60 \leqslant P < 65$	2106
$65 \leqslant P < 70$	733

a Calculate an estimate for the mean age of employees in the UK in 2020.

b Determine the median age of employees in the UK in 2020.

c Does the data show a fair representation of different ages in the workforce? Explain your answer.

5 As part of a similar survey, in 2020, the UK government gathered data on the age ranges of the workforce who had been unemployed for 1 year or more. The data is given in the table.

Age range	Number of UK unemployed (thousands)
$16 \leqslant P < 18$	4.9
$18 \leqslant P < 25$	54.0
$25 \leqslant P < 50$	86.0
$50 \leqslant P < 70$	87.0

a Calculate an estimate for the mean age of unemployed people in the UK.

b Determine the median age group of unemployed people in the UK.

c Does the grouping of data in this table give a fair representation of the unemployment data?

Fact

range of a data set = maximum value − minimum value

19.2.3 Describing data: Measures of dispersion

A statistic is a measure that represents a data set. Statistics that represent the spread of data are called **measures of dispersion**: for example, the **range**, the **interquartile range** and the **variance** and **standard deviation** (**section 19.4**).

Communication skills

Investigation 19.1

Molly and Max were discussing the time their friends spend watching shows online. Molly thought that she and her friends spend less time than Max and his friends. They both decide to ask their friends to record the total number of the hours they spend watching shows online during one week.

The two sets of data are shown in the table.

	Hours spent watching online shows										
Max's friends	1	3	11	12	14	15	15	17	17	20	32
Molly's friends	9	10	10	11	11	12	13	13	16	16	18

Determine the mean and the median for both sets of data.

Determine the range of both sets of data.

Max thinks that the range of his friends' data is affected by the two extreme values at each end of the table. He believes these are outliers, so he suggests that the spread of his data would be represented more accurately by measuring the range of the middle section of his data. He divides each data set into two equal sections, on either side of the median, and then he identifies the middle value of the lower half and the middle value of the upper half of each data set.

Identify the middle value of the lower half of each set of data.

Identify the middle value of the upper half of each set of data.

Is the difference between these two values a better representation of the spread of each set of data?

Explain your reasoning clearly.

Fact

An **outlier** is an observation which seems to lie outside of the general pattern of the rest of the data.

The median value divides a data set into two sections with an equal number of data values on each side.

Quartiles divide a data set into four equal sections. These values are called the first quartile (Q1), the second quartile (Q2) and the third quartile (Q3)

Interquartile range of a data set (IQR) = third quartile − first quartile

The second quartile is the same data value as the median.

Fact

The quartiles are also labelled as the **lower quartile** (Q1) and **upper quartile** (Q3).

When we use the median as a control measure, there are two measures of spread that quantify the range of a data set.

The **range** is the maximum value – the minimum value. This is affected by outliers.

The **interquartile range** is the range of the middle 50% of the data. This is not affected by outliers.

In the example in Investigation 19.1, Max thought that two values in his data set were outliers because they seemed to be very different from the rest of the data.

Outliers can be identified using the following calculations:

Boundary for lower outlier = Q1 – (1.5 × IQR)

Boundary for upper outlier = Q3 + (1.5 × IQR)

Any value lower than the lower boundary or higher than the upper boundary is considered to be an outlier.

For Max's data:

Boundary for lower outlier = $11 - (1.5 \times 6) = 2$

Boundary for upper outlier = $17 + (1.5 \times 6) = 26$

So Max is correct in his assumption that the data values 1 and 32 are outliers.

Worked example 19.4

Research skills

Communication skills

Sam is conducting a study into the weekly wages earned by students at his school who have part-time jobs. He believes that the spread of the data will be quite small as the students should be earning similar amounts. Their working hours will be similar, and they have similar skill sets. He conducts a survey of the two grades who are following the IB Diploma Programme. The weekly wages (AUD) are summarised in the table.

Wages per week (AUD)	Number of students in Sam's school
$0 \leqslant P < 10$	22
$10 \leqslant P < 20$	18
$20 \leqslant P < 30$	14
$30 \leqslant P < 40$	14
$40 \leqslant P < 50$	3
$50 \leqslant P < 60$	1
$60 \leqslant P < 70$	1

a Use a GDC to determine an estimate for:

i the mean

ii the median

iii the lower quartile

iv the upper quartile.

b Calculate an estimate for the interquartile range of Sam's data.

c Determine whether there are any outliers in the data.

d Is Sam correct in thinking that the spread of the data is small? Justify your answer.

Solution

The data is grouped so we must use the midpoint as a representative value of each group. All of the statistic values will be estimates. A GDC can be used to find the data values so we will focus on interpreting these.

We will determine the midpoint of each group and enter these into the GDC list or spreadsheet. The key statistics will be given by the GDC.

The key statistics can be used to find the interquartile range:

interquartile range of a data set = upper quartile − lower quartile

Outliers can be identified once the IQR is known:

boundary for lower outlier = Q1 – (1.5 × IQR)

boundary for upper outlier = Q3 + (1.5 × IQR)

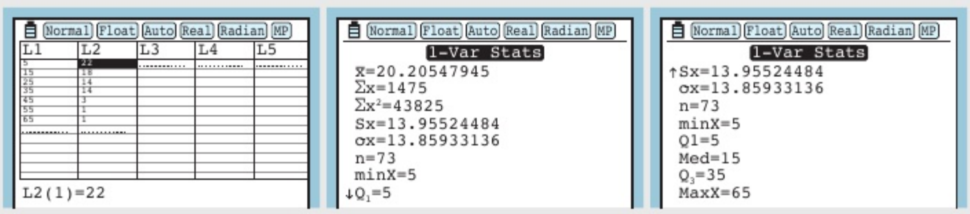
When we have found the values of the IQR and the value of any outliers we can discuss whether Sam's hypothesis is correct.

a

Wages per week (AUD)	Midpoint value (AUD)	Number of students in Sam's school
$0 \leqslant P < 10$	5	22
$10 \leqslant P < 20$	15	18
$20 \leqslant P < 30$	25	14
$30 \leqslant P < 40$	35	14
$40 \leqslant P < 50$	45	3
$50 \leqslant P < 60$	55	1
$60 \leqslant P < 70$	65	1

We enter the midpoint and the frequency of each wage range into the GDC.

One output example:



The key statistics are obtained:

i Estimate of mean is AUD 20.21

ii Estimate of median is AUD 15

iii Estimate of lower quartile is AUD 5

iv Estimate of upper quartile is AUD 35

b The estimate of the interquartile range (IQR) is therefore:

$\text{IQR} = 35 - 5 = 30$

c Outliers can be identified as outside of these boundaries:

boundary for lower outlier $= 5 - (1.5 \times 30) = -40$

boundary for upper outlier $= 35 + (1.5 \times 30) = 80$

So, any value less than –40 or greater than 80 is an outlier.

Therefore, there are no outliers identified in Sam's data.

d As Sam's data is grouped, we can only estimate the spread of the data.

The IQR is 30.

We can estimate the range by subtracting the lowest possible value from the highest possible value.

range $= 70 - 0 = 70$

The estimates for the mean and median are 20.21 and 15. These are much smaller than the range, which suggests that Sam is incorrect. Although he expected the spread of the weekly wages to be small, the data suggests that this might not be true.

The frequency table shows higher frequencies towards the lower end of the wage ranges. There is only one student in each of the upper wage bands. This supports the fact that the estimates of the mean and median are lower than the range. This suggests that the data has a larger range than Sam anticipated.

Reflect

In Australia, there are different minimum wage rates for young people at 16, 17, 18 and 19 years of age. Would this make a difference to Sam's interpretation of his data?

The data in this example is grouped. Is this method an accurate way of finding the quartiles? Is there another way that would be more accurate?

19.3 Representing and analysing data

19.3.1 Representing and analysing discrete data

Communication skills

Explore 19.4

The following data was used to investigate the time that Max and Molly's friends spend watching online shows.

	Hours spent watching online shows										
Max's friends	1	3	11	12	14	15	15	17	17	20	32
Molly's friends	9	10	10	11	11	12	13	13	16	16	18

In order to get as much information as possible and use that information to compare the two data sets, Molly suggested that the data can be represented by a back-to-back stem-and-leaf plot. Can you check what information she can get from doing that?

Can you suggest for Max another type of representation that he can use?

Representing data in a stem-and-leaf plot can help to identify or calculate the values of the key statistics: mean, mode, median, lower quartile, upper quartile, maximum and minimum. These can be compared for different data sets.

There is another type of diagram that can be used to visualise the key statistics that represent the spread of a set of data.

A **box-and-whisker plot** can be used to represent five key statistics of a data set: maximum, minimum, median, lower and upper quartiles. This helps to identify patterns in the spread of the data.

Fact

The name box-and-whisker plot is often shortened to box plot.

Worked example 19.5

a Represent the statistics determined from Max's and Molly's data in the form of two box plots above the same horizontal axis.

b Use your box plots to compare the spread of each set of data.

	Hours spent watching online shows										
Max's friends	1	3	11	12	14	15	15	17	17	20	32
Molly's friends	9	10	10	11	11	12	13	13	16	16	18

Solution

The data is discrete and organised in a table. We can find the maximum value, minimum value, median, lower quartile and upper quartile and represent them in a box plot with a suitable horizontal scale.

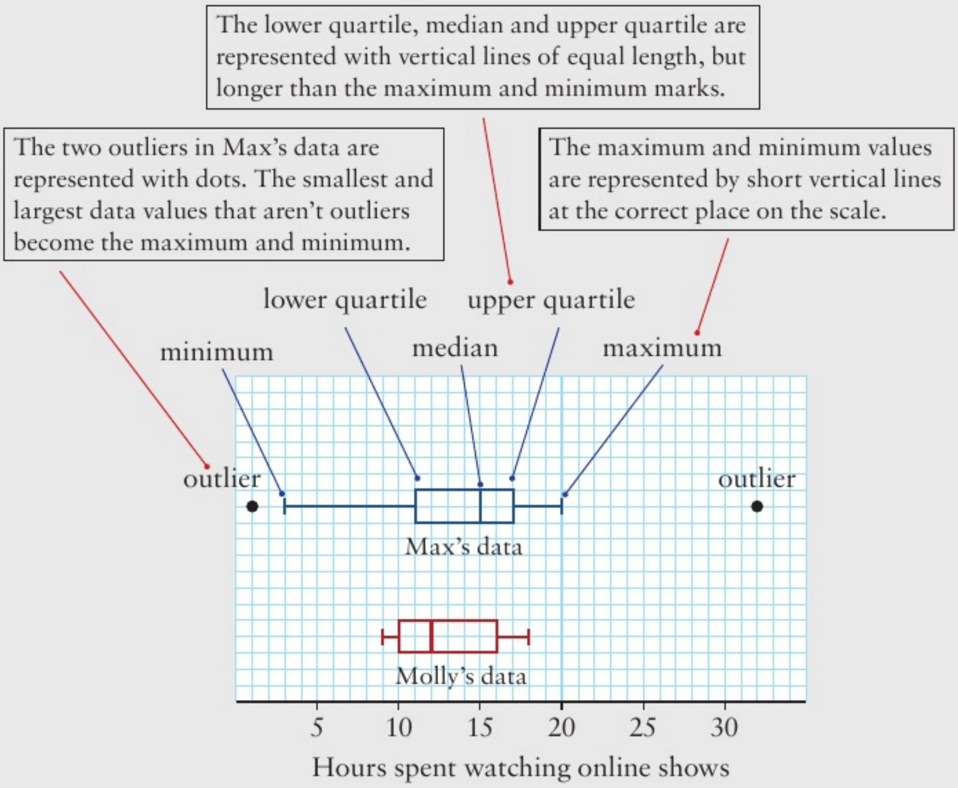
We start by collecting and organising the key statistics in a table. Then we identify an appropriate horizontal scale for the box plots. Finally, we draw box plots to represent the key statistics for each data set.

a Summary of the key statistics:

Key statistics	Max's friends	Molly's friends
Minimum	1	9
Lower quartile	11	10
Median	15	12
Upper quartile	17	16
Maximum	32	18

The smallest value is 1 and the largest is 32, so our horizontal scale needs to go from 1 to 32.

Plotting the data values:



Hint

You may already have found these values during the Investigation in Section 19.2.3 or during Explore 19.4.

Hint

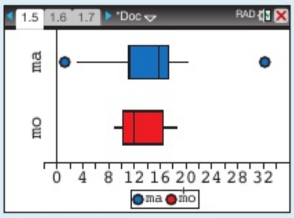
We can use a GDC to find the key statistics in the order we need them for a box plot.

Hint

The vertical position of each box plot is arbitrary. There is no scale on the vertical axis.

Hint

A GDC or suitable software can be used to represent your data in box plot form.



The central box is completed by drawing a horizontal line above and below, joining the lower quartile, median and upper quartile. The whiskers are formed by drawing a line from the centre of the upper quartile line to the maximum point and the centre of the lower quartile to the minimum point.

When the two sets of data are represented by box plots above the same scale, we can compare the spread of each set. The box plots show that Molly's data is less spread out than Max's data; the overall range is smaller, but the middle values have the same spread since the interquartile ranges are equal. The box plots also show the difference between the median values: the median value for Molly's data (12) is smaller than for Max's data (15), which means that on average more than 50% of Molly's friends spend less time watching online shows than Max's friends.

Practice questions 19.3.1

1 Draw a box plot to represent the following five key statistics.

Key statistics	
Minimum	5
Lower quartile	12
Median	21
Upper quartile	28
Maximum	30

2 Draw two box plots above the same horizontal scale to represent the key statistics for the data sets shown in the table. Use your box plots to compare the data sets.

Key statistics	Set 1	Set 2
Minimum	5	1
Lower quartile	12	10
Median	21	16
Upper quartile	28	18
Maximum	30	24

3 Draw two box plots above the same horizontal scale to represent the key statistics for the data sets shown in the table. Use your box plots to compare the data sets.

Key statistics	Set 1	Set 2
Minimum	12	6
Lower quartile	18	19
Median	21	25
Upper quartile	38	32
Maximum	42	38

4 For the data set below, determine the five key statistics for a box plot, identify any outliers and represent the data with a box plot.

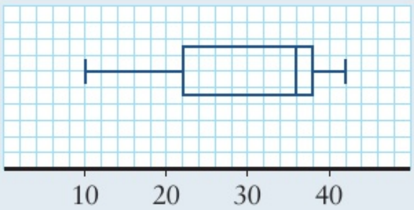
3 10 15 20 21 23 23 24 25 25 29 30 31 33 65

5 For each data set below, determine the five key statistics for a box plot, identify any outliers and represent the data with two box plots above the same horizontal scale.

6 7 9 10 15 16 18 20 45

6 17 18 19 20 20 20 23 24 25 50

6 Identify the five key statistics for the data set from the box plot below.



19.3.2 Representing and analysing grouped data

In Worked example 19.4 we explored how Sam's data for his friends' weekly wages could be analysed using the values of the key statistics. We were able to estimate the mean, median and quartiles for the grouped data by using the midpoints of the groups and treating the data as discrete. However, this gave only estimates of the statistics, and the accuracy of this method should be questioned. Is there another more accurate way to estimate the median and quartiles for grouped and continuous data?

Explore 19.5

In Worked example 19.4 we used a GDC to estimate the median, the lower quartile and the upper quartile.

Wages per week (AUD)	Number of students in Sam's school
$0 \leqslant P < 10$	22
$10 \leqslant P < 20$	18
$20 \leqslant P < 30$	14
$30 \leqslant P < 40$	14
$40 \leqslant P < 50$	3
$50 \leqslant P < 60$	1
$60 \leqslant P < 70$	1

GDC values:

```
Normal Float Auto Real Radian MP
1-Var Stats
 x̄=20.20547945
 Σx=1475
 Σx²=43825
 Sx=13.95524484
 σx=13.85933136
 n=73
 minX=5
↓Q₁=5
```

```
Normal Float Auto Real Radian MP
1-Var Stats
↑Sx=13.95524484
 σx=13.85933136
 n=73
 minX=5
 Q1=5
 Med=15
 Q₃=35
 MaxX=65
```

Could we use the midpoint of the median group as an estimate of the median value?

Does this estimation method give the same value as the GDC?

Explore the values of the key statistics given by the GDC and your estimates based on using midpoints. Are they similar?

Reflect on your findings. Will this always be true?

Is representing a group by using the midpoint an accurate method?

A **bar chart** is usually used to represent categorical data. It is drawn with separate, distinct bars that correspond to one value on the horizontal axis.

When we draw a bar chart for continuous data, the bars touch, to represent the continuity (the fact that all values are possible). The bars correspond to a range of values on the horizontal axis, called **classes**. The **class width** is the difference between the upper and lower boundary of each class. This type of diagram is called a **histogram**.

When a bar chart represents grouped discrete data, it is sometimes drawn with gaps between the bars. If the gaps between the bars are small, it will be drawn with the bars touching. Again, this is called a histogram.

Investigation 19.2

Use this table to answer the questions on the following page.

Wages per week (AUD)	Number of students in Sam's school
$0 \leqslant P < 10$	22
$10 \leqslant P < 20$	18
$20 \leqslant P < 30$	14
$30 \leqslant P < 40$	14
$40 \leqslant P < 50$	3
$50 \leqslant P < 60$	1
$60 \leqslant P < 70$	1

1 Draw a histogram to represent Sam's data about students' weekly wages.

2 Add two columns to the right of the table. Use one to record the cumulative group and one to record the cumulative frequency of each cumulative group.

3 On a separate set of axes, draw a second histogram for Sam's data with wages on the horizontal axis and cumulative frequency on the vertical axis.

4 Using your graph from part 3, identify the coordinate points at the top right-hand corner point of each bar.

5 Plot your coordinate points from part 4 on a new set of axes and join them with a smooth curve. What does this curve represent?

6 Use this curve to find an estimate of the median, the lower quartile and the upper quartile.

7 Compare your estimates from part 6 with those obtained using a GDC in Explore 19.5.

By plotting the end point of a cumulative group as the horizontal variable, and the corresponding cumulative frequency as the vertical variable, we can generate a **cumulative frequency graph**, which can be used to make more accurate estimations of the quartiles and the median values.

A cumulative frequency graph is formed by plotting cumulative frequency on the vertical axis, against the upper boundary of each cumulative group on the horizontal axis, to give a set of discrete coordinate points.

A **cumulative frequency curve** is formed by joining these coordinate points with a smooth curve.

A **cumulative frequency polygon** is formed by joining the coordinate points with straight-line segments.

Worked example 19.6

The data in the frequency table represents the scores of the students in the mid-year exam from Explore 19.3.

Test score (%)	Frequency
$0 \leqslant x < 10$	2
$10 \leqslant x < 20$	16
$20 \leqslant x < 30$	25
$30 \leqslant x < 40$	7
$40 \leqslant x < 50$	7

Test score (%)	Frequency
$50 \leqslant x < 60$	8
$60 \leqslant x < 70$	7
$70 \leqslant x < 80$	16
$80 \leqslant x < 90$	27
$90 \leqslant x < 100$	5

a By first calculating the cumulative frequency for each group, determine the five key statistics: the minimum value, lower quartile, median, upper quartile and maximum value for the data.

b Draw a box plot to represent the values.

Solution

The data is grouped and presented in a frequency table, so the median and the quartiles are not easily identified. We can make a cumulative frequency polygon using the cumulative frequency of each group and use it to find the values of the five key statistics.

We can then represent these on a box plot.

We extend the data table to include a column for the cumulative groups and the cumulative frequency values, and a column with the end point (upper boundary) of each group.

Then we identify the coordinate points required for the cumulative frequency graph and plot them on a coordinate grid. We then join the points with straight-line segments.

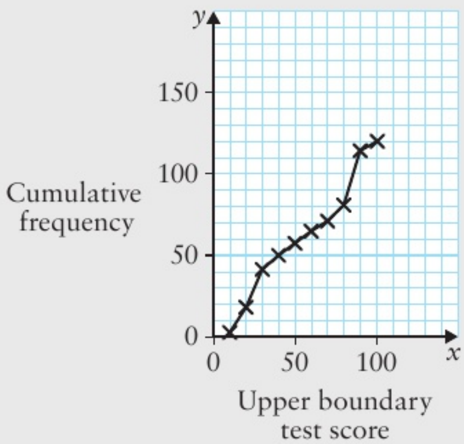
We use the frequency polygon to estimate the median, lower quartile and upper quartile before representing the statistic values on a box plot.

a The cumulative frequency and upper boundary values are shown in the table.

Test score (%)	Frequency	Cumulative groups	Upper boundary of the cumulative group (x)	Cumulative frequency (y)
$0 \leqslant x < 10$	2	$0 \leqslant x < 10$	10	2
$10 \leqslant x < 20$	16	$0 \leqslant x < 20$	20	18
$20 \leqslant x < 30$	25	$0 \leqslant x < 30$	30	43
$30 \leqslant x < 40$	7	$0 \leqslant x < 40$	40	50
$40 \leqslant x < 50$	7	$0 \leqslant x < 50$	50	57
$50 \leqslant x < 60$	8	$0 \leqslant x < 60$	60	65
$60 \leqslant x < 70$	7	$0 \leqslant x < 70$	70	72
$70 \leqslant x < 80$	16	$0 \leqslant x < 80$	80	88
$80 \leqslant x < 90$	27	$0 \leqslant x < 90$	90	115
$90 \leqslant x < 100$	5	$0 \leqslant x < 100$	100	120

The upper boundary of each group gives the x-coordinates, and the cumulative frequencies give the y-coordinates.

Plotting the points and joining with straight-line segments generates the following graph.



We can estimate the median by reading the test score on the horizontal axis that corresponds to the middle value of the data set. The position of the middle value can be found by:

median position = $\frac{\text{total frequency}}{2}$

In this example:

median position = $\frac{120}{2} = 60$

The lower quartile can be identified by reading the test score on the horizontal axis that corresponds to the value $\frac{1}{4}$ of the way along the data set.

The position of the lower quartile value can be found by:

lower quartile position = $\frac{\text{total frequency}}{4}$

In this example:

lower quartile position = $\frac{120}{4} = 30$

The lower quartile is the 30th value.

The upper quartile can be identified by reading the score on the horizontal axis that corresponds to the value $\frac{3}{4}$ of the way along the data set.

The position of the upper quartile value can be found by:

upper quartile position = $\frac{3}{4}$(total frequency)

In this example:

upper quartile position = $\frac{3 \times (120)}{4} = 90$

Hint

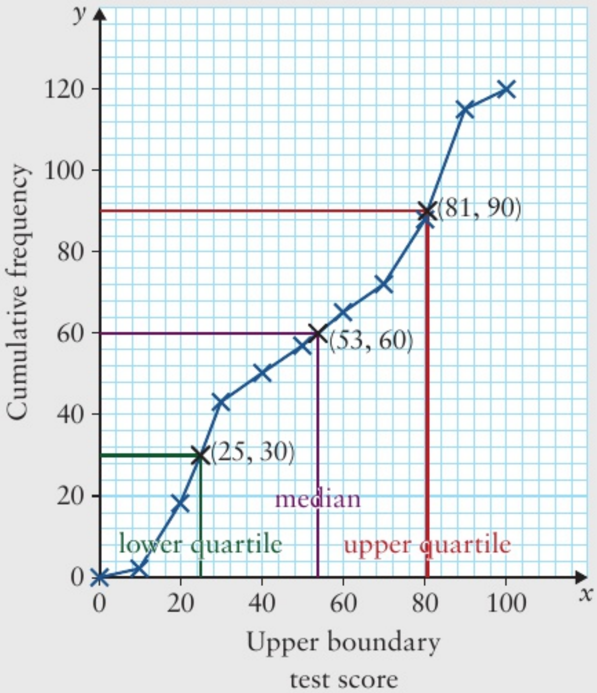
Your GDC can represent data as a cumulative frequency graph by plotting the upper boundary as the horizontal variable and the cumulative frequency as the vertical variable.

Hint

The median and quartile positions are estimates. Thus, your estimate may slightly differ from estimates made by your GDC or software. In fact, if the total frequency is odd, then the median position would be at $\frac{n+1}{2}$ and if it is even then it will be located between $\frac{n}{2}$ and $\frac{n+2}{2}$.

The slight differences are usually tolerated.

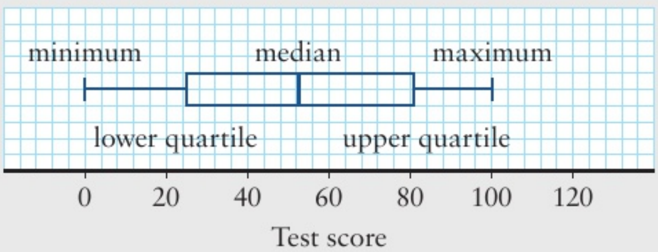
The upper quartile is the 90th value.



Reading from the graph, the upper quartile is 81, the median is 53 and the lower quartile is 25.

As the data is grouped, we can estimate the minimum and maximum values. The minimum estimate is 0 as this is the lower boundary of the first group. The maximum estimate is 100 as this is the upper boundary of the last group.

The box plot shows the five key statistics for this data.



The final cumulative frequency value corresponds to the sum of the frequencies, which suggests they are correct. The cumulative frequency curve has the typical shape of a cumulative frequency polygon. As it is representing the running total of the frequencies it should continue to increase until it reaches the total frequency value, rather than increasing and decreasing as a graph showing the actual frequencies would do. The gradient of the cumulative graph should be greater when there is a higher group frequency. This can be seen for the intervals $20 \leqslant x < 30\%$ and $80 \leqslant x < 90\%$. The original estimate of the median obtained in Section 19.2.2 was 55.5%. This is close to the estimate of 53% obtained from the cumulative frequency graph.

Reflect

Half (or 50%) of the data has a value that is less than the median value.

How can a cumulative frequency graph be used to determine how many data points have a value of more than a given amount?

Practice questions 19.3.2

1 a Construct a cumulative frequency graph for the following set of data.

Data group	Frequency
$0 \leqslant T < 20$	5
$20 \leqslant T < 40$	12
$40 \leqslant T < 60$	23
$60 \leqslant T < 80$	7
$80 \leqslant T < 100$	3

b Use your graph to find the median value for the data.

c Use your graph to find the lower and upper quartiles.

d Calculate the interquartile range.

In questions 2–5, you will revisit questions from Practice questions 19.2, using the new methods you have learned in this section.

2 As an indicator of fairness and development, the number of students with access to university education in each country across the world is monitored. The table shows the percentage of the population with a university education in 1970. The frequency is the number of countries with a value within the given band.

% of population with university education (P)	Number of countries (frequency)
$0 \leqslant P < 1$	65
$1 \leqslant P < 2$	32
$2 \leqslant P < 3$	14
$3 \leqslant P < 4$	14
$4 \leqslant P < 5$	7
$5 \leqslant P < 6$	4
$6 \leqslant P < 7$	4
$7 \leqslant P < 8$	1
$8 \leqslant P < 9$	0
$9 \leqslant P < 10$	1
$10 \leqslant P < 11$	1
$11 \leqslant P < 12$	1

a Construct a cumulative frequency polygon for the data.

b Use your cumulative frequency polygon to estimate the median of the data.

c Compare your estimated value with that from your original solution in Practice questions 19.2. Does it lie within the median group? Is it close to the midpoint of the median group?

d Use your cumulative frequency polygon to estimate the lower and upper quartiles.

e Calculate the interquartile range (IQR) for the data.

f Identify whether or not there are any outliers in the data.

3 The table below shows the percentage of the population with a university education in the year 2010. As with question 2, the frequency is the number of countries with a value within the given band.

% of population with university education (P)	Number of countries (frequency)
$0 \leqslant P < 5$	63
$5 \leqslant P < 10$	31
$10 \leqslant P < 15$	24
$15 \leqslant P < 20$	16
$20 \leqslant P < 25$	6
$25 \leqslant P < 30$	3
$30 \leqslant P < 35$	1

a Construct a cumulative frequency polygon for the data.

b Use your cumulative frequency polygon to estimate the median of the data.

c Compare your estimated value with that from your original solution in Practice questions 19.2. Does it lie in the median group? Is it close to the midpoint of the median group?

d Use your cumulative frequency polygon to determine an estimate for the interquartile range (IQR).

e Identify whether or not there are any outliers.

f Compare the data for 2010 with the data for 1970 in question 2.

4 In 2020, the UK government conducted a survey to determine the impact of a campaign to promote diversity in the workplace. One of the factors they focused on was the age range of the workforce.

The table shows the number of employees (in thousands) in each age range.

Age range	Number of UK employees (thousands)
$16 \leqslant P < 25$	2321
$25 \leqslant P < 30$	4582
$30 \leqslant P < 35$	4703
$35 \leqslant P < 40$	5311
$40 \leqslant P < 45$	5983
$45 \leqslant P < 50$	5983
$50 \leqslant P < 55$	4914
$55 \leqslant P < 60$	3877
$60 \leqslant P < 65$	2106
$65 \leqslant P < 70$	733

a Construct a cumulative frequency polygon for the data.

b Use your cumulative frequency polygon to estimate the median of the data.

c Compare your estimated value with that from your original solution in Practice questions 19.2. Does it lie in the median group? Is it close to the midpoint of the median group?

d Governments usually quote percentile values rather than quartiles. Given that the median is the 50th percentile, calculate the 20th percentile and the 80th percentile for the data.

5 In 2020, the UK government gathered data on the age ranges of the workforce who had been unemployed for 1 year or more.

The data is shown in the table.

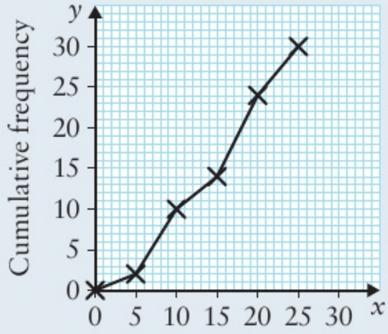
Age range	Number of UK unemployed (thousands)
$16 \leqslant P < 18$	4.9
$18 \leqslant P < 25$	54.0
$25 \leqslant P < 50$	86.0
$50 \leqslant P < 70$	87.0

a Construct a cumulative frequency polygon for the data.

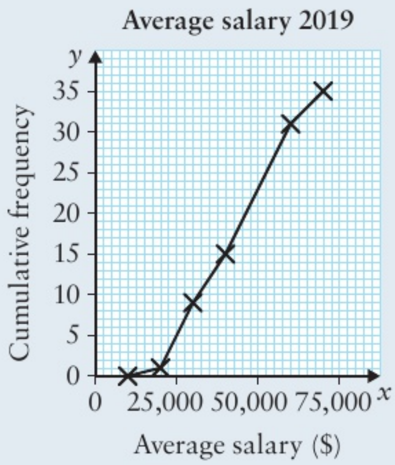
b Use your cumulative frequency polygon to estimate the median of the data.

c Calculate the 40th and the 60th percentiles.

6 Draw a box plot to represent the data given in the cumulative frequency graph.



7 The cumulative frequency plot below represents the world average salary figures, in US$, for 2019. The cumulative frequency represents the number of countries with an average salary up to each amount.



a State the minimum and maximum average salaries.

b Estimate the median salary.

c Estimate the number of countries that have an average salary of less than $30,000.

d Estimate the number of countries that have an average salary of greater than $60,000.

19.4 Analysing bivariate data

Univariate data concerns the analysis of one variable, for example the heights of students in a mathematics class. We can compare the heights of two different classes.

Bivariate data concerns the analysis of two paired variables, for example the heights and the arm span lengths of students in a mathematics class. This is bivariate because it involves two variables. We are usually trying to identify a relationship between the two variables.

Explore 19.6

Communication skills

The World Health Organization has published data on the levels of access to drinking water and sanitation services for each country in the world. The data in the table shows the percentage of the population that have access to the basic level of drinking water and sanitation. The data has been summarised into regions.

By representing the data in various ways, compare the two data sets.

Are there similarities? Are there differences?

World regions	Access to basic drinking water (%)	Access to basic sanitation services (%)
East Asia and Pacific	93	84
Europe and Central Asia	98	97
Eastern Europe and Central Asia	96	94
Western Europe	100	99
Latin America and Caribbean	97	87
Middle East and North Africa	94	91
North America	99	100
South Asia	92	59
Sub-Saharan Africa	61	31
Eastern and Southern Africa	58	31
West and Central Africa	64	30
Least developed countries	65	34

> **Fact**
>
> A scatter graph represents the relationship between paired data. The points are plotted with crosses and they are not joined by a line.

19.4.1 Scatter graphs

When we compare two different variables it is called **bivariate analysis**.

When the two sets of variables are represented separately, for example in back-to-back stem-and-leaf plots or in box plots, we can find similarities and differences.

If we are looking for a **relationship** between the two variables then the bivariate data must be paired, as in Explore 19.6, where the two variables are for the same region.

We can draw a **scatter graph** to investigate the relationship. If one variable is dependent on another variable, the **independent variable** is plotted on the horizontal axis and the **dependent variable** is plotted on the vertical axis.

Worked example 19.7

Plot a scatter graph to represent the data about access to basic drinking water and basic sanitation from Explore 19.6. Plot access to basic drinking water on the horizontal axis and access to basic sanitation on the vertical axis.

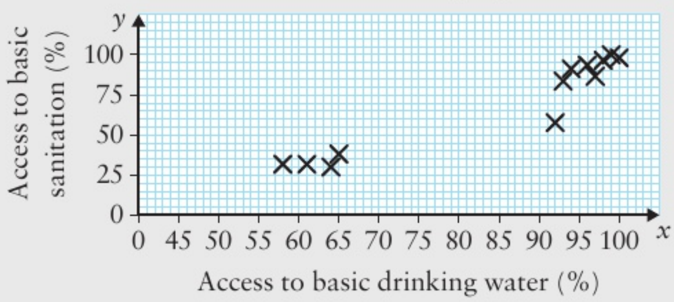
What conclusions can you make about the relationship between the data sets?

Solution

The two sets of data are paired, as there are two values for each region. They are not necessarily dependent and independent variables, but a value of one may imply a value for the other. By plotting the data on a coordinate grid we can identify any relationship.

We plot the data as coordinate points and observe any pattern that is formed.

Access to basic drinking water and basic sanitation by region



From the scatter graph, it appears that regions that have a lower level of access to basic drinking water also have a lower level of access to basic sanitation.

Similarly, regions that have a higher level of access to basic drinking water also have higher access to basic sanitation.

One data value seems to lie outside the pattern as it has a high percentage of access to basic drinking water, but a lower percentage of access to basic sanitation.

The graph also demonstrates that there is a large gap between the regions that have a high level of access to both basic drinking water and basic sanitation and those that have a low level of access.

The data from the table is consistent with the graphical representation. The relationship between the two sets of data can be seen easily from the graphical form.

The graphical form enables conclusions to be made regarding the connection between the two data sets.

Reflect

Can we claim that a low level of access to basic drinking water 'causes' a low level of access to basic sanitation?

Even if a relationship is identified between two sets of data, it cannot be assumed that one variable causes an effect on the other. **Causation** can only be determined by further investigation.

Worked example 19.8

The population of some villages, at different distances from a city centre, are given in the table.

District	I	II	III	IV	V	VI	VII	VIII	IX
Distance (km)	0.6	2.6	2.4	0.3	2	1.5	1.8	3.4	4
Population of the village	900	400	330	1200	600	500	800	200	500

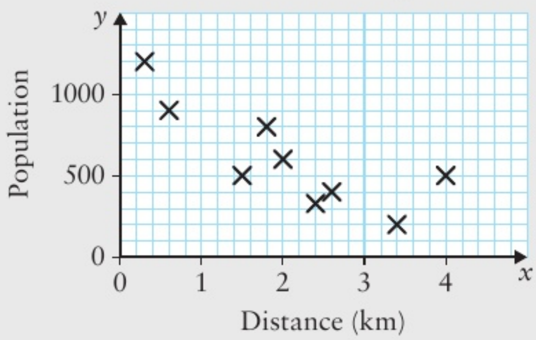
Draw a scatter graph to represent the data. Comment on the relationship between the two variables.

Solution

The two sets of data are paired since there is a pair of values for each village. There may be a dependency between the variables, since the population of each village may be related to how far away from the city centre it is. In this example, the distance will be the independent variable and the population will be the dependent variable. By representing the data in a scatter graph, we can investigate the relationship between the two variables.

We will plot the data as coordinate points, with distance on the horizontal axis and population on the vertical axis. Then we will observe any pattern that is formed.

Population of a village and its distance from the city centre



The scatter graph shows that the population appears to decrease as the distance from the city centre increases. We cannot claim that the distance causes the population to decrease, but we can identify the pattern.

The data from the table is consistent with the graphical representation and the relationship between the two sets of data can be seen easily from the graphical form.

The graphical form has allowed us to make conclusions about the two sets of data.

Reflect

Can you think of a reason why the last data point on the graph, representing a village at a distance of 4 km from the city centre, has a higher population than some of the villages that are closer to the city centre?

Practice questions 19.4.1

1 Draw a scatter graph to represent the data in the table and describe the relationship between the variables.

x	2	3.2	4.5	5	6.2	7.4	8	9.4	10.7
y	18	16.3	15	13.1	12.1	9	7.8	6.5	5.1

2 The following data shows the time for which a patient is free from pain after taking a painkiller tablet, Drug A.

Dosage (mg)	2	3	4	5	6	7	8	9
Relief time (hours)	1	1.6	2	2.7	3.2	4.1	4.8	5.3

a Draw a scatter graph to represent the data.

b Comment on the relationship between the dosage and the time for which the patient is free from pain.

3 A new painkiller, Drug B, has been developed. The same patient as in question 2 reports the following relief times for each dose.

Dosage (mg)	2	3	4	5	6	7	8	9
Relief time (hours)	2.2	2.9	3.1	3.8	4.2	4.2	4.2	4.2

a Draw a scatter graph to represent the data.

b Comment on the relationship between the dosage and the time for which the patient is free from pain.

c Which drug would you recommend for the patient, Drug A or Drug B? Give a reason for your answer.

4 The data below shows the number of ice creams sold per day in a local store over a ten-day period and the temperature for that day.

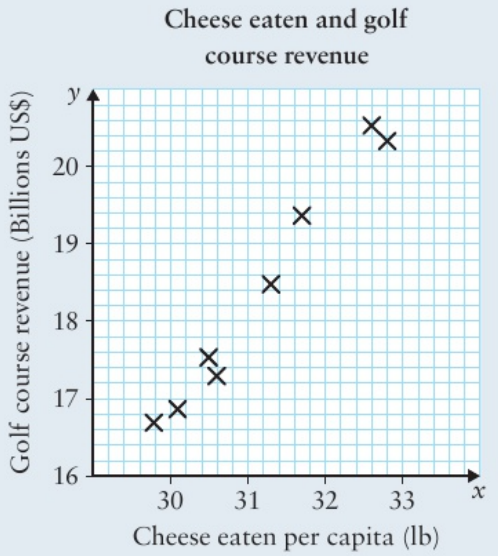
Temp °C	19	26	28	32	24	22	33	28	15	30
Ice cream sales	18	32	38	42	29	25	44	37	15	39

a Draw a scatter graph to represent the data.

b Comment on the relationship between the variables.

5 The scatter graph represents the amount of cheese (in lb) eaten per capita in the USA in 2020 on the horizontal axis and the revenue for golf courses (in billions of $) for the same year on the vertical axis.

Comment on the relationship.



Fact Q5

'Per capita' means 'per person'.

19.4.2 Correlation

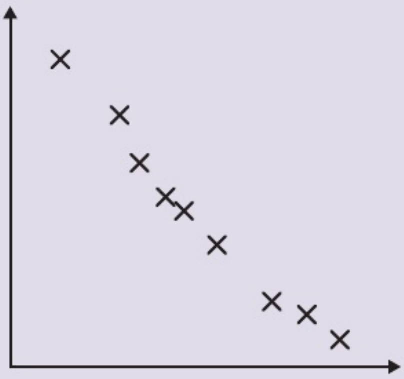
Representing bivariate data with scatter graphs identifies whether there is a relationship between the two sets of data. We can describe the relationship in terms of trends. For example, the population and distance from the city centre data shows a downwards trend.

Explore 19.7

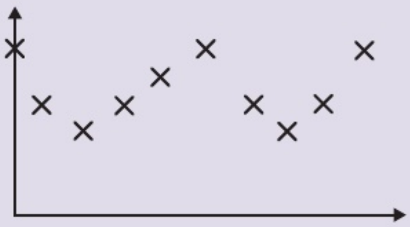
Describe each scatter graph below in terms of its trend.

Suggest two variables that each graph could represent.

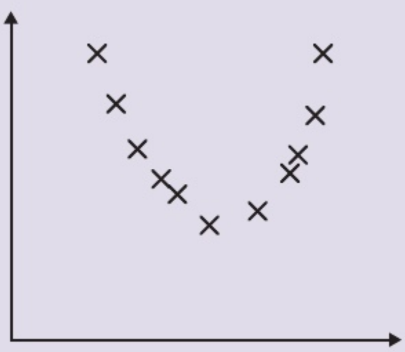
a



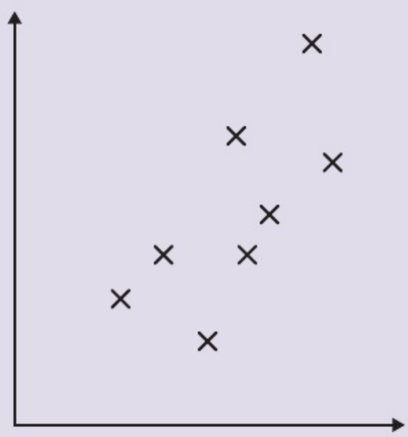
b



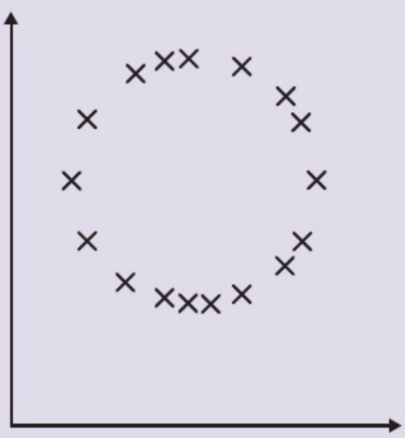
c



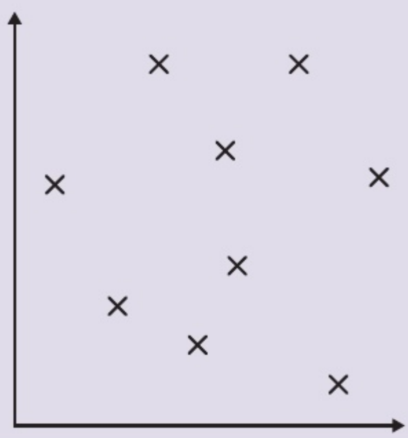
d



e



f

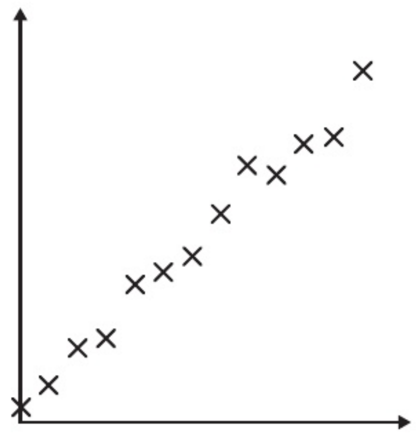


If the relationship between two data sets follows a straight-line pattern, then it is called a **linear trend**.

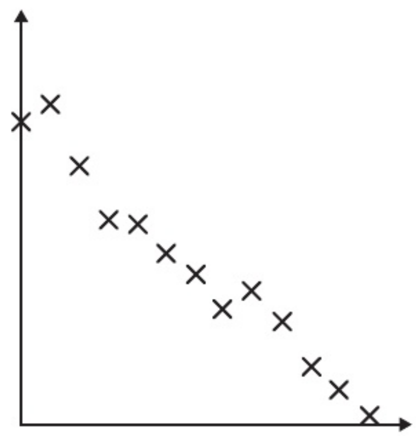
Sometimes the data points appear to be closely following a trend line; sometimes they do not follow the trend line closely. The degree of closeness to the trend line is called **linear correlation**. It allows us to quantify how closely related two variables are.

Relationships between data sets do not have to be linear. They can have different underlying patterns.

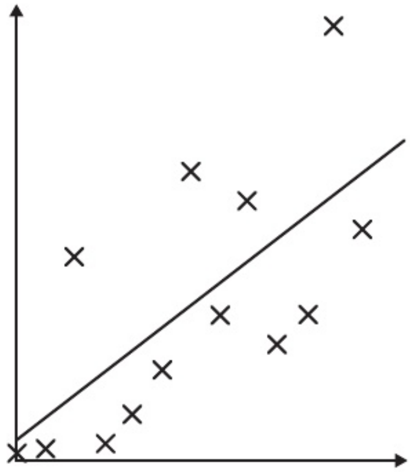
An upward linear trend has a **positive correlation** because it is moving in the positive direction.



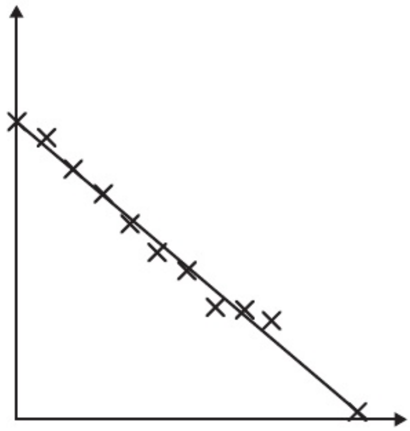
A downward linear trend has a **negative correlation** because it is moving in a negative direction.



Data that is not close to the trend line is **weakly correlated**.



Data that is very close to the trend line is **strongly correlated**.



Social skills

Research skills

Investigation 19.3

Leonardo da Vinci is famous for many things, one of which is the drawing of the Vitruvian Man. It is a diagram which is meant to depict the perfect proportions of man: if a man's arm span is equal to his height, then he is said to be in perfect proportion.

1 Research information about the Vitruvian Man.

2 Measure the arm span and height of each person in your group and record your data in a table.

3 Draw a scatter graph to represent your data.

4 Describe the relationship between the two variables, arm span and height, for your group.

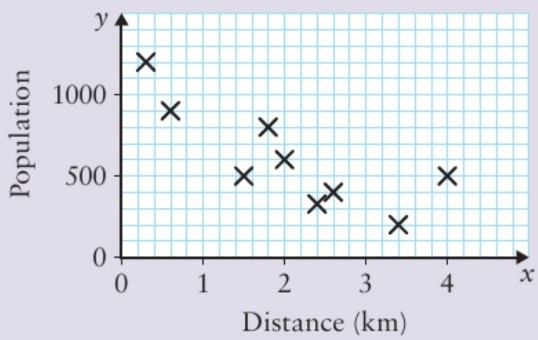
5 Describe the correlation for your group's data.

19.4.3 The line of best fit

Explore 19.8

Look back at the scatter graph showing the population and distance from the city centre in Worked example 19.8. There appears to be a linear trend. Is it possible to draw a line on the graph that represents the trend of the data?

Population of a village and its distance from the city centre

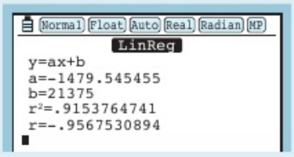


Make a copy of the graph and draw where you think the trend line should be. Explain why you decided to place your line in the position you did.

If the scatter graph identifies a linear trend in the data, we can draw a **line of best fit** that represents the trend. We can draw it by eye, but it must pass through the coordinate point that represents the mean of both sets of the data.

Hint

Your GDC can find the equation of the line of best fit by using a regression method, LinReg, that minimises the distance of the data points from the line.



Worked example 19.9

The data below represents the expected depreciation in the value of a car each year for 10 years. For example, the table shows that the car was bought new for £25,000 and was worth £18,000 after 1 year.

Years of ownership	0	1	2	3	4	5
Value of car (£)	25,000	18,000	17,500	17,000	14,500	13,000

Years of ownership	6	7	8	9	10
Value of car (£)	12,000	10,500	10,500	8000	7750

a Draw a scatter graph to represent the data. Describe the relationship between the value of the car and the time for which it has been owned.

b Use technology to determine the line of best fit. Plot the line on your scatter graph.

c Use your line of best fit to predict the value of the car after 3.5 years.

d Explain why you cannot use your linear model to predict the value of the car when it is 15 years old.

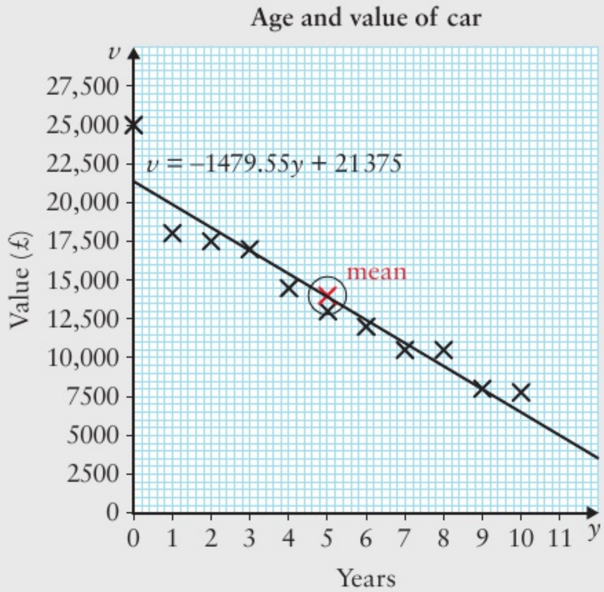
Solution

The problem involves representing the trend of the data with a line of best fit. This requires the data to be represented in a scatter plot. As the value is dependent on the year, we will put the year on the horizontal axis and the value of the car on the vertical axis. We can identify the relationship by the trend that the data shows.

The line of best fit must go through the point (mean year, mean value). We can use technology to calculate these values and add them to the graph. Then we can use technology to determine the equation of the line of best fit and the coordinates of a point on this line and draw the line of best fit going through this point and the mean point.

We can use the line of best fit to determine the value of the car after 3.5 years, then substitute the year value of 3.5 into the equation of the line of best fit, or read the y-value from the graph to determine the value of the car.

We can then critically evaluate the line of best fit to determine why it cannot be used to predict a value after 15 years.



a The data from the table is plotted with crosses. The value of the car decreases as the years increase, so there is a strong negative correlation between the variables.

b Using the GDC, the mean values are 5 years and £13,977.27. We plot the point (5, 13 977.27) (circled).

Using the GDC, the equation of the line of best fit is $v = -1479.55y + 21\,375$, where v is the value in £ and y is the age of the car in years. When $y = 0$, this equation gives the point of intersection with the vertical axis to be (0, 21 375). We join the points (0, 21 375) and (5, 13 977.27) to draw the line of best fit.

Connections Qb

You can also find the equation of the line of best fit using your understanding of straight-line graphs from Chapter 10.

c Substituting the value $y = 3.5$ into the equation of the line of best fit gives:

$$v = -1479.55(3.5) + 21\,375$$
$$= £16{,}196.59$$

d Using the equation of the line of best fit, after 15 years the car would have a negative value, which is not possible.

$$v = -1479.55(15) + 21\,375$$
$$= -£818.25$$

We plotted the data, on a coordinate grid, as a scatter graph. The trend shows a decrease in the value of the car over time which agrees with the data in the table. The line of best fit passes through the mean coordinate point and it represents the trend of the data by eye. This implies that the equation of the line is correct. The trend line slopes downwards towards the horizontal axis and this implies that at some point the value of the car will cross the horizontal axis and become negative. This is confirmed by calculating the value of the car after 15 years and identifies one limitation of the model.

Practice questions 19.4.3

1 Emily is investigating the changes in the gender pay gap over time. The pay gap measures the % difference between male and female average salaries. Emily is curious to see whether there is a difference between the rate at which the gap is closing in countries that have male leaders and those that have female leaders. She has sourced the data below for Norway, a country that has had female leaders, and for Japan, which has not.

Year	Norway % pay gap	Year	Japan % pay gap
2007	10.8	2007	31.7
2008	9.6	2008	30.7
2009	9.0	2009	28.3
2010	8.4	2010	28.7
2011	8.1	2011	27.4
2012	6.7	2012	26.5
2013	7.3	2013	26.6
2014	6.6	2014	25.9
2015	7.5	2015	25.7
2016	6.8	2016	24.6
2017	6.4	2017	24.5
2018	5.8	2018	23.5
2019	4.9	2019	23.5

a Plot the two sets of data on a scatter graph and draw the lines of best fit for both Norway and Japan.

b Comment on the relationship between the gender pay gap and time for each country.

c Compare the trend of Norway's data with that of Japan's.

d Can you predict the year when the pay gap for each country should be 0?

e Can Emily claim that there is a difference in the trend of the gender pay gap in countries led by males and females?

2 Elizabeth wanted to know whether the mean amount of sleep a student has affects their predicted IB score. She surveyed her year group and gathered the following data.

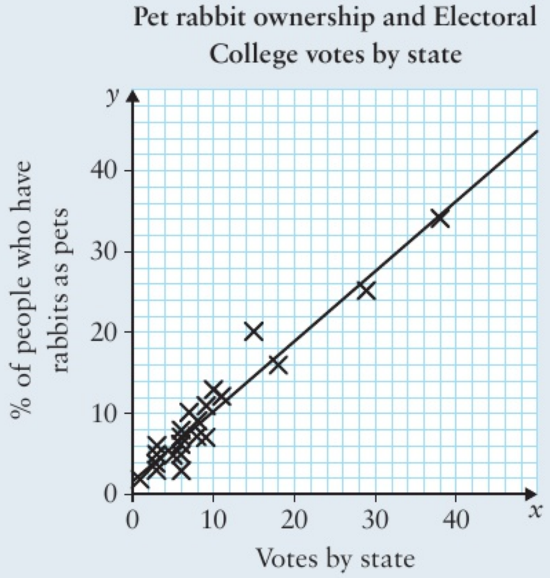
Average number of hours of sleep	Predicted IB score (out of 45)
6	34
7	30
5	27
11	36
5	41
5	25
8	32
10	32
11	29
6	40
6	43
10	33
7	31
9	27
9	36
5	35
9	39
9	32
7	42
5	32

a Plot the data on a scatter graph, using hours of sleep as the independent variable and predicted IB score as the dependent variable.

b Comment on the relationship between the two variables and describe the correlation.

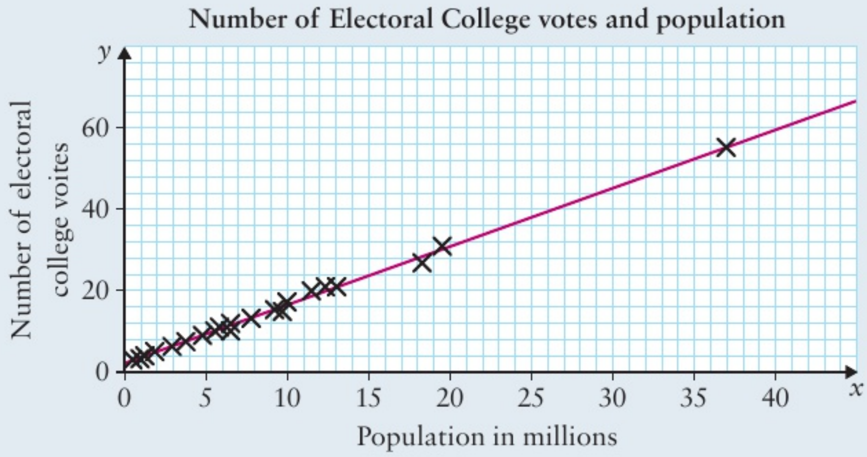
c What conclusions can Elizabeth make?

3 Theo has found information online that gives the following scatter graph and trend line. The independent variable, on the horizontal axis, is the number of Electoral College votes gained in one state by President Trump in the 2020 USA election. The variable on the vertical axis is the percentage of households in the same state that own a pet rabbit.



Interpret the graph and explain the conclusions that Theo can make.

4 Theo was curious about the USA electoral system and researched the number of Electoral College allocations versus the population of the state (millions). He finds the following scatter graph for 2008.



Fact Q3

The Electoral College is the name for a group of people called 'presidential electors', whose role is to choose the President and the Vice President of the United States.

The presidential electors are made up of the two Congress members each state has and additional Representatives, the number of which is determined in proportion to the population of the state.

Each political party nominate their own candidates and the public cast their votes on election day.

48 of the 50 states will award the presidential elector position to the candidates with the most public votes. (Maine and Nebraska have a slightly different system).

Once the presidential electors have been assigned they meet to cast their votes for the President and the Vice President. The winner is the presidential candidate to gain 270 or more Electoral College votes.

Challenge Q5

Hint Q5

One source for the USA election data archives can be found here:

Communication skills

Research skills

a Describe the relationship between the population and the number of Electoral College votes. Is the system fair?

The line of best fit is determined as $V = 1.44P + 2.03$, where V is the number of Electoral College votes allocated to the state and P is the population of the state in millions.

b In 2008, Florida had a population of 18.3 million and 27 Electoral College votes. Was this fair? Justify your answer.

c What was the allocation of Electoral College votes for Florida in 2020? Did this affect the result of the 2020 US Presidential election?

5 During the 2016 USA elections, Ms Clinton gained more votes overall, but President Trump gained more Electoral College votes so was declared the winner of the election. Given that the Electoral College votes are expected to represent the correct proportion of the population, can you explain why this happened?

Self assessment

- I can identify the difference between a census and a sample.
- I can correctly select a sample from a population.
- I understand the different types of sampling methods.
- I can organise data into a frequency table.
- I can identify the mode from a frequency table.
- I can calculate the mean of data from a frequency table.
- I can determine the median of data from a frequency table using the cumulative frequencies.
- I know that the mean, median and mode are measures of central tendency.
- I can find the range of a data set.
- I know that the range is a measure of dispersion.
- I can identify the modal group of a set of grouped data from a frequency table.
- I can estimate the mean of a set of grouped data from a frequency table.
- I can estimate the median of a set of grouped data from a frequency table by using cumulative frequencies.
- I can find the quartiles of a discrete data set and calculate the interquartile range.
- I have refreshed my knowledge of how to represent discrete data in the form of a stem-and-leaf plot.
- I have refreshed my knowledge of how to represent two sets of data on a back-to-back stem-and-leaf plot and compare them.
- I have refreshed my knowledge of how to determine the median and mode of a data set from a stem-and-leaf plot.
- I have refreshed my knowledge of how to explain why a stem-and-leaf plot is not appropriate for grouped data.
- I can represent grouped data with a histogram.
- I can explain why a histogram is drawn with bars touching.
- I can generate the cumulative frequency polygon for a set of grouped data from a frequency table.

- I can use the cumulative frequency polygon to estimate quartile and percentile values of grouped data.
- I understand the difference between univariate and bivariate data.
- I can compare univariate data sets when they are represented by box plots.
- I can describe the relationship between two variables when they are plotted on a scatter graph.
- I can describe the correlation of bivariate data.
- I understand the difference between correlation and causation.
- I can draw the line of best fit to represent the trend in a data set.
- I can determine the equation of the line of best fit using technology.

Check your knowledge questions

1 For each of the examples below, explain which would be suitable for a census and which would be suitable for a sample.

  a A computer manufacturer would like to test the maximum temperature that their processors can withstand before they no longer work.

  b A company is asked to predict the results of an election.

  c A school would like to determine whether they should change the time of the morning break.

2 Patricia is conducting a survey about mental health support for the IB Diploma students in her school. She would like to take a stratified sample to ensure that she has the correct proportion of males and females. She needs a sample size of 50. The number of male and female students in Grades 11 and 12 are shown in the table.

Show working to explain how she would do this.

	Girls	Boys
Grade 11	65	70
Grade 12	54	61

3 a Organise the following raw data into a frequency table.

8	6	8	4	1	7	5	2	9	3
2	4	1	3	6	8	1	1	9	8
2	4	9	2	5	1	2	2	7	1
7	9	5	3	2	5	4	1	8	8
5	1	5	1	2	7	5	6	8	4

b State the mode.

c Determine the median.

d Calculate the mean.

4 Represent the data from question 3 with a suitable graph.

5 Using the stem-and-leaf diagram below, identify the median and mode, and calculate the range for the data.

Stem	Leaf
0	1 2 3 4
1	2 3 3 3 3
2	1 2 2 3 4 4
3	2 2 2 3 9
4	1 1 1 4 6
5	7 9
6	5 6 7 7
7	2 3 9
8	6 9 9 9
9	5 7 7 8

Key: 2|4 means 24

6 Ms Choi has two mathematics classes. Both classes completed the same end-of-unit quiz. The percentage scores are represented in the back-to-back stem-and-leaf diagram. Compare the results of the two classes.

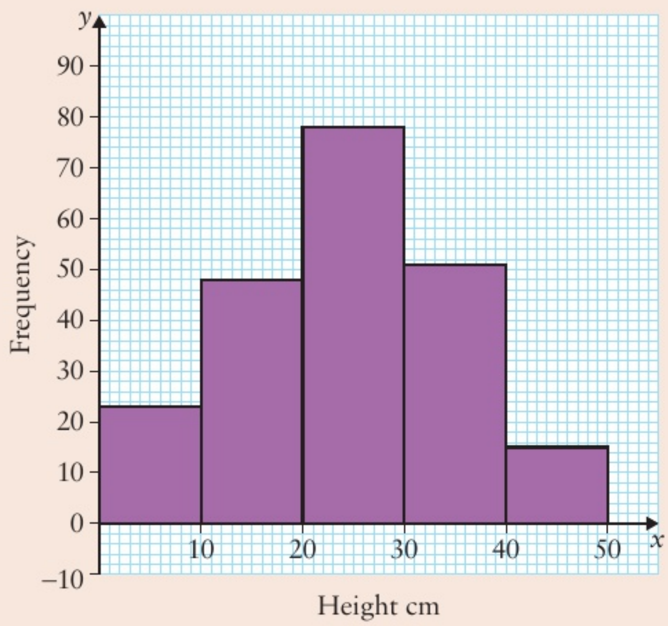
Red		Blue
	0	
6	1	
5 5 4 4 4 4	2	
6	3	
5 5 3 1	4	3 4
9 2 0	5	3 5 7
8 8 4 1	6	1 5 6
7 4	7	1 7 7 7 8
	8	6 7 9
3	9	6
	10	0 0

key: 1|6 means 61% key: 7|1 is 71%

7 Alex measured the heights of the plants he grew for a biology experiment. He recorded his results in this table. Draw a histogram to represent the data.

Height of plants (cm)	Frequency
$0 \leqslant P < 10$	12
$10 \leqslant P < 20$	45
$20 \leqslant P < 30$	52
$30 \leqslant P < 40$	78
$40 \leqslant P < 50$	37

8 Raul also experimented with growing plants. He represented his data in a histogram but could not find his original data collection sheet. Using the histogram below, represent Raul's data in a frequency table.



9 a Represent Alex's data from question 7 and Raul's data from question 8 by two cumulative frequency polygons on the same set of axes.

b Use the polygons to estimate the median and the interquartile range for each set of data.

c Using the polygon for Alex's plants, estimate how many of the plants are less than 32 cm in height.

d Using the polygon for Raul's plants, estimate how many plants are taller than 12 cm.

10 Draw box plots above the same scale to represent the values of the key statistics for Alex's and Raul's plants, from question 9. Compare the heights of Alex's and Raul's plants.

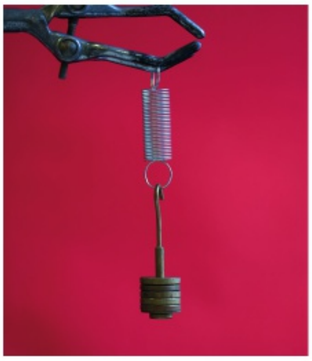
11 Jenny read an article in her student biology magazine that suggested that the number of cricket chirps per minute is related to the atmospheric temperature. The article gave the following results for a monitoring experiment in El Paso, Texas.

Temperature (°F)	66	71	74	81	85	89	93	99	105
Number of chirps per minute	25	30	35	40	45	50	55	60	65

a Draw a scatter graph to represent the data. Use technology to determine the line of best fit.

b Comment on the relationship between the atmospheric temperature and the number of cricket chirps.

c Is this an example of correlation or causation or both? Explain your reasoning.



12 Sara is performing a physics experiment and is loading weights onto a spring.

She collects the following data.

Weight (g)	20	25	35	55	105
Length of spring (cm)	3.63	3.82	4.14	4.80	6.43

a Draw a scatter graph to represent the data. Use technology to determine the line of best fit.

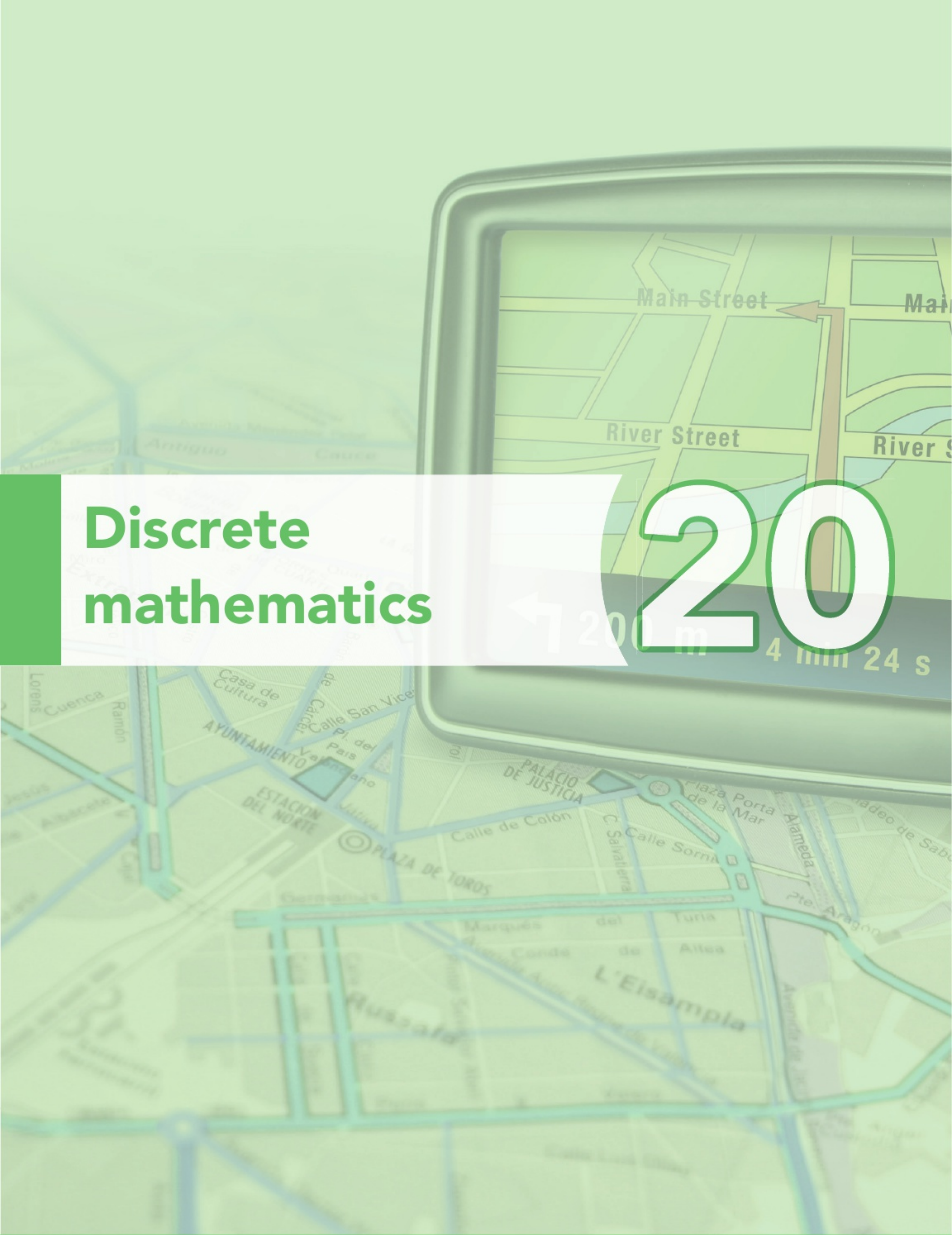
b Comment on the relationship between the weight added and the amount each spring is stretched.

c Is this an example of correlation or causation or both? Explain your reasoning.

d Use your line of best fit to determine the original length of the spring with no weight attached.

e Can your model be used to predict the amount of stretch for a weight of 1 kg? Explain your reasoning.

20 Discrete mathematics



20 Discrete mathematics

KEY CONCEPT

Form

RELATED CONCEPTS

Models, Representation, Systems

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Representing complex spatial problems in graphical form enables us to develop efficient systems that allow us to solve them more easily.

Factual

- What are algorithms and how can they be used in different contexts?
- What are minimum spanning trees and how can they be used in solving real problems?

Conceptual

- How can weighted graphs be used to represent real-world situations and solve related problems?
- What are divisibility rules?

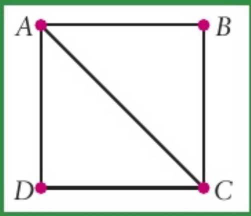
Debatable

- Are algorithms always the most efficient way to solve a problem?
- Do algorithms designed to find minimum paths always find the optimal paths?

Do you recall?

1 Consider the graph below.

a What is the degree of each vertex?



b Is the graph simple? Give a reason for your answer.

2 Is the following statement true or false?
A tree is a connected simple graph without any cycles.

3 Describe and draw a complete graph, K_5.

4 How many edges does a complete graph of order n have?

5 How does a subgraph relate to the original graph?

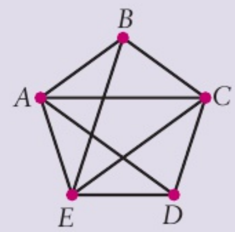
6 Find the greatest common divisor of 1856 and 258.

20.1 Graph theory review: Graphs, subgraphs, and trees

Explore 20.1

Find the sum of the degrees of the vertices in this graph.

Can you suggest how this relates to the number of edges?



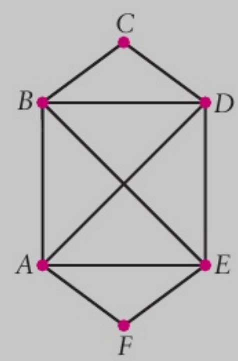
Worked example 20.1

Consider the given graph.

a List the degrees of the vertices.

b What do you notice about the sum of the degrees, compared to the number of edges? Can you explain?

c Can you generalise your observation from part b?



Fact

When we list the degrees of all vertices of a graph in ascending order, we call the list the **degree sequence** of the graph. In this case it is: $\{2, 2, 4, 4, 4, 4\}$

Solution

a The degrees of the vertices are as follows:

$\deg(A) = \deg(E) = \deg(B) = \deg(D) = 4$, $\deg(C) = \deg(F)=2$

b The sum of the degrees is $4 + 4 + 4 + 4 + 2 + 2 = 20$. The number of edges is 10, so the sum of the degrees is twice the number of edges. This is because each edge is counted twice. For example, edge AB is counted both at A and B.

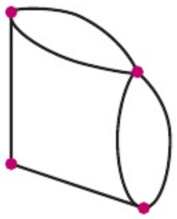
c Since an edge connects two vertices, every edge creates one degree to each vertex. Thus, every edge creates two degrees. If the number of edges is n then the number of degrees is $2n$.

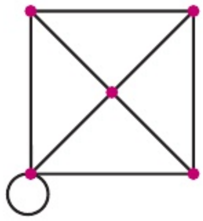
The types of graph that we will study in this chapter are **connected** and **simple**. Remember that connected means that each vertex is connected (adjacent) to another. In other words, there are no vertices of degree 0.

In simple graphs there is no more than one edge connecting each pair vertices.

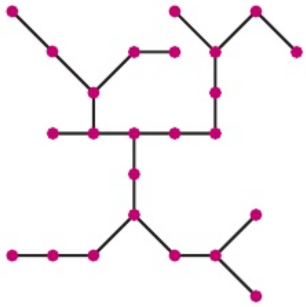
The diagram below shows two graphs that are not simple:

In the first graph there are two vertices with two edges between them while the second graph contains a loop.



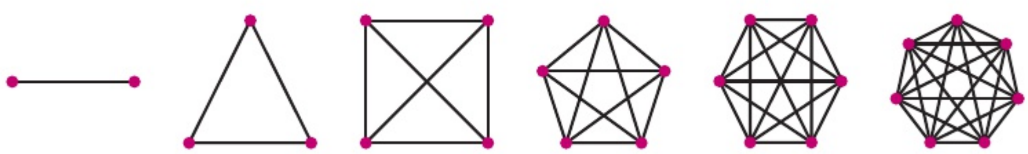


Remember that **trees** are simple connected graphs, where there is only one way of getting from one vertex to another. That is, a tree is a simple, connected graph with a unique simple path between any pair of vertices. Thus, a tree has no cycles.



A graph with cycles can be made into a tree by deleting appropriate edges to remove the cycles while keeping the graph connected. Because a tree will feature all the vertices, but not all the edges, of the original graph, we can say that the set of elements of the tree is a subset of the set of elements of the original graph. Therefore, we call it a **subgraph**.

A simple connected graph where each vertex is connected to all other vertices is called a **complete graph**. Below are the first six complete graphs:



K_2 K_3 K_4 K_5 K_6 K_7

Challenge

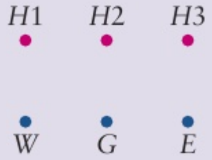
Can you explain why the number of edges in a complete graph equals the binomial coefficient ${}^nC_2$?

Reminder

The number of edges in a complete graph equals $\frac{n(n-1)}{2}$

Explore 20.2

Three houses are to be connected to the utilities of water (W), gas (G) and electricity (E) as shown in the diagram.



Each connection will be an edge connecting a house to a utility. Is it possible to connect each of the houses to each of the utilities without any of the edges crossing?

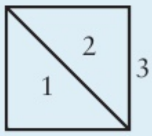
A **planar graph** is a graph where the edges do not cross.

In Explore 20.2 you discovered that the given graph is not planar, as you cannot draw the last edge without crossing any of the other ones.

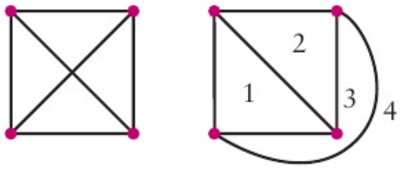
The graph in Explore 20.2 is known as a **bipartite** graph, because the two sets of vertices are not connected amongst themselves. It is known as $K_{3,3}$

Hint

A face in a planar graph is a region bounded by a set of edges and vertices. The outer region is considered as one face. For example, the graph below has 3 faces.



By contrast, K_4 is planar, although the usual presentation shows two edges crossing. We can, however, redraw it as shown in the diagram.



K_4 has 6 edges and 4 vertices. In its planar form it has 4 faces, as shown in the same diagram: 3 of the faces are inside the graph and the fourth is outside it.

Practice questions 20.1

1 For each graph:

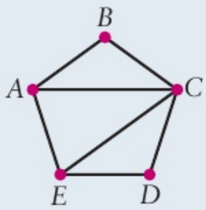
i state the number of edges

ii state the number of vertices

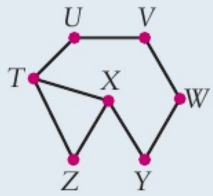
iii state the degrees of the vertices

iv check that the sum of the degrees is twice the number of edges.

a



b



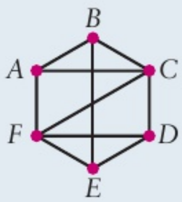
2 From the graph shown, draw a subgraph that:

a is a not a tree

b has more vertices than edges

c is disconnected

d is complete.

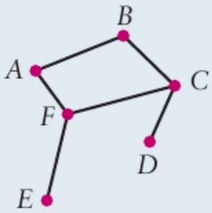


3 The graph shows glass fibre cables connecting houses B to F to the main source at A.

a Is this graph a tree? Explain your answer.

b Why is it important that the graph is connected?

c Which cable (edge) could you remove for the graph to remain connected?



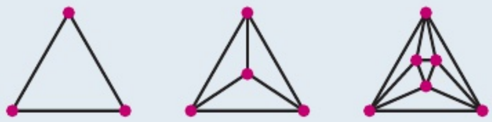
4 Watch the TED-Ed video on the Control Room Riddle, in which you are asked to find out on what floor of a pyramid the Control Room can be found. Try to solve it by yourself, before you watch the solution.

Hint Q4

Each room can be represented by a vertex and each door by an edge. What is the degree of each room?

Challenge Q5

5 For each graph, write down the number of edges, vertices and faces. Put your results in a table in that order. Can you find a relationship between the three? Research Euler's formula and check your result.

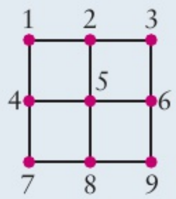


Thinking skills

Research skills

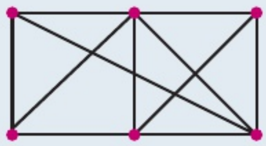
6 How many more edges are there in the complete graph K_7 than in the complete graph K_5?

7 Show that the graph below is bipartite.



Challenge Q8

8 Consider the graph below.



a This graph is planar. Redraw it so that no two edges cross.

b List the number of vertices v, edges e and faces f in the redrawn graph.

c Find a relationship between v, e and f found in part b.

d Generalise the relationship in part c.

Challenge Q9

9 Suppose that in a group of five people, Alex, Bert, Cathy, Dan and Eric, the following pairs of people are acquainted with each other.

- Alex and Cathy
- Alex and Dan
- Bert and Cathy
- Cathy and Dan
- Cathy and Eric

a Draw a graph, G, to represent this information.

b In a meeting of this group, new acquaintances will result. Draw a graph, H, to represent the new acquaintances.

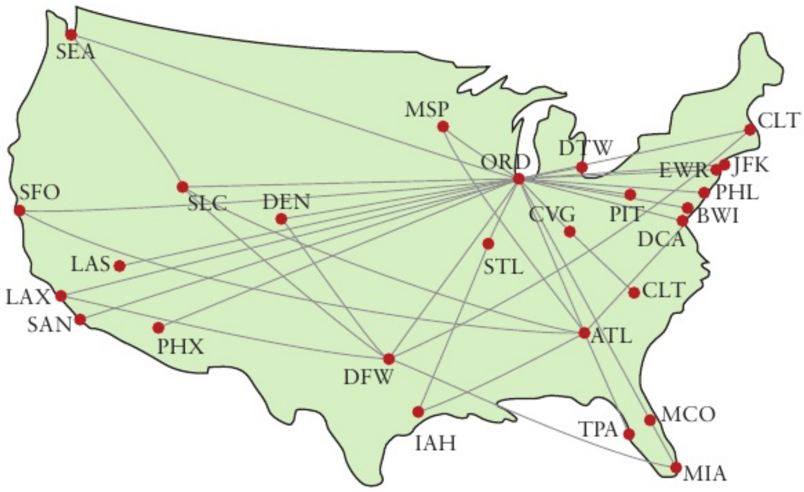
c What is the graph of $G \cup H$?

Hint Q9b

H is called the **complement** of G.

20.2 Weighted and directed graphs

20.2.1 Weighted graphs



In this section, we will introduce different kinds of problems related to weighted graphs. A **weighted graph** is a graph in which each edge has a number associated with it. This number can represent distance, travel time, cost, and so on.

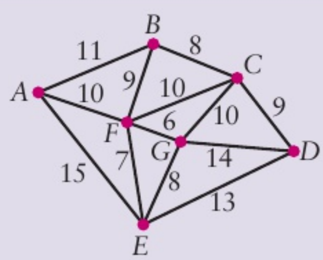
There are two famous types of problem linked to weighted graphs. In this section we are going to solve them using trial and error. In the next section we will introduce a more systematic approach to solving these problems.

In the first type of problem, the edges represent roads or travel routes, and the numbers represent distances. We need to find the shortest travelling distance while visiting all vertices.

Explore 20.3

The vertices in this diagram represent different landmarks in a city. The numbers on the edges represent travelling times between them.

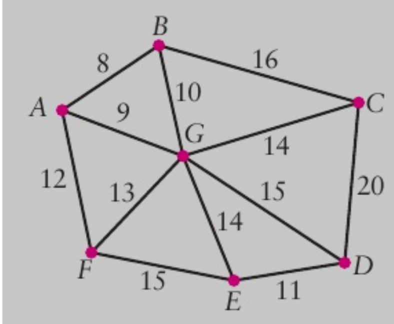
What would be the fastest way to travel from A to D, visiting each landmark (vertex) on the way?



The second type of problem deals with **minimum spanning trees**. These are trees that make sure all vertices are connected, while minimising the cost. When graphs are relatively simple, we can, quite easily, find the solution using trial and error, as shown in Worked example 20.2.

Worked example 20.2

The diagram shows the cost, in hundreds of dollars, of connecting a number of computers to a main server. Each vertex represents a computer and the numbers on the edges represent the costs of the connections.



What is the minimum cost of building this network?

Solution

Understand the problem

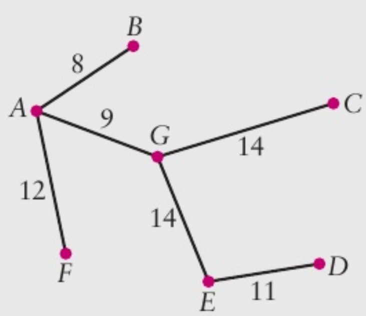
We need to connect all the computers to the server (which could be at any of the vertices) at minimum cost.

Make a plan

We will start at A and connect it to the next computer using the cheapest connection. We will then do the same at that computer and so on, without creating a cycle, until we have connected all the computers.

Carry out the plan

Starting at A, the cheapest connection (8) is to B. Now A and B are in the network. The next cheapest connection (9) is from A to G. The next cheapest is the edge between B and G, but since G is already in the network we'll not use that. Instead we will choose the edge between A and F (12). Next are GE and GC (in any order) and finally ED. The resulting minimal spanning tree is shown in the diagram.



The total weight is $8 + 9 + 12 + 14 + 14 + 11 = 68$, meaning that the cheapest solution will cost \$6800.

Look back

Since we have 7 vertices, any spanning tree must have 6 edges. Looking at the tree we have, the numbers we added are mostly the smallest weights available in the graph. A smaller sum can be $8 + 9 + 10 + 11 + 12 + 13 = 63$ However, such a graph is not a tree. For example, $8 + 9 + 10$ represents a cycle ABG. We also see that the graph we get is not connected.

By following the relatively simple plan outlined in Worked example 20.2 instead of randomly choosing edges, we are following an **algorithm**.

For example, the steps of an algorithm for adding two digit numbers is:

1 Add the tens digits.

2 Add the ones digits.

3 Add the numbers from steps 1 and 2.

So to add 15 and 32 using that algorithm:

1 Add 10 and 30 to get 40.

2 Add 5 and 2 to get 7.

3 Add 40 and 7 to get 47.

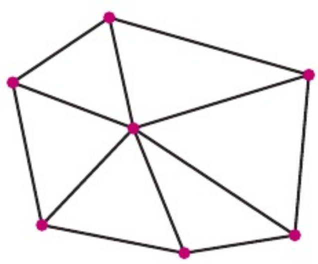
Long division is another example of an algorithm; when you follow the steps you get the answer.

In Worked example 20.2 we made the network on the assumption that the server could be anywhere. In real life, this is often not the case, as servers need cooling and an extremely powerful and reliable source of electricity. Environmental impact is therefore also a consideration. Where would you put the server, if you could choose any of the vertices?

A variation on the problem outlined in Explore 20.3 is the **travelling salesman problem** (TSP). The name comes from a job title. The person holding this job would travel from town to town to sell a particular product, before returning home. The salesperson would likely travel the shortest possible distance to visit all the towns before returning home. For example, pharmaceutical companies send their sales representatives to different doctors, introducing them to new products to prescribe to their patients.

Fact

An **algorithm** is a sequence of precise instructions, typically to solve a class of problems or to perform a computation. It is a step-by-step solution where each step has clear instructions, like a recipe.



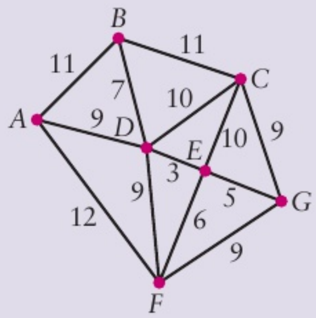
Fact

There is currently no proven solution for the TSP. Available solutions are approximate, and, when a large number of cities are involved, it takes a lot of computing time to calculate a solution.

Explore 20.4

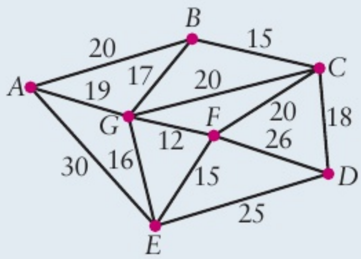
Travelling from city A, a salesperson must travel to all cities once before returning home.

The distances between the cities are given by the weights of the edges, which are the only possible connections between the cities. Can you find the minimum total distance to be travelled?



Practice questions 20.2.1

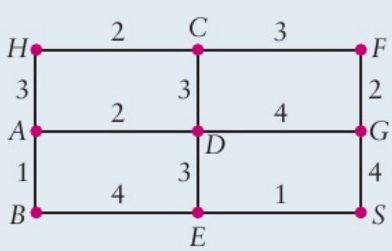
1 **a** Find the minimum spanning tree of the given graph, starting at A and using the method outlined in Worked example 20.2.

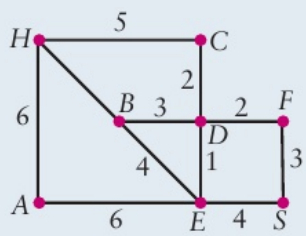


b Now find the minimum spanning tree, starting at vertex D.

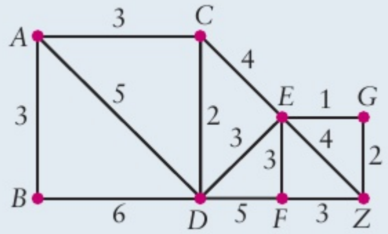
c Did you get the same result in part a and part b? Explain why that might be the case.

2 The diagrams show two street networks. The first shows Jason's walk from home (H) to school (S). The second shows Justin's walk from home (H) to school (S). The weights of the edges represent the time taken in minutes. Who can get to school the quickest? Explain your answer.

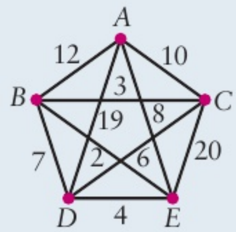




3 Find the minimum spanning tree for the graph below.



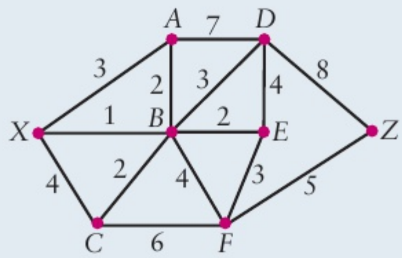
4 **a** What is the shortest travelling salesman trip, starting at City E and returning to the start, in this network?



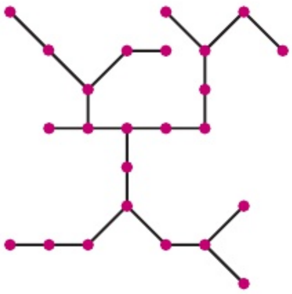
b Find the minimum spanning tree of the network.

5 Find the difference in weight between the longest and the shortest path between X and Z in the graph below, where each vertex is visited exactly once.

Challenge Q5

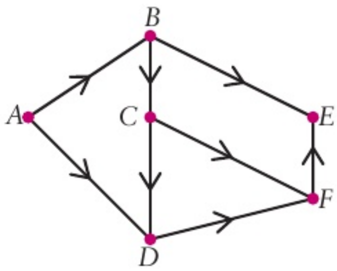


6 Find the minimum spanning tree for the graph in question 5.



20.2.2 Directed graphs

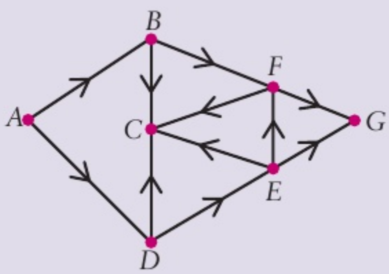
Remember that a **directed graph** is a graph in which the direction of travel from one vertex to the next is fixed. The direction of travel is indicated by arrows, as in the diagram. In a directed graph, the edges are called **arcs.** Vertex A is called a **source**, as there are no arcs entering A. Vertex E is called a **sink** as there are no arcs leaving it.



Restricting the direction of travel has an impact on how we solve the graph problems outlined in the previous section. In this section, we will look at exactly how the limitations on the direction of travel impact the problems we have looked at earlier in this chapter.

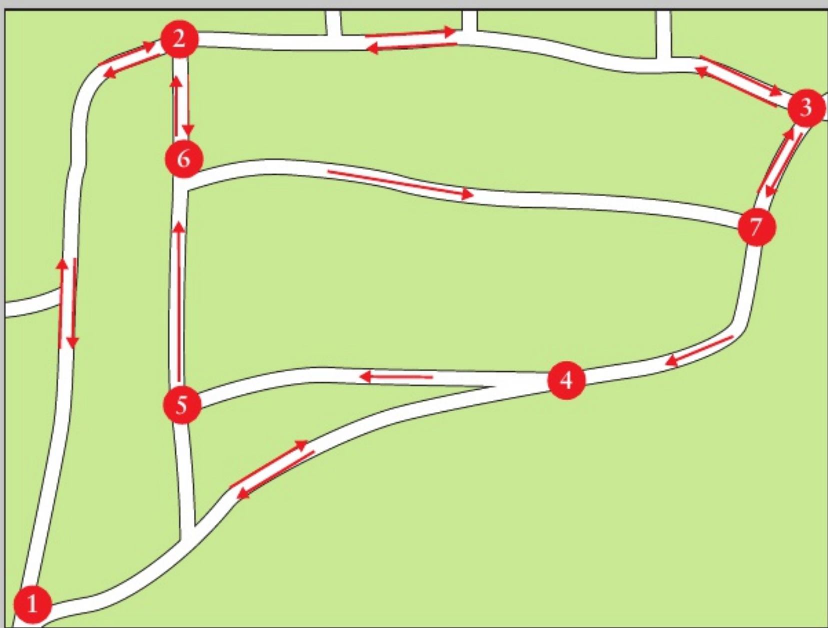
Explore 20.5

In the given graph, how many distinct pathways are there leading from A to G? Can you identify any sinks or sources?



Worked example 20.3

Every week, rubbish is collected from a part of the city shown below. The arrows show whether streets are one-way or two-way and the numbers show street intersections. Draw a graph that will enable you to plan the rubbish collection by one truck that will attempt to avoid passing a street more than once.

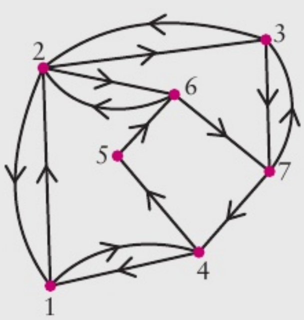


Solution

We need to make a directed graph that shows the connections between the different intersections.

The intersections can be represented by vertices and the streets by edges.

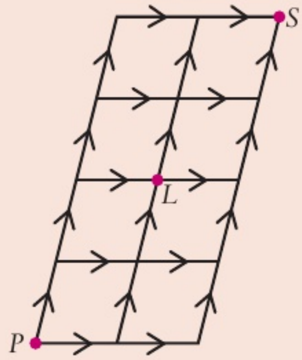
The vertices do not have to be at the same geographic position as in the map. The important thing is to make sure that adjacent vertices are correctly identified in our graph. Here is one such graph.



By trial and error, we find that the truck must pass through some streets twice. One such tour is 1–2–6–7–4–5–6–2–3–7–4–1.

Investigation 20.1

The directed graph shows the roads that Pete (P) can take to visit his friend Suhani (S). Along the way, he may or may not visit Liam (L).



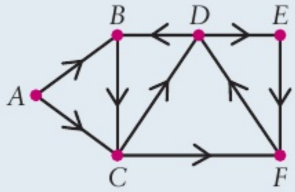
1 How many ways are there for Pete to get to Suhani's house?

2 How many of these pass by Liam's house?

3 How many of these avoid Liam's house?

Practice questions 20.2.2

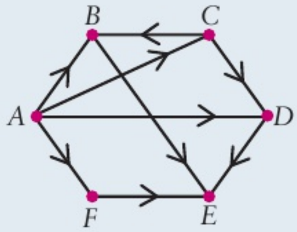
1 For the given graph:



a list the paths you can take to get from B to F

b list the paths you can take to get from A to F.

2 For the given graph:



a how many vertices are there

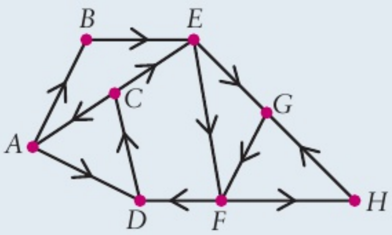
b how many edges are there?

c List the different paths from:

i A to C

ii A to D.

3 Consider the given graph.



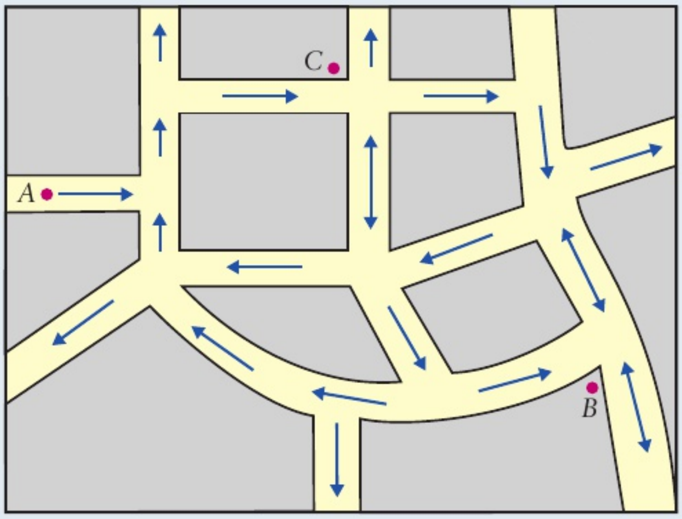
a Verify that it is not possible to make a circuit visiting each vertex only once.

b What change would make this possible?

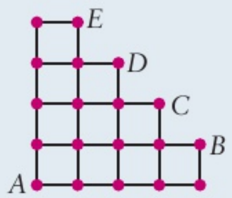
c Is it possible to go around each edge once and return to where you started?

4 The diagram shows a map of a city centre. The arrows indicate the direction of traffic. A courier car is at point A and must pick up a package from B and deliver it to C. What is the best route for the car to drive?

Challenge Q4



5 In the given graph, you start at A and you can move only right and up. How many ways are there to get to B, C, D and E?



20.3 Graph algorithms

20.3.1 Prim's algorithm

Fact

The word algorithm has the same origin as the word algebra, namely the 9th-century Middle-Eastern mathematician Muḥammad ibn Mūsā al-Khwārizmī, whose name is Latinised as Algoritmi

Hint

There are other algorithms to find the minimum spanning tree. Another famous algorithm is Kruskal's algorithm.

An **algorithm**, as we saw earlier, is a clearly defined set of instructions which, when followed in order, always leads to the solution of a particular type of problem.

Prim's algorithm aims to find the shortest spanning tree in a weighted graph. Here is a simplified form of this algorithm:

Step 1: Start with any vertex.

Step 2: Connect this vertex to the nearest vertex, using the edge with the smallest available weight.

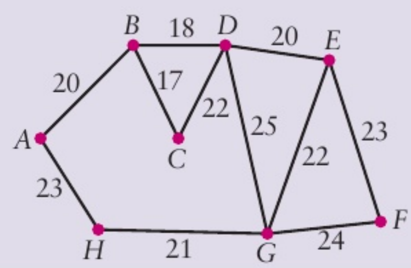
Step 3: Connect the next vertex that is nearest to any of those already connected.

Step 4: Repeat until all vertices are connected.

Step 5: Add the weights of all edges included. This is a minimum spanning tree.

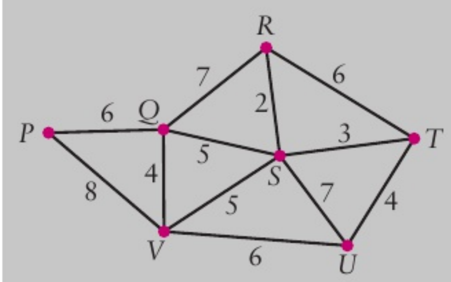
Explore 20.6

Can you apply Prim's algorithm twice to the given graph, starting at two different vertices? Compare your two results.



Worked example 20.4

Apply Prim's algorithm to the given weighted graph to find the minimum spanning tree.



Solution

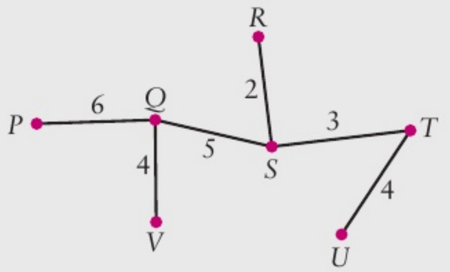
Step 1: We choose to start at P.

Step 2: We connect P to Q using the edge with weight 6 (the smallest possible weight).

Step 1: Next we connect Q to V (4 is the smallest possible weight).

Step 1: We connect Q to S (5), then S to R (2), then S to T (3) and finally T to U (4).

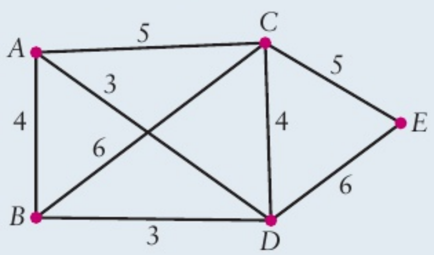
Step 5: The minimum spanning tree is as shown. Its weight is:
$6 + 4 + 5 + 2 + 3 + 4 = 24$



If we had started at R, S or T, would we have got the same minimum weight?

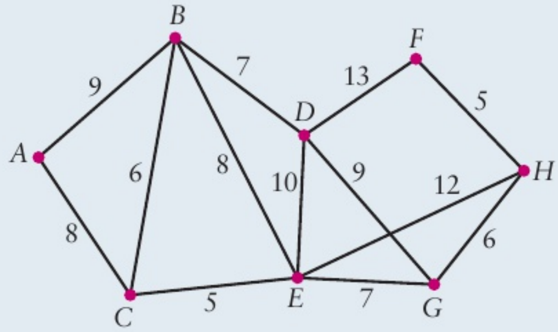
Practice questions 20.3.1

1 Find the minimum spanning tree of the following graph by inspection, starting at any point.

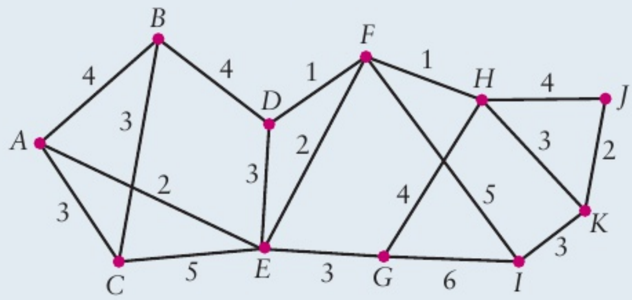


2 Find the minimum spanning tree of the graph in question 1 using Prim's algorithm. Did you get the same result?

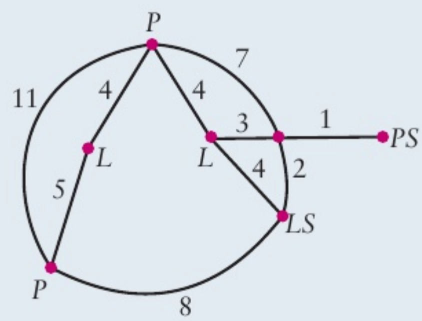
3 Find the minimum spanning tree of the following graph using Prim's algorithm.



4 Find the minimum spanning tree of the following graph using Prim's algorithm and starting at A.



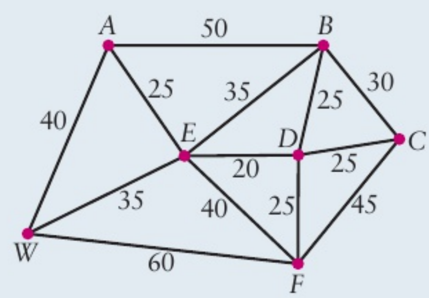
5 Yanik is an electrician. He needs to install two ceiling lights (L), two power points (P) and a light switch (LS) in a room, by connecting them all to the existing power supply (PS). The diagram shows the relevant dimensions of the room, in metres.



What is the minimum length of cable that Yanik must use to connect all the elements to the power supply?

Challenge Q6

6 In a tourist area in Central Italy, tourists with caravans can rent a position in specified caravan sites. A caravan site has positions as shown in the diagram, with distances in metres between them. Determine how these positions should be connected so that the total length of pipe to supply water required is a minimum.

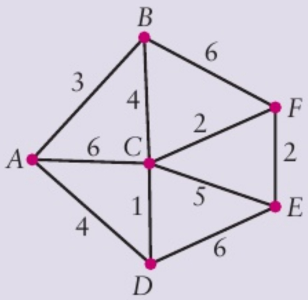


20.3.2 Dijkstra's algorithm

Dijkstra's algorithm is designed to find the shortest path between two given vertices of a weighted graph. This algorithm is used in mapping software to find the shortest route from your current location to your destination. You might have used this function on your phone.

Explore 20.7

In the given graph, use any method you can think of to find the shortest path between vertices A and E.



You might have found, in Explore 20.7, that there are a number of paths in this graph leading from A to E. If you do not have a better method, the best thing you can do is to try them all out and check which path has the smallest total weight. This is fine in a relatively small graph, but the more vertices it has, the longer it will take. You will only be asked to find paths in small graphs.

The following algorithm is a simplified modification of the original Dijkstra's algorithm.

Dijkstra's algorithm

Step 1: Circle the starting vertex. Examine all edges incident to this vertex. Darken the edge with the shortest length and circle the vertex at the other endpoint.

Step 2: Examine all uncircled vertices that are adjacent to the circled vertices in the graph.

Step 3: Using only circled vertices and darkened edges between the vertices that are circled, find the length of all paths from the starting vertex to each vertex being examined. Choose the vertex and edge that yield the shortest path. Circle this vertex and darken this edge. (Ties are broken arbitrarily.)

Step 4: Repeat steps 2 and 3 until all vertices are circled. The darkened edges of the graph form the shortest routes from the starting vertex to every other vertex in the graph, including the destination vertex.

Hint

Look at Worked example 20.5 and then go back to the steps to see if they make more sense. Go back and forth between the algorithm and the example as often as you need, until you thoroughly understand how it works. It may also help you to go through this worked example on GeoGebra:

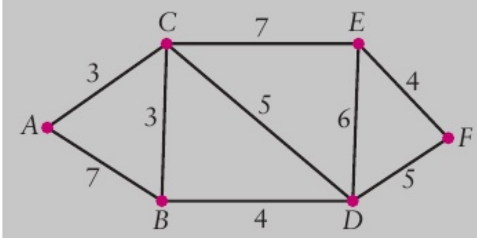
It will take you through the algorithm step by step, using an example graph of similar size.

Hint

It can be helpful to organise your work in a table, as in Worked example 20.5.

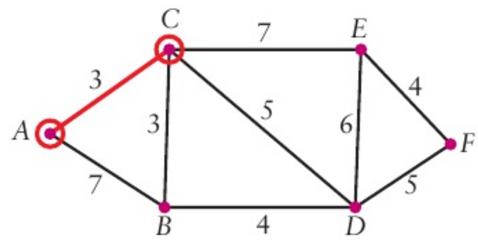
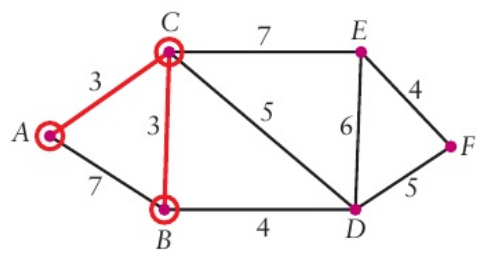
Worked example 20.5

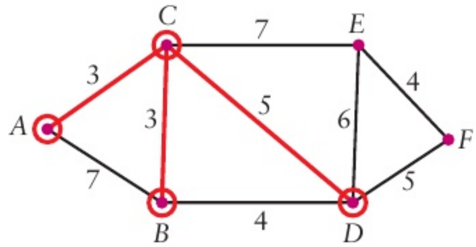
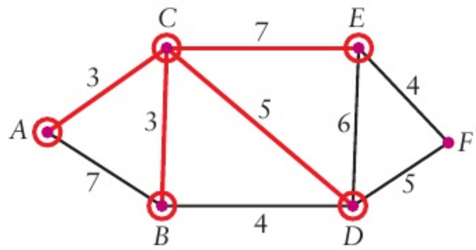
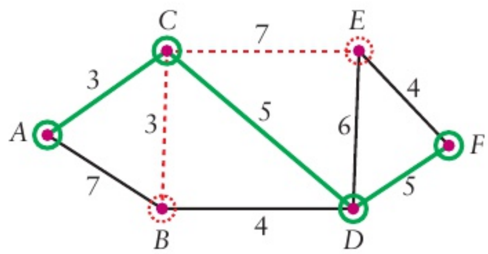
Use Dijkstra's modified algorithm to find the shortest path from A to F in this graph.



Solution

To find the path, we begin by circling vertex A and examining all vertices adjacent to it, namely B and C. We will use a table to track our steps.

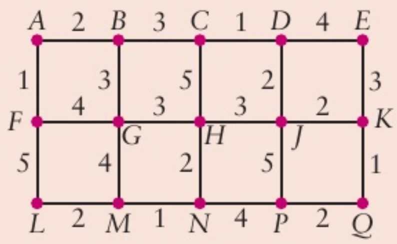
Step	Adjacent vertices	Path from A to vertex	Length of path	
1. Circle A. Circle C and darken AC	B C	AB AC	7 3	
Adjacent to A Adjacent to C	B B D E	AB ACB ACD ACE	7 $3 + 3 = 6$ $3 + 5 = 8$ $3 + 7 = 10$	
1. Circle B and darken CB				
Adjacent to B Adjacent to C	D D E	$ACBD$ ACD ACE	$3 + 3 + 4 = 10$ $5 + 3 = 8$ $3 + 7 = 10$	

Step	Adjacent vertices	Path from A to vertex	Length of path	
3. Circle D and darken CD				
Adjacent to C Adjacent to D	E E F	ACE $ACDE$ $ACDF$	$3 + 7 = 10$ $3 + 5 + 6 = 14$ $3 + 5 + 5 = 13$	
1. Circle E and darken CE				
Adjacent to D Adjacent to E	F F	$ACDF$ $ACEF$	$3 + 5 + 5 = 13$ $3 + 7 + 4 = 14$	
5. Circle F and darken DF.				

The shortest paths from A to all vertices are shown. Specifically we need the path to F, which is $ACDF$ with weight 13.

Investigation 20.2

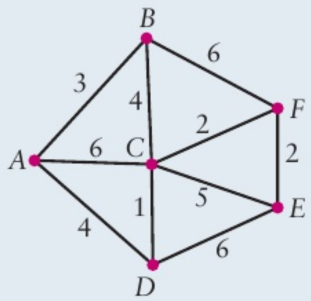
For the network in the given diagram, find the shortest path from A to Q. What is the length (weight) of this path?



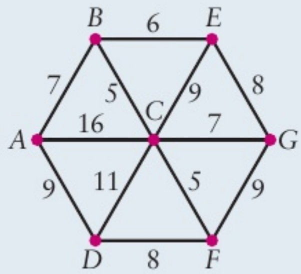
Practice questions 20.3.2

1 Use Dijkstra's algorithm to find the shortest path from A to E in the graph from Explore 20.7.

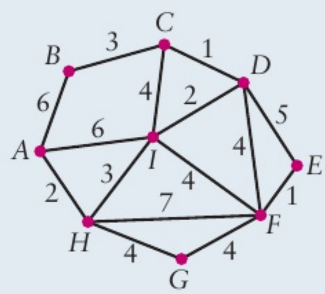
Compare your answer to your earlier answer. Is there a difference?



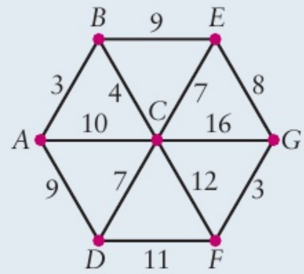
2 Use Dijkstra's algorithm to find the shortest path from A to G.



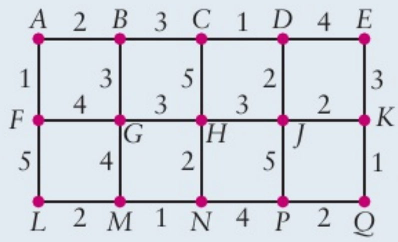
3 The graph shows the pathways in a park. The weights represent the time it takes to walk from one point to the next. What is the fastest way to walk from A to E? Which paths should you take?



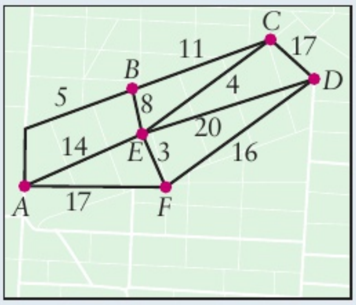
4 The diagram shows a map of six ports and the travelling times between them in days. What is the fastest way to travel from A to G? How long will the journey take?



5 Find the shortest distance from L to E.



6 The diagram shows a map of part of a city. The points are road intersections and the numbers are the times it takes to cover those stretches. A courier driver is at point A. Find, using Dijkstra's algorithm, the fastest time that this driver can take to deliver a parcel to point D.



Challenge Q6

20.4 Introduction to number theory

20.4.1 Algorithms in number theory

In this section we will look at algorithms related to numbers.

Fact

Long division and long multiplication are based on algorithms.

Explore 20.8

Use long division to divide 5473 by 13. Can you formulate the different steps of the algorithm?

Fact

This is an example of reducing a complex problem into a few simpler ones. You will find this technique everywhere in mathematics.

Fact

There are several forms of the algorithm. You will not be required to produce this in calculations. It is given here for demonstration only.

Worked example 20.6

Use long multiplication to find the answer to 37×852 and identify the steps of the underlying algorithm.

Solution

Step 1: Arrange the numbers one on top of the other and line up the place values in columns.

Step 2: Start with the ones digit of the bottom number. Multiply it by the ones digit in the top number. Write the answer below the equals line. If that answer is greater than nine, write the ones place as the answer and carry the tens digit.

Step 3: Proceed right to left. Multiply the ones digit of the bottom number by the next digit to the left in the top number. If you carried a digit, add it to the result and write the answer below the equals line. If you need to carry again, do so.

Step 4: When you've multiplied the ones digit by every digit in the top number, move to the tens digit in the bottom number. Multiply as above, but this time write your answers in a new row, shifted one digit place to the left.

Step 5: When you finish multiplying, draw another answer line below your last row of answer numbers. Use long addition to add your number columns from right to left, carrying as you normally do for long addition.

			8	5	2
		×		3	7
		5	9	6	4
+	2	5	5	6	
	3	1	5	2	4

Reflect

Think about other ways to work out 37×852. For example, knowing that $852 = 800 + 50 + 2$ can lead to another algorithm for multiplication. That is, first multiply 37 by 800, then by 50, then by 2. Finally, add the results.

There can be more than one algorithm to solve a problem. However, some algorithms are more efficient than others. This means they solve the problem with the fewest number of steps or using the least amount of memory. This is particularly important when creating algorithms for computers to use.

Practice questions 20.4.1

1 Follow the algorithm steps in Worked example 20.6 to multiply 325 by 17.

2 a Add 3475 and 8173 without the use of a calculator.

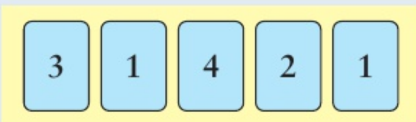
b Identify the different steps of the algorithm you used in part a.

3 a Subtract 3475 from 8173 without the use of a calculator.

b Identify the different steps of the algorithm you used in part a.

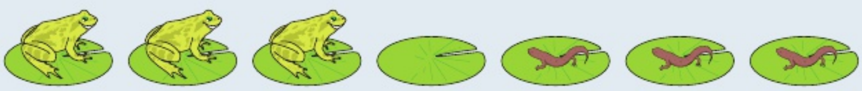
In questions 4–6, you need to investigate the problem before designing an algorithm to solve it.

4 On the table in front of you are five cards numbered 1 to 5.

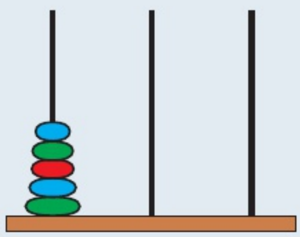


You need to sort the cards from smallest to largest, starting at the left, using a simple algorithm. Design an algorithm and check that it works.

5 There are three frogs and three newts on the seven lily pads below. The frogs want to go to the right and the newts want to go to the left. Frogs and newts can only slide one lily pad to the side or jump over a single frog or newt. Design an algorithm that allows both the frogs and the newts to move to their desired side.



6 The towers of Hanoi are a famous game in which you are asked to move a set of rings from the peg on the left to the peg on the right, in the same order as they started. You can move the rings to any peg you like, but only one at a time and you cannot put a bigger ring onto a smaller ring. Each ring should be on a peg, except the one moving. Can you find an algorithm to do this?



Hint Q4

You can use moves such as COMPARE, KEEP and SWAP. When you use SWAP, you will need to compare the swapped card to the one to the left of it.

Challenge Q6

Hint Q6

Start with three rings. Pay attention to where you should put the smallest one first. Test your algorithm by using Geogebra.

20.4.2 Absolute value and the division algorithm

Developing algorithms to solve common problems saves having to solve the same problem over and over again. It is like having the formula for the circumference of a circle and applying it, rather than starting from scratch and developing it all over again.

In this section we are going to look at two algorithms. The first is the absolute value algorithm and the second is the algorithm for division.

Fact

The absolute value is also called the **modulus**.

The absolute value algorithm

Remember that the absolute value algorithm is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Explore 20.9

Can you find the value of x if $|x - 3| = 5$?

This is a simpler algorithm than long multiplication. We can express this in words: If a number is non-negative, leave it as is. If it is negative, make it positive.

The division algorithm

Explore 20.10

5 does not divide into 17. If you divide 17 by 5, you will find a quotient of 3 and a remainder of 2.

We can write that as an equation: $\frac{17}{5} = 3 + \frac{2}{5}$

Can you write this equation without fractions?

Can you do the same for 68 and 11? 42 and 7?

Can you generalise your result?

In general, for any two integers a and b, where $b > 0$, when we divide a by b we can write

$$\frac{a}{b} = q + \frac{r}{b}$$

where q is the quotient and r is the remainder.

When simplified, this gives us the **division algorithm**.

You may have to get used to the language here, but when you have read the example, you will see that this is really what you have been doing with long division all these years.

Fact

For any two integers a and b, where $b > 0$, there exist unique integers p and q such that

$a = bq + r$ where $0 \leqslant r < b$

where a is the dividend, b the divisor, q the quotient and r the remainder.

Hint

This statement is the same as saying $\frac{a}{b} = q$ with remainder r.

Worked example 20.7

For each of the following, find the quotient and remainder when a is divided by b and write the division algorithm in the form $a = bq + r$ where $0 \leqslant r < b$

a $a = 113, b = 4$

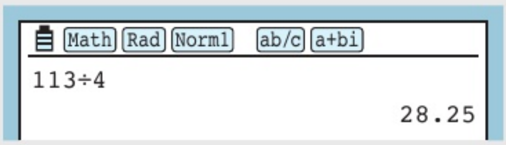
b $a = 110, b = 3$

c $a = 208, b = 4$

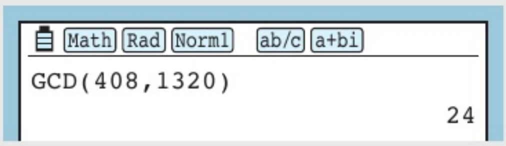
Solution

a Using long division we find that $q = 28$ and $r = 1$, so we can write $\frac{113}{4} = 28 + \frac{1}{4}$, or equivalently, $113 = 4 \times 28 + 1$

We could also have done this calculation with a calculator:



We see that $q = 28$. To find r, we multiply the decimal part by 4: $0.25 \times 4 = 1$



b $110 \div 3 = 36$, remainder 2. That is, $110 = 3 \times 36 + 2$

c $\frac{208}{4} = 52$, remainder 0. That is, $208 = 4 \times 52 + 0$

Investigation 20.3

1 7, 8, and 9 are three consecutive integers. Is the product of these numbers divisible by 2? Is the product divisible by 3? What about 6?

2 Take any three other consecutive integers and answer the same questions.

3 Let n be an integer. What are the two consecutive integers that follow n?

4 Let P be the product of these three integers. Work out the value of this product for any three values of n.

5 Can you state a general property of P that you discovered through this investigation?

Practice questions 20.4.2

1 Find $|x|$ for each of the following:

a $x = -12$

b $x = \sqrt{5}$

c $x^2 = 4$

d $x^2 = -7$

2 Solve the following equations for x.

a $|x| = 3$

b $2|x| - 1 = 17$

3 For each of the following, find the quotient and remainder when a is divided by b.

a $a = 37, b = 5$

b $a = 89, b = 7$

c $a = 183, b = 13$

4 For the number pairs in question 3, rewrite your answers as $a = bq + r$

5 A number divided by 7 gives 13 with a remainder of 2. What is the number?

6 Is it true that when you divide two numbers, the divisor should always be smaller than the quotient? Justify your response.

7 Claim: All integers are either multiples of 3 or leave a remainder of 1 or 2 when divided by 3.

Is the claim true? Justify your response.

Challenge Q9

8 Using the result of question 8, or otherwise, show that for a positive integer p one of the following will be divisible by 3: $p, p + 2, p + 4$

20.4.3 Divisibility, GCD and the Euclidean algorithm

Explore 20.11

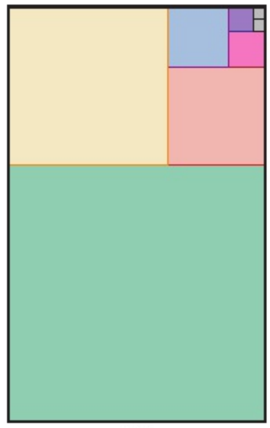
7 is a divisor of 21, since $21 = 3 \times 7$

7 is also a divisor of 35, since $35 = 5 \times 7$

So 7 is a common divisor of 21 and 35. In fact, GCD(21, 35) = 7

a What can you say about 7 and 35 'plus or minus' 21? Or 7 and 6 'times' 35 'plus or minus' 5 'times' 21?

b If g is a common divisor of a and b, what can you say about g and $m \times a \pm n \times b$ for any integers m and n?



A visual demonstration of Euclidean algorithm for GCD(34, 21) = 1.

> **Fact**
>
> Remember that 2, 3, 4, 6 and 12 are common divisors (factors) of 24 and 36. We call 12 the **greatest common divisor** (GCD) or **highest common factor** (HCF) of 24 and 36.

Remember that $36 = 4 \times 9 = 2^2 \times 3^2$, and $24 = 8 \times 3 = 2^3 \times 3$
Thus $\text{GCD}(36, 24) = 2^2 \times 3 = 12$
This is one way of finding the GCD of two integers. However, when the numbers are large it can take a long time to find the prime factors. Another approach is to use the Euclidean algorithm.

The **Euclidean algorithm** is based on the observation that if integers a and b are divisible by g, then $a - b$ is also divisible by g.

This is so because if a and b are divisible by g, then we can write $a = mg$ and $b = ng$, where m and n are integers.
Therefore, it follows that $a - b = mg - ng = (m - n)g$, which shows that $a - b$ is also a multiple of g.

Worked example 20.8

a Verify that 408, 1320 and 1320 – 408 are multiples of 12.

b Using the observation above, find the GCD of 408 and 1320.

Solution

a $408 = 34 \times 12$, $1320 = 110 \times 12$

$1320 - 408 = 110 \times 12 - 34 \times 12 = (110 - 34) \times 12 = 76 \times 12$

b We use the observation outlined above.
Let g be the GCD of 1320 and 408.

We can subtract 408 from 1320: $1320 - 408 = 912$
So g is a divisor of 912.

Also, g will divide $912 - 408 = 504$ as well as $504 - 408 = 96$
This is equivalent to saying that g divides $1320 - 3(408) = 96$

We can now repeat the operation with 408 and 96: $408 - 4(96) = 24$

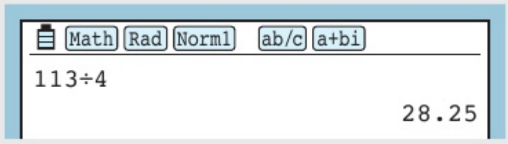
And again, with 96 and 24: $96 - 4(24) = 0$

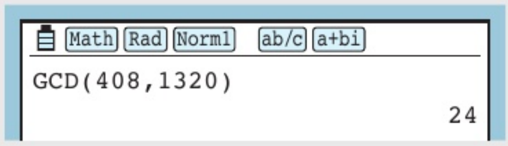
This is where we stop, as this means that 24 divides 96 and 408.

However, since $96 = 4 \times 24$ and $408 = 17 \times 24$, then it turns out that 24 is the greatest common divisor of 96 and 408.
Thus, by backtracking it is also the GCD of 408 and of 1320.

Looking back at our calculation, we observe that $1320 = 55 \times 24$ and $408 = 17 \times 24$. That is, since there is no common divisor between 55 and 17, 24 will be the GCD.

We can check this on a calculator:





By comparison, using the prime factors of 1320 and 408 we have:

$408 = 2^3 \times 3 \times 17$

$1320 = 2^3 \times 3 \times 5 \times 11$

The prime factors they have in common are $2^3 \times 3 = 24$

So, $\mathrm{GCD}(408, 1320) = 24$

Investigation 20.4

The Euclidean algorithm can also be illustrated using GeoGebra. Use the link below to investigate what this might look like. Can you illustrate the example above in this way?

Euclidean algorithm

Fact

In this chapter we will consider a and b as positive integers.

The basic idea is to repeatedly use the fact that $\mathrm{GCD}(a,b) = \mathrm{GCD}(b, a - q) = d$

In the previous section, you saw that $a = qb + r$ with some integer q, so, if d divides both a and b, it must divide both a and qb and, thereby, their difference $r = a - qb$. Similarly, if d divides both b and r, it should divide a as well. Thus, the greatest common divisors of a and b and of b and r coincide: $\mathrm{GCD}(a,b) = \mathrm{GCD}(b,r)$. But the pair (b, r) consists of smaller numbers than the pair (a, b), so we have reduced our task to a simpler one. And we can do this reduction again and again until the smaller number becomes 0.

For example, suppose we wish to find GCD(84, 36). Using the division algorithm we find:

$84 = 2 \times 36 + 12$

Now we divide the quotient by the remainder to get:

$36 = 3 \times 12$. Here the remainder is zero.

The greatest common divisor is the last non-zero remainder. In this case it is 12.

$\therefore$ GCD(84, 36) = 12

Often this division process needs to be repeated to get a remainder of zero.

Worked example 20.9

Find:

a GCD(210, 336)

b GCD(51, 109).

Solution

a $336 = 1(210) + 126$

$210 = 1(126) + 84$

$126 = 1(84) + 42$

$84 = 2(42) + 0$;
Thus, since 42 is the last non-zero remainder, then:
GCD(210, 336) = 42

b $109 = 2(51) + 7$

$51 = 7(7) + 2$

$7 = 3(2) + 1$

$2 = 2(1) + 0$

Since 1 is the last non-zero remainder, GCD (51, 109) = 1
Note that in this case the last step was unnecessary. Also, numbers like 51 and 109 with a GCD of 1, are said to be relatively prime.

Practice questions 20.4.3

1 Find the GCD of the following number pairs by:

i listing their factors first

ii finding the prime factors of each

iii using the Euclidean algorithm.

a 32 and 56

b 35 and 98

c 70 and 105

2 Find the GCD of 182 and 378 using your preferred method.

3 Find the GCD of 425 and 1071 using the Euclidean algorithm.

4 GCD(12, 18) = 6 and 18 – 12 = 6. Can you find another pair of numbers where the GCD is equal to the difference between the two?

5 Do you agree that the sum of two numbers has the same GCD as those two numbers? Explain your answer.

Challenge Q6

Hint Q6

Use backward substitution in the Euclidean algorithm.

6 GCD(78, 348) = 6 and 6 = 9(78) – 2(348). That is, the GCD of 78 and 348 can be expressed as a linear combination of the two numbers. Use your answer to question 2 to express the GCD of 182 and 378 as a combination of both numbers.

Challenge Q7

7 A florist has 124 roses and 279 carnations. She wants to create as many similar flower bouquets as possible while using all of the flowers she has.

a What is the biggest number of similar flower bouquets she could create?

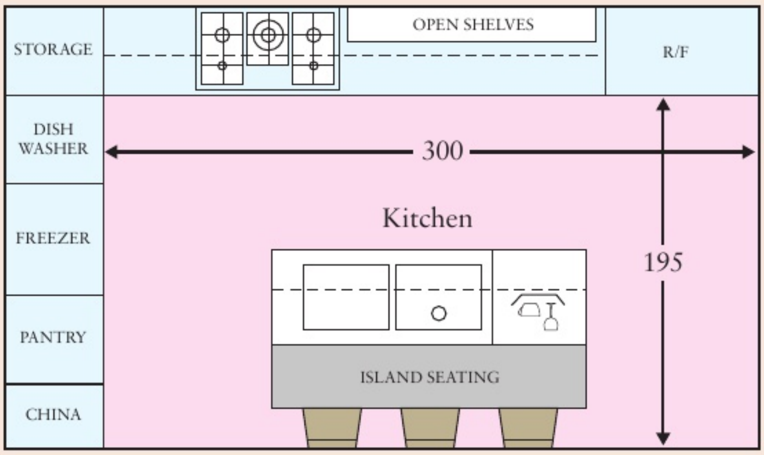
b How many roses and how many carnations does each bouquet contain?

Self assessment

- I understand the definition of vertex, edge and degree of a vertex.
- I understand what a subgraph and a tree are.
- I know the number of edges a tree with n vertices has.
- I know what a complete graph is and how to find the number of its edges it contains.
- I understand what an algorithm is.
- I can find the minimum spanning tree by applying Prim's algorithm for weighted graphs.
- I can find the shortest path by applying Dijkstra's algorithm for weighted graphs.
- I understand how the divisibility algorithm works.
- I know what the GCD of two numbers signifies.
- I can find the GCD of two numbers by using Euclid's algorithm.

Check your knowledge questions

1 A new kitchen has a rectangular floor that measures 300 cm × 195cm. Before installing furniture it has to be covered with tiles.



a What is the size of the largest square tile that can be used to cover the floor exactly?

b How many such tiles will be needed?

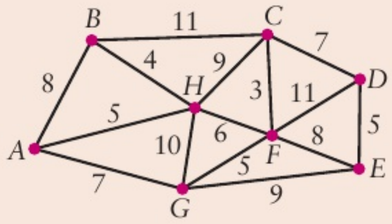
2 a Make a sketch of K_5, the complete graph with 5 vertices.

b How many edges are leaving from each of the vertices?

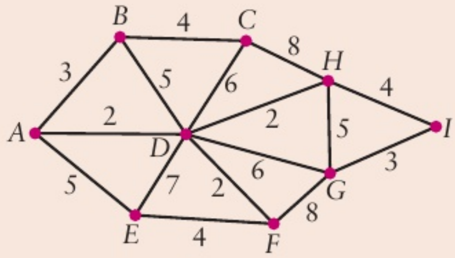
c Use your answer to part b to state the total number of edges in K_5.

d Does this method also work to find the total number of edges in K_n? Explain your answer.

3 Use Prim's algorithm to find the minimum spanning tree and state its total weight.



4 Use Dijkstra's algorithm to find the shortest distance from A to I.

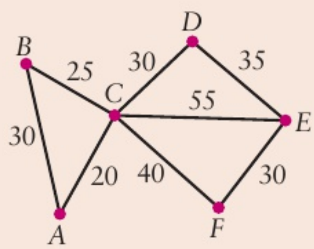


5 Find the GCD of 384 and 1080 using:

a Euclid's algorithm

b factor trees.

6 Look at this graph, which represents a network of paths in a park connecting six sculptures. The lengths are given to the nearest 5 m.



a Find the minimum spanning tree and find its total weight.

b The city wants to signpost a tour visiting each sculpture, starting from and returning to A. Which route would you advise the city to use? Explain your answer.

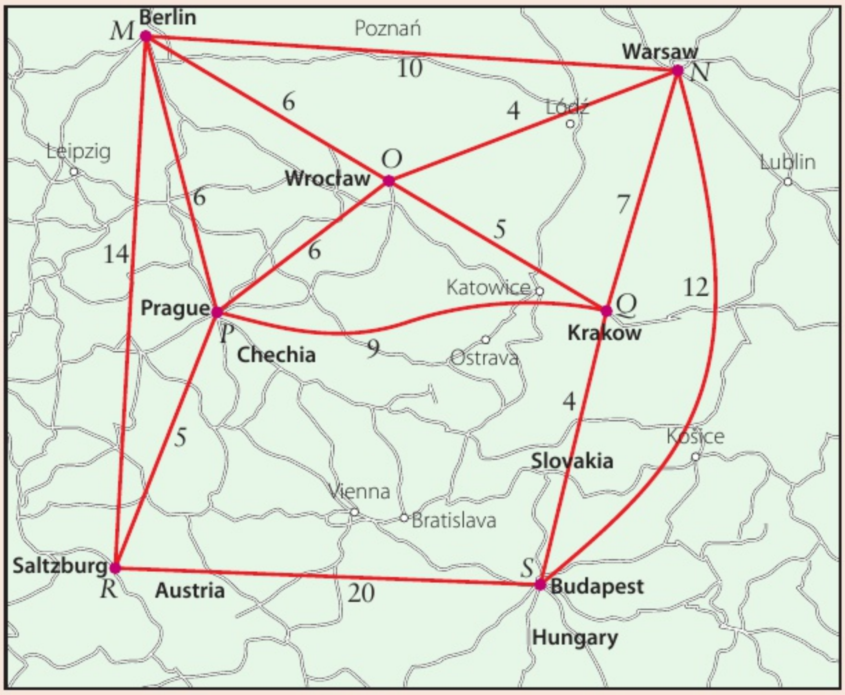
c If you could build an additional path to improve on the tour, which path would you build? Explain your answer.

7 The map shows a network of train routes connecting some cities. The weight on each edge represents the duration of travel on that route in hours. A group of travellers wants to visit each city at least once and then return to their starting point at M.

a Which route would you recommend for the group?

b Which, if any, routes would they have to travel more than once?

c What would be the total time for the whole tour?



Chapter 1 answers

Do you recall?

1 a $\frac{25}{3}$ b 0 c 0.99 d 0.045

2 $\frac{a^2}{b}$

3 €42

Practice questions 1.1

1 a 0.275

b $\frac{12}{18} = \frac{2}{3} = 0.666... = 0.\dot{6}$

c 2.66

d 1.35

2 a $\frac{23}{10}$ b $\frac{3}{4}$

c $\frac{14}{9}$ d $\frac{33}{50}$

3

Fraction	Decimal	Percentage
$\frac{35}{40}$	0.875	87.5%
$\frac{28}{9}$	$3.\dot{1}$	$311.\dot{1}$%
$\frac{29}{50}$	0.58	58%
$2\frac{3}{8}$	2.375	237.5%
$\frac{63}{100}$	0.63	63%

4 €112.50

5 a $\frac{14}{3}$ b 2.42 c $\frac{4}{9}$ d 0.12

6 a 6 : 13 b 2 : 5 c 7 : 10 d 15 : 11

7 a $x = 27$ b $y = 126$

c $z = 45$ d $w = 8, t = 4$

8 $x = 40°$ and $y = 50°$

9 a 8 km/h = 133.333… m/min

b 12 km/litres = 12 m/ml

c 1200 g/h = 0.02 kg/min

d 12 500 cm³/g = 12.5 m³/kg

10 a $30a^3b^5$ b $16x^2y^6$

c $4ab$ d $\frac{5x^2}{y}$

Practice questions 1.2

1 a $-2x + 7a$

b $-3x + 5y$

c $5ab + 2bc$

d $-3xy$

e $\frac{3x}{w}$

f $\frac{13c^2}{a^2}$

2 a $\frac{5x}{6}$ b y

c $\frac{8x^5}{15}$ d $\frac{y^2}{2x^2}$

3 a Area $= 4x^2 + 6x$ Perimeter $= 8x + 6$

b Area $= 5x^2 + 2$ Perimeter $= 10x + 5$

4 a $x = 12$ b $x = 9$

c $y = 14$ d $x = 12$

5 a $x \leqslant 3$, the graphical solution:

b $x \geqslant 3$, the graphical solution:

c $x > 6$, the graphical solution:

d $\frac{a}{3} - \frac{a}{2} < 3$, $a > -18$, the graphical solution:

6 a $6x^2 + 9x$ b $a^2 - 4$

7 a $3x(x - 2)$

b $y(y + 5)$

c $-6xy(x + 2y)$

d $(x - 13)(x + 1)$

8 a Let x cm be one side, so the other is $(10 - x)$ cm. Then $x(10 - x) = 16$, so $x = 8$ or 2. Hence length is 8 cm

b 15 cm

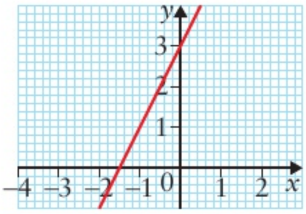
c width = 15 cm, length = 25 cm

d 22, 24 and 26

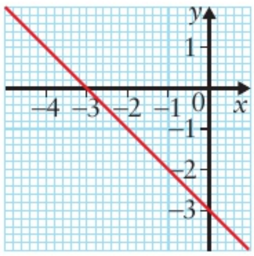
e 105

Practice questions 1.3

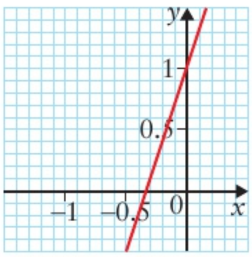
1 a



b



c



2 Line 1: $x = -2$, Line 2: $y = 3x$, Line 3: $y = -3$, Line 4: $y = -0.5x + 2$

3 a $y = 5x + 2$ b $y = -2x + 1$

c $y = 1.5x - 3$ d $y = -x + 2$

4 The point A is not on the line as it does not satisfy the equation: $4 \neq 3(3) - 2$

5 a $P = 30\text{ cm}$, $A = 30\text{ cm}^2$

b $P = 6\pi\text{ cm}$, $A = 9\pi\text{ cm}^2$

6 a $\frac{45}{8}\pi\text{ cm}^2$ b 22 units^2

7 a 666 m^2 b 216 m^2

Practice questions 1.4

1 a {*Saturday*, *Sunday*}

b {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

c {*January*, *June*, *July*}

d {*Head*, *Tail*}

2 a $\frac{2}{11}$ b $\frac{4}{11}$ c $\frac{1}{11}$ d 1

3 a $\frac{9}{20}$ b $\frac{3}{6}$ c 71% d 0.05

4 a $\frac{3}{5}$ b $\frac{22}{25}$ c $\frac{7}{25}$

5 a 2 b 2.18 c 5 d 2

6 a

Outcome (x)	Frequency (f)	Cumulative frequency
0.5	3	3
1.5	5	8
2.5	7	15
3.5	9	24
4.5	2	26
5.5	4	30

b 5

c Mode = 3.5, Median = 3.0, Mean = 2.97

Check your knowledge questions

1

Fraction	Decimal	Percentage
$\frac{12}{25}$	0.48	48%
$\frac{10}{9}$	$1.\dot{1}$	$111.\dot{1}$%
$\frac{9}{20}$	0.45	45%
$\frac{12}{5}$	1.4	140%
$\frac{89}{100}$	0.89	89%

2 a 11.03 b 50 c 57 d $0.11 = \frac{11}{100}$

3 a 1 : 3 b 40 km/h

c $\frac{29}{33}$ d 5 : 8 : 12

4 a $x = 5$, $y = 8$ b $x = 51$

c $y = 7$ d $z = 40$

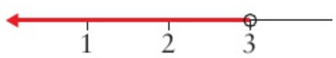
5 450 km

6 5

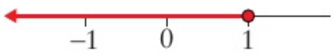
7 a $\frac{x}{6}$ b $240a^3b^5c^2$

8 a $x = 2$ b $x = 161$

9 a $x < 3$

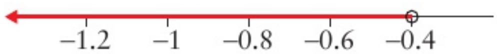


b $x \leqslant 1$



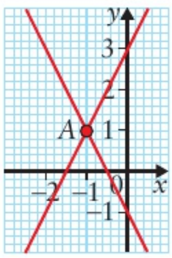
c $x \leqslant -12$

d $x < -0.4$



10 18π cm

11 $A(-1, 1)$ is the intersection of the two lines.



12 A: $y = \frac{1}{2}x + 1$, B: $y = -4x + 4$

13 a 26.5 **b** $16 + 10\pi$

14 a 2800 **b** $(156 + 60\sqrt{2})$ cm^2

15 a $\frac{1}{26}$ **b** $\frac{3}{26}$

16 a

Outcome (x)	Frequency (f)	Cumulative frequency
5	5	5
6	6	11
7	6	17
8	4	21
9	7	28
10	2	30

b 5

c Mode = 9, median = 7, mean = 7.27

Chapter 2 answers

Do you recall?

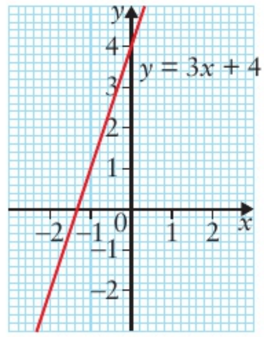
1 $\frac{2}{7}$

2 $\frac{8}{11}$

3 $4k^2 + 4k + 1$

4 $(3a - 4)(2a - 1)$

5



Practice questions 2.1.1

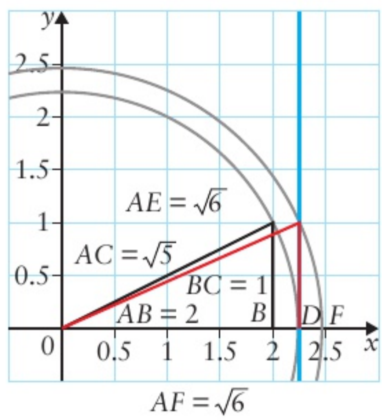
1 a rational **b** irrational
c rational **d** rational
e irrational **f** rational
g irrational **h** rational
i irrational **j** rational

2 a 1.414 **b** 1.732 **c** 2.236
d 3.162 **e** 3.873

3 a 7 and 8 **b** 9 and 10 **c** 11 and 12
d 13 and 14 **e** 16 and 17

4 $\frac{\pi}{2}, \sqrt{5}, 3.12, \frac{5}{\sqrt{2}}, 3\sqrt{2}, 2^3$

5



6 a False: $\sqrt{5} \times \sqrt{5} = 5$

b False: $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{4} = 2$

c False: $\sqrt{3} + (-\sqrt{3}) = 0$

d False: $\sqrt{7} - \sqrt{7} = 0$

Practice questions 2.1.2

1 a $(3 \times 4) \times 5$ **b** $0.2 + (0.5 + 1.2)$

c $(xy)z$ **d** $\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4}$

2 a $9 \times 12 - 9 \times 3 = 81$

b $1.2 \times 0.2 + 1.2 \times 0.3 = 0.6$ c $wx + zx$

d $6x + 12 - 5x + 15 = x + 27$

3 a -123 b -12.35

c $-\sqrt{5}$ d $-\pi - 2$

4 a $-\frac{1}{32}$ b $\frac{1}{0.3} = \frac{10}{3}$

c $\frac{4}{\pi}$ d $\frac{1}{\sqrt{15}}$

5 a -15 b $1\frac{11}{12} = \frac{23}{12}$

c $46x + 16y$

6 No, subtraction does not have closure property for natural numbers. For example; 2 and 3 are both natural numbers but, $2 - 3 = -1$ is not a natural number.

7 There is no identity element for division. For example; $5 : 1 = 5$ but $1 : 5 \neq 5$.

Practice questions 2.2.2

1 a $[-19, 2)$ b $\left[\frac{2}{3}, 12\right]$

c $(0, 19)$ d $\left(\frac{\pi}{2}, 4\pi\right]$

2 a $\left\{x \mid -\frac{1}{2} < x < 5\right\}$ b $\{x \mid 2.3 \leqslant x \leqslant 5.9\}$

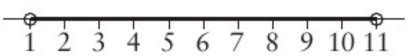
c $\{x \mid -1 \leqslant x < 1\}$ d $\left\{x \mid -5 < x \leqslant \frac{11}{2}\right\}$

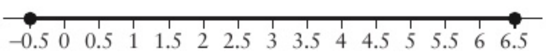
3 a $\{x \mid x < 15\}$

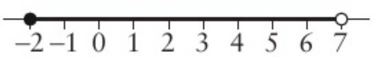
b $\{x \mid x > 1\}$

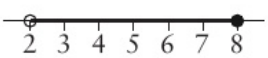
c $\{-5, -4, -3, -2, -1, 0\}$

d $\{x \mid x \leqslant 0 \cup x \geqslant 5\}$

4 a 

b 

c 

d 

5 $\{5, 7, 11\}$

6 $(3.5, \infty)$

7 Empty set, { }, because there is no number which is an odd and even number at the same time.

Practice questions 2.2.3

1 a π b 2

c $\frac{3}{2}$ d $\sqrt{2}$

2 a 18 b 5 c $\frac{8}{9}$ d 18

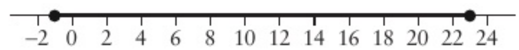
3 a $x = 1$ or $x = 11$ b $x = -3$ or $x = 8$

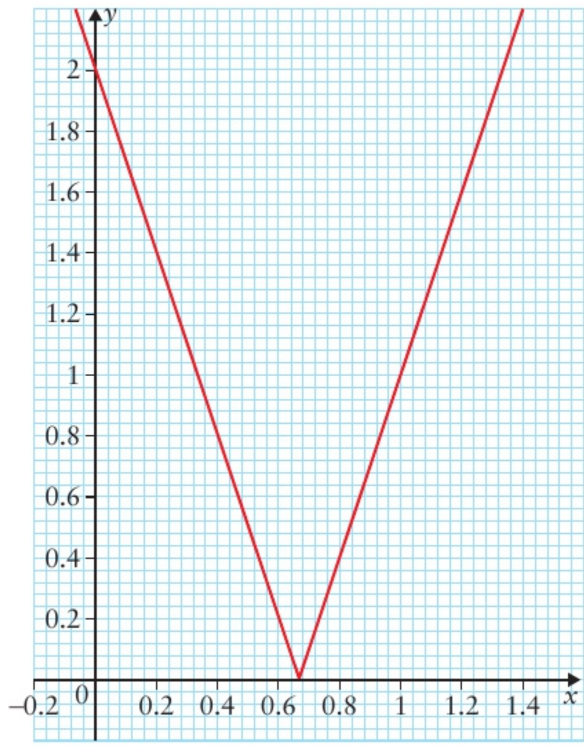
c $x = 6$ or $x = -6$ d $x = -7$ or $x = 17$

4 a $8 < x < 14$ b $x \geqslant 0$ or $x \leqslant -6$

c $-15 \leqslant x \leqslant 15$ d $x < -14$ or $x > 10$

5 $-1 \leqslant x \leqslant 23$



6 

Practice questions 2.2.4

1 a 2 b 4 c 3 d 5

2 a x^2 and 2 b $-x^4$ and -1

c $2x^4$ and 0 d $-x^3$ and 4

3 a not a polynomial b polynomial

c not a polynomial d not a polynomial

4 a $3x - 5$

b $x^3 - x^2 - 2x + 4$

c $x^3 - 3x^2 + 7x - 5$

5 a $2x^2 - 6x + 4$

b $x^4 - 2x^3 + 2x^2 - 2x + 1$

c $2x^4 - 6x^3 + 6x^2 - 6x + 4$

6 Yes, $P(x)$ is a factor of $R(x)$, because $R(1) = 0$

7 $R(2) = 5$

Check your knowledge questions

1 a rational b irrational c rational

d irrational e irrational

2 a 4 and 5 b 6 and 7 c 7 and 8

d 9 and 10 e 8 and 9

3 a $-\frac{13}{11}$ b $\frac{1}{8}$ c $\frac{1}{\sqrt{5}}$

d $\frac{\pi}{2}$ e $\frac{1}{2 - x}$

4 a -16 b 0.5 c $2 - \pi$

d $-\sqrt{2}$ e $x - \frac{\pi}{2}$

5 a $3y - 7$ b $1\frac{1}{2} = \frac{3}{2}$

c $16x + 16y$

6 a $[-1, 5)$ b $\left[\frac{12}{5}, 14\right]$ c $(-4, 9)$

d $\left(\frac{3\pi}{2}, 2\pi\right]$

7 a $\left\{x \middle| -\frac{2}{3} < x < 1.5\right\}$ b $\{x | 2 \leqslant x \leqslant 13\}$

c $\{x | -1 \leqslant x < 1 \text{ or } 1 < x \leqslant 5\}$

d $\left\{x \middle| -\frac{\pi}{2} < x \leqslant \pi\right\}$

8 $(8, \infty)$

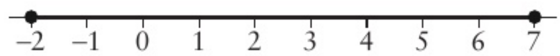
9 $x = \frac{47}{7}$

10 a $\frac{\pi}{2}$ b 2 c $\frac{1}{6}$ d $2\sqrt{2}$

11 a $x = -8$ or $x = 14$ b $x = -\frac{14}{3}$ or $x = 6$

c $x = -55$ or $x = 55$ d $x = -14$ or $x = 16$

12 $\{x | -2 \leqslant x \leqslant 7\}$



13 a $x^2 + 2x + 1$

b $x^3 + 4x^2 + 5x + 2$

c zero

Chapter 3 answers

Do you recall?

1 $y = 2x + 3$

2 $x + y = 4$ (or $y = -x + 4$)

3 28

4 yes

5 no, $(-4)^2 = 16$

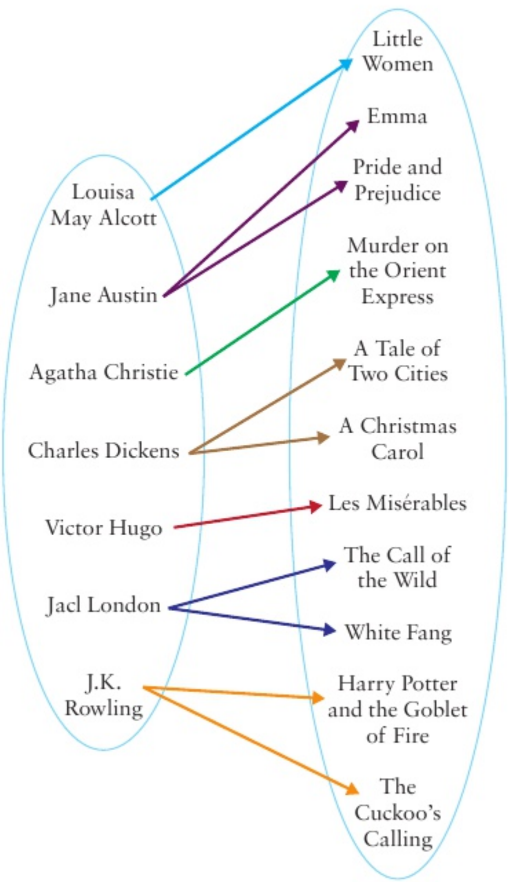
6 The set (or type) of numbers should be given. For example there are ten integers, five primes, but an infinite number of rationals.

Practice questions 3.1.1

1 a

is the author of	A Christmas Carol	A Tale of Two Cities	Emma	Harry Potter and the Goblet of Fire	Les Misérables	Little Women	Murder on the Orient Express	Pride and Prejudice	The Call of the Wild	The Cuckoo's Calling	White Fang
Louisa May Alcott						×					
Jane Austin			×					×			
Agatha Christie							×				
Charles Dickens	×	×									
Victor Hugo					×						
Jack London									×		×
J.K. Rowling				×						×	

b



2

	passed/will pass		
	D	E	H
A	×	×	×
B	×	×	×
C		×	×
D		×	×
E			×
F		×	×
G		×	×
H		×	
I		×	×
J		×	×
K		×	×

3 a i Domain: $\{-2, -1, 0, 1, 2\}$,
Range: $\{-1, 0, 1, 2, 3\}$

ii $(-2, -1), (-1, 0), (0, 1), (1, 2), (2, 3)$

iii $y = x + 1$ (where x is in the domain and y is the corresponding element in the range)

iv e.g. $(-4, -3), (3, 4), (0.5, 1.5), (100, 101)$

b i Domain: $\{-2, -1, 0, 1, 2\}$,
Range: $\{-4, -2, 0, 2, 4\}$

ii $(-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4)$

iii $y = 2x$

iv e.g. $(-4, -8), (3, 6), (0.5, 1), (100, 200)$

c **i** Domain: $\{-2, -1, 0, 1, 2\}$,
Range: $\{0, 1, 4\}$

ii $(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)$

iii $y = x^2$

iv e.g. $(-4, 16), (3, 9), (0.5, 0.25), (100, 10\,000)$

d **i** Domain: $\{0, 1, 4\}$,
Range: $\{-2, -1, 0, 1, 2\}$

ii $(4, -2), (1, -1), (0, 0), (1, 1), (4, 2)$

iii $y^2 = x$

iv e.g. $(16, -4), (9, 3), (0.25, -0.5), (100, 10)$

e **i** Domain: $\{1, 2, 3, 4\}$,
Range: $\{5, 4, 3, 2\}$

ii $(1, 5), (1, 4), (1, 3), (1, 2), (2, 5), (2, 4), (2, 3), (3, 5), (3, 4), (4, 5)$

iii $y > x$

iv $(5, 6), (0, 3), (3, 7), (7, 9)$

f **i** Domain: $\{1, 2, 3, 4, 5\}$,
Range: $\{6, 5, 4, 3, 2\}$

ii $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2)$

iii $x + y = 7$

iv e.g. $(9, -2), (0, 7), (0.5, 6.5), (100, -93)$

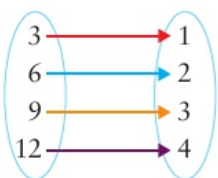
g **i** Domain: $\{2, 3, 4\}$,
Range: $\{1, 2, 3, 4, 5\}$

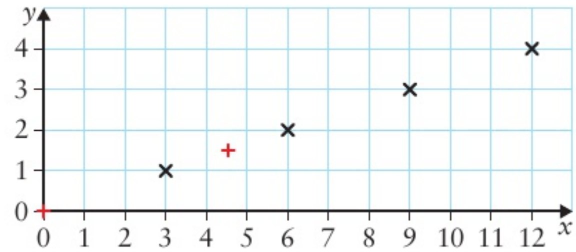
ii $(2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5)$

iii $y = x \pm 1$

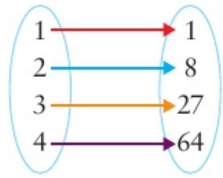
iv e.g. $(0, -1), (0, 1), (0.5, -0.5), (0.5, 1.5), (100, 99), (100, 101)$

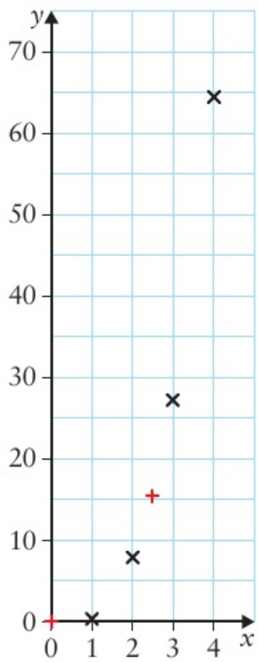
4 **a** $y = \dfrac{x}{3}$



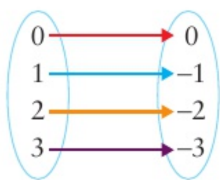


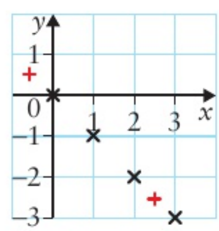
b $y = x^3$



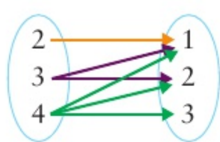


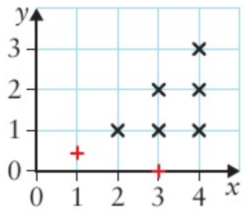
c $x + y = 0$



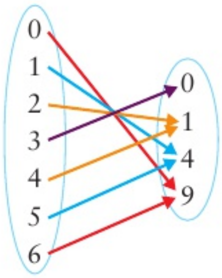


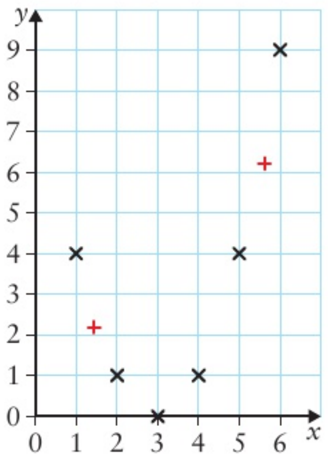
d $y < x$





e $y = (x - 3)^2$





Practice questions 3.1.2

1 a one-to-one b one-to-one
c one-to-one d not one-to-one
e not one-to-one f one-to-one
g one-to-one

2 a one-to-one b one-to-one
c not one-to-one d not one-to-one
e not one-to-one f one-to-one
g not one-to-one

3 a one-to-one b one-to-one
c one-to-one d not one-to-one
e not one-to-one

Practice questions 3.2.1

1 Only part c

2 a many-to-one function
b not a function
c one-to-one function
d one-to-one function
e many-to-one function
f one-to-one function
g not a function
h many-to-one function
i not a function
j many-to-one function
k not a function

3 not a function. For example 5 chocolates remaining could imply the presence of 27 people or 9 people.

4 a For example, 11, 12, 13, 14, 15 are all greater than 10 and therefore possible values of y

b No. For example if $x = 11$ and $y = 13$, then $y > x$

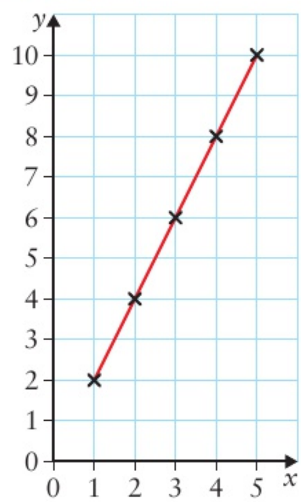
c No. For a single value of y, more than one value of x is possible.

d No

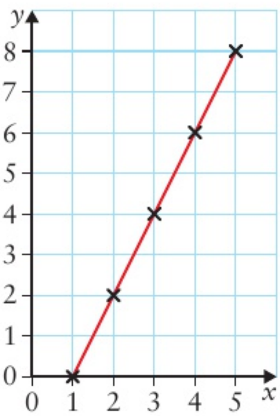
e In that case, y is a function of x as the value of y is certain. But x is not a function of y as, for example, $y = 3$ could imply both $x = 2.9$ and $x = 2.8$, among other possibilities.

Practice questions 3.2.2

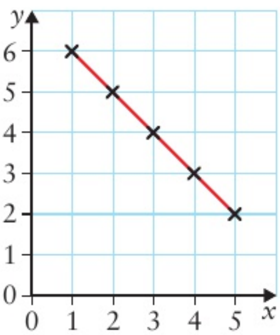
1 a $y = 2x$



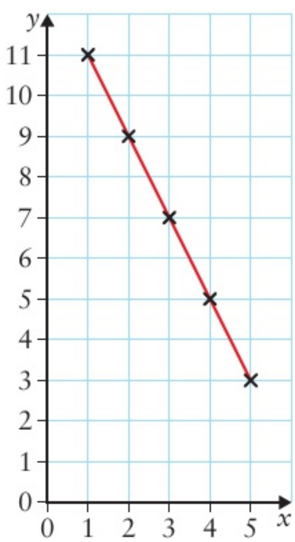
b $y = 2x - 2$



c $x + y = 7$

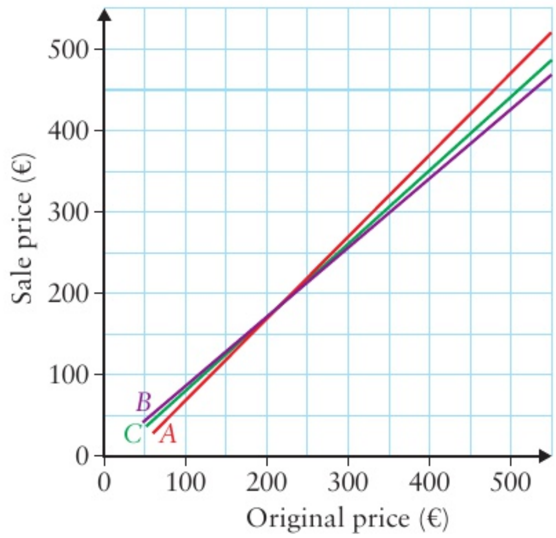


d $y = 13 - 2x$



2 **a** 2.2 **b** 0.2 **c** 5.9 **d** 10.8

3

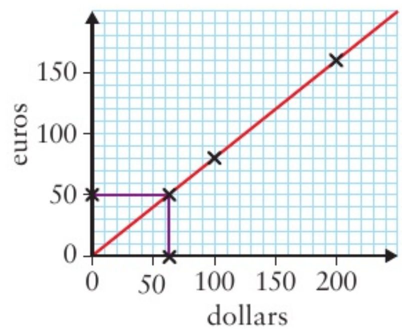


4 **a** yes

b €80.75

c €165.75

d



e \$63.80 from graph (\$63.82 calculated)

Practice questions 3.2.3

1 **a** $f(3) = 12$ and $f(-1) = -4$

b $f(4) = 2$ and $f(2) = 0$

c $f(10) = 14$ and $f(-10) = -6$

d $f(2.2) = 2, f(1.9) = 2, f(-1.3) = -1$

e $f(4) = 3, f(2) = 6, f(12) = 1$

f $f(0) = 0, f(2) = 16, f(-2) = 16$

2 **a** $f(1) = 4$

b $f(6) = 4$

c $f(0) = 4$

d $f(4.4) = 4$

e $f(3) = 4$

f $f(-1) = 4$

3 d and f

4 e, since $f(0)$ cannot be evaluated

5 a

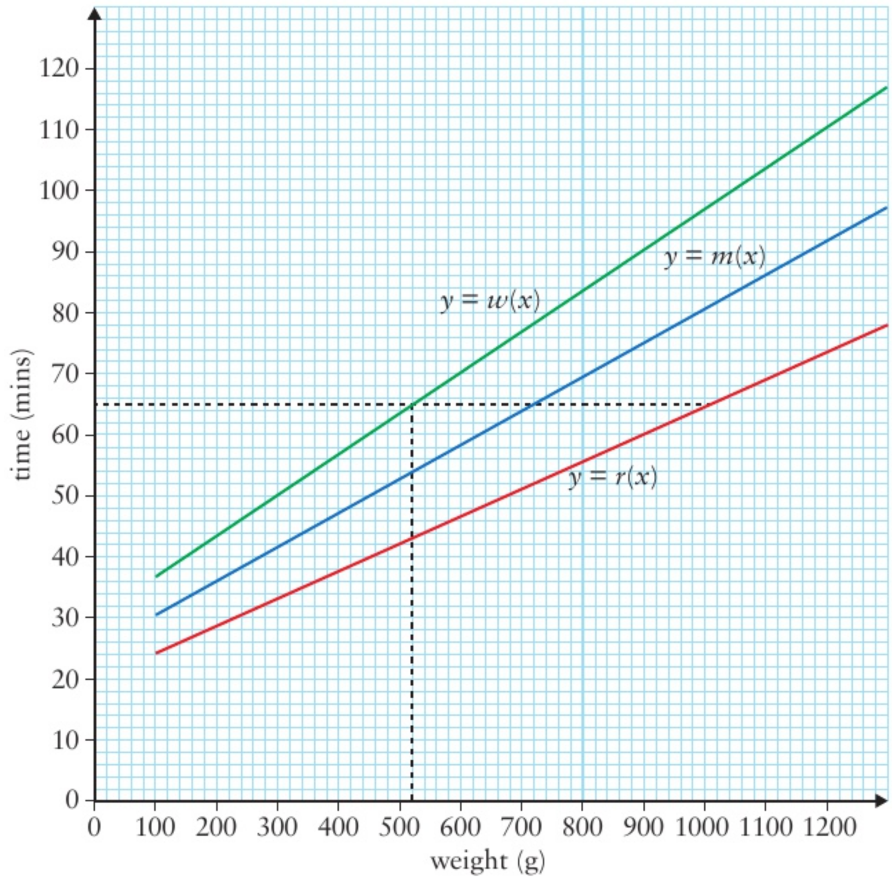
rare	56 minutes
medium	69 minutes
well-done	83 minutes

b $r(x) = \dfrac{2x}{45} + 20$

$m(x) = \dfrac{x}{18} + 25$

$w(x) = \dfrac{x}{15} + 30$

c



d 520 grams (see dashed lines on graph, 517 by calculation)

Practice questions 3.2.4

1 a, c and e

2 a mpg measures distance for a given quantity, litres/100 km measures quantity for a given distance.

b 5.65 litres/100 km

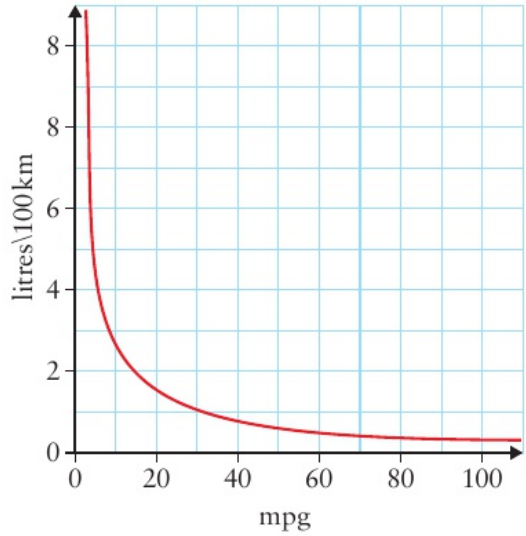
c $L(x) = \dfrac{454\,600}{1609x}$

d $M(x) = \dfrac{454\,600}{1609x}$

e $\dfrac{454\,600}{(1609 \times 5.65)} = 50$

f There is a reciprocal relationship between the two conversions.

g The graph of $y = M(x)$ will be identical but with the axes labels reversed.



h No. It would require division by zero.

Practice questions 3.2.5

1 **a** **i** yes **ii** yes **iii** no
iv many-to-one **v** no

b **i** yes **ii** yes **iii** yes
iv one-to-one **v** yes

c **i** no **ii** no **iii** yes
iv – **v** yes

d **i** yes **ii** yes **iii** yes
iv one-to-one **v** yes

e **i** yes **ii** yes **iii** yes
iv one-to-one **v** yes

f **i** no **ii** no **iii** no
iv – **v** no

2 **a** $f^{-1}(x) = x - 1$ **b** $f^{-1}(x) = x + 1$

c $f^{-1}(x) = \frac{x}{3}$ **d** $f^{-1}(x) = 5x$

e $f^{-1}(x) = \frac{x + 1}{2}$ **f** $f^{-1}(x) = 1 - x$

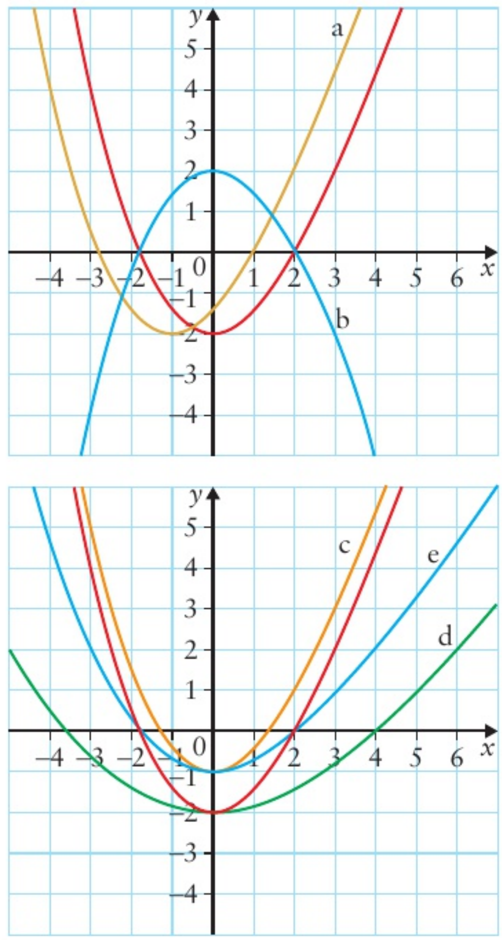
g $f^{-1}(x) = \frac{x - 4}{3}$

3 $f(2) = 16$ and $f(-2) = 16$, so f is many-to-one and therefore does not have an inverse function.

4 $x \geqslant 0$ and $x \leqslant 0$

Practice questions 3.2.6

1



2 **a** $y - 2 = f(x)$

b $y = f(-x)$ or $-y = f(x)$

c $y = f(x + 3)$ **d** $3y = f(x)$

3 **a** $y - 3 = f(x)$ **b** $y = f(x + 5)$

c $y = f\left(\frac{x}{4}\right)$ **d** $4y = f(x)$

4 A and C, B and D

5 **a** Phase 3 **b** None

c Phase 2 **d** Phase 2

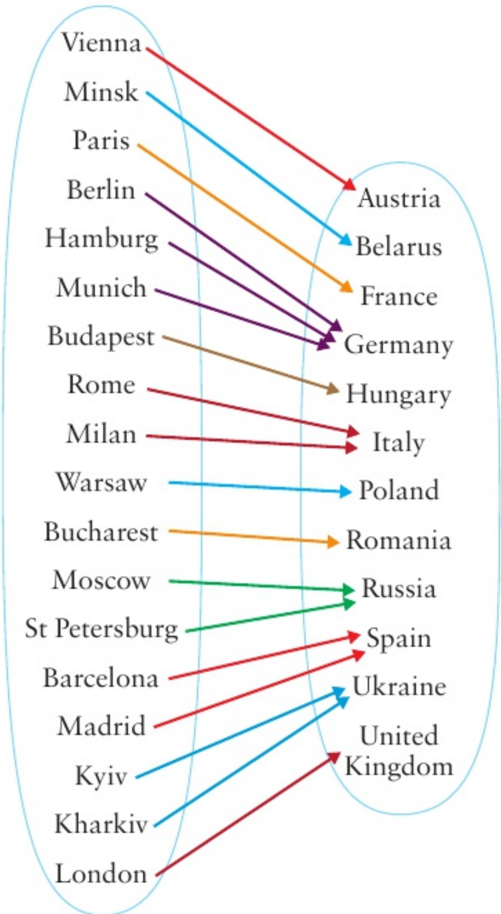
e Phase 1 **f** Phase 1

6 $y - 2 = f(x - 3)$

7 $(x - 4)^2 + (y + 1)^2 = 4$

Check your knowledge questions

1



2 **a** (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)

b (1, 1), (1.2, 1), (1.4, 1), (1.6, 2), (1.8, 2), (2, 2)

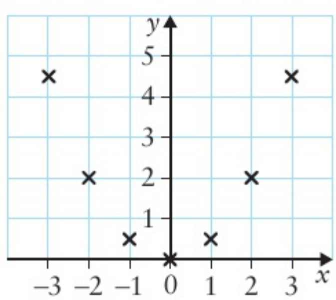
c (−2, 6), (−1, 2), (0, 0), (1, 0), (2, 2)

3 **a** many-to-one **b** many-to-one

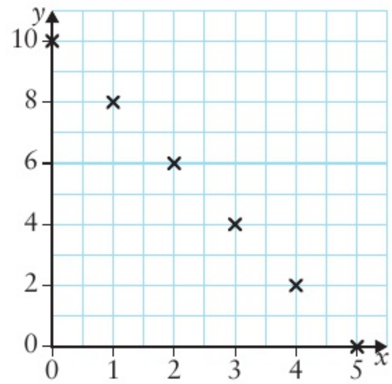
c one-to-many **d** one-to-one

e many-to-many

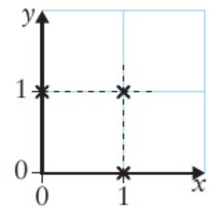
4 **a** many-to-one



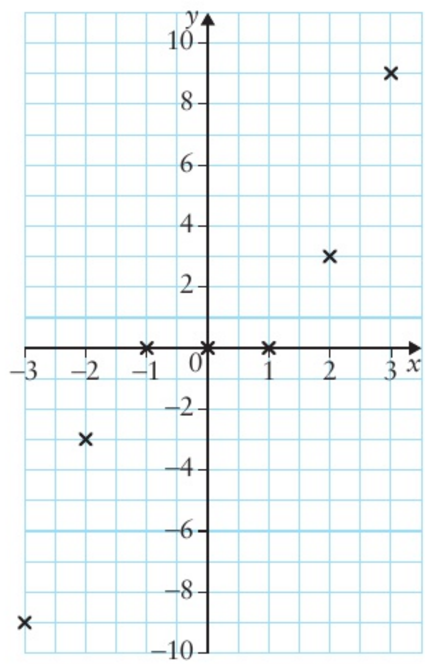
b one-to-one



c many-to-many



d many-to-one



5 b and e are functions (d is not since the range can include negative integers).

6 **a** 15

b no

c yes, it would then be a function

7 **a** a function with inverse

b not a function

c a function with inverse

d a function without inverse

e not a function

f not a function

8 $G: y - 1 = f(x)$, $H: 2y = f(x)$

9 a a stretch of scale factor $\frac{1}{5}$ with the x-axis as the line of invariant points

b a translation of 3 to the right (direction positive x)

c a stretch of scale factor 2 with the y-axis as the line of invariant points

d a translation of 10 down (direction negative y)

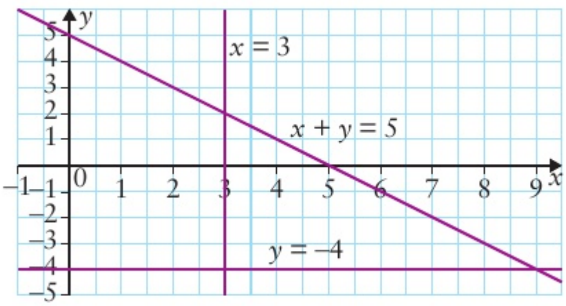
e a reflection in the y-axis

Chapter 4 answers

Do you recall?

1 Not parallel. They don't have the same gradient.

2



3 (3, 0); (0, 2)

4 $a = 2$

5 a $y = 2$ b $x = -2$ c $y = x + 2$

d $y = 0$ e $x = 0$

6 $y = 2x - 1$

7 $y = -\frac{1}{2}x - \frac{3}{2}$

8 $x < 5$;

Practice questions 4.1

1 a $y = -\frac{1}{2}x + \frac{11}{2}$ b $y = -\frac{1}{9}x + \frac{5}{3}$

c $y = -3$ d $y = \frac{4}{3}x + 4$

e $2x - 3y = -13$ f $3x + 2y = 0$

2 a $y = \frac{1}{2}(x + 9)$ b $y = \frac{1}{m}(x - b)$

c Gradient of inverse is the reciprocal of the gradient of the function; y-intercept is $-\frac{b}{m}$

d $Bx + Ay = C$

3 a $m = -\frac{q}{p}$ b $b = q$

c $y = -\frac{q}{p}x + q$ or $\frac{x}{p} + \frac{y}{q} = 1$

4 a Solve for y. Gradient $= -\frac{A}{B}$, y-intercept $= \frac{C}{B}$

b Gradient $= -\frac{3}{4}$, y-intercept $= 3$

c x-intercept $= \frac{C}{A}$; y-intercept $= \frac{C}{B}$

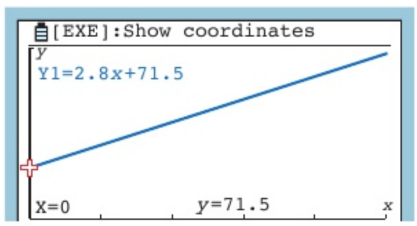
d x-intercept $= \frac{12}{3} = 4$; y-intercept $= \frac{12}{4} = 3$

5 a $y = 4x + 25$ b \$225 c 44 baskets

6 a 8.25 b $y = 8.25x + 0.94$

c Approximately 1.1 seconds

7 a



b 31.6 cm

c A female with humerus zero would be 71.5 cm tall. This is not a reasonable number.

Reasonable domain approximately between 28 cm and 40 cm.

8 a $v = 0.606T + 331$ b 347.36 m/s

c −3.3 °C

9 a $T = -6.5h + 20$ b $T = -8.8$ °C

c $T = -6.5h + 30$

d Temperature at the summit was at 5 °C, which means that the snow was melting.

10 a $C = 5.18n + 9750$

b i $13,635 ii 1476

Practice questions 4.2

1 A: $y = 3x + 2$ B: $x = 3$ C: $y = 4$

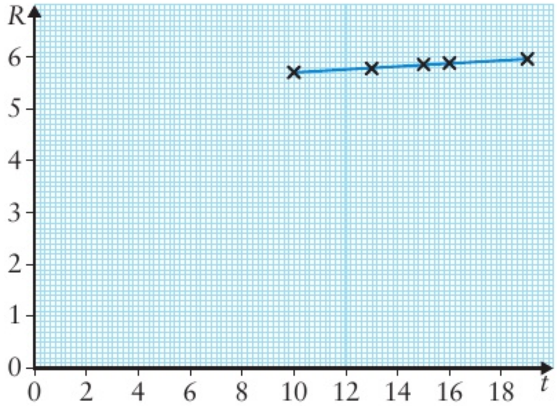
D: $y = -x + 3$ E: $y = \frac{1}{2}x - 1$

2 a $y = -\frac{2}{5}x - \frac{11}{5}$ b $y = \frac{5}{2}x - 8$

3 a neither b perpendicular

c neither

4 a



b $R = 0.03t + 5.4$

5 a $y = -3x + 7$ b $y = \frac{1}{3}x + \frac{5}{3}$

c $\left(\frac{8}{5}, \frac{11}{5}\right)$ d 12

6 a $y = 2x - 6$ is of the form $y = mx + b$

b 2 is the gradient

c The point is (5, 4). This is a point-gradient form.

7 a $y = -x + 8$

b No. The gradient of first 2 points is $-\frac{3}{2}$ while the gradient for the second and third points is $-\frac{13}{8}$.

c $y = \frac{3}{2}x + \frac{5}{2}$

8 a a is the x-intercept and b is the y-intercept.

b i $5x + 3y = 15$ ii $y = -\frac{5}{3}x + 5$

c $\frac{x}{3} - \frac{y}{12} = 1$, x-intercept is 3 and y-intercept is -12.

9 $A\left(-\frac{37}{25}, \frac{6}{5}\right)$; $B\left(2, \frac{6}{5}\right)$; $C\left(0, -\frac{5}{2}\right)$;

$D\left(2, -\frac{17}{10}\right)$; $E(-1, 0)$

10 a Use gradient and y-intercept:

$y = -\frac{1}{2}x$; $y = -\frac{1}{2}x + 4$; $y = 2x - 4$

b $y = 2x$, $y = 2x - 8$, $y = 2x - 20$, $y = 2x + 12$

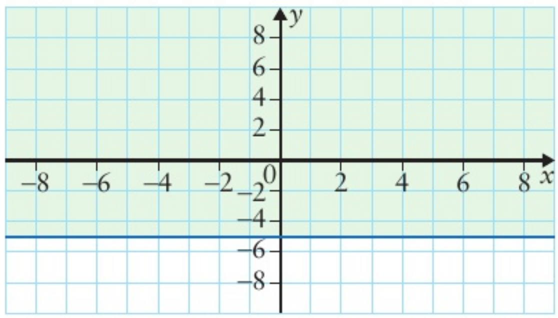
11 a C_1 and C_2 are not necessarily connected

$\frac{A_2}{B_2} = -\frac{B_1}{A_1}$

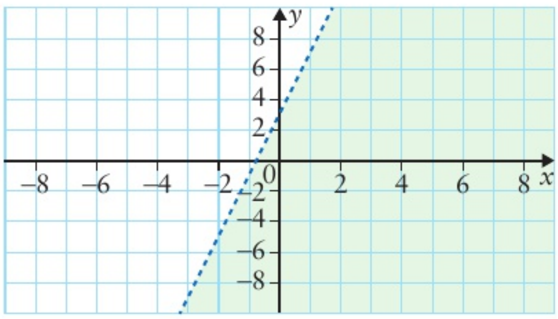
b One such equation can be $2x - 3y + 5 = 0$

Practice questions 4.3

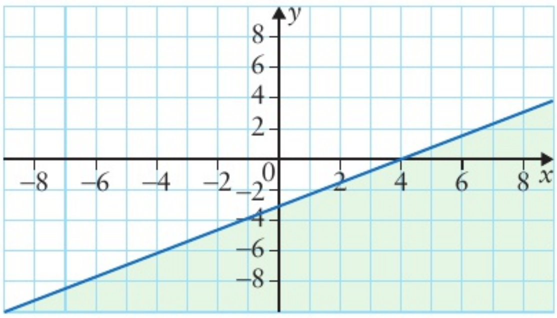
1 a



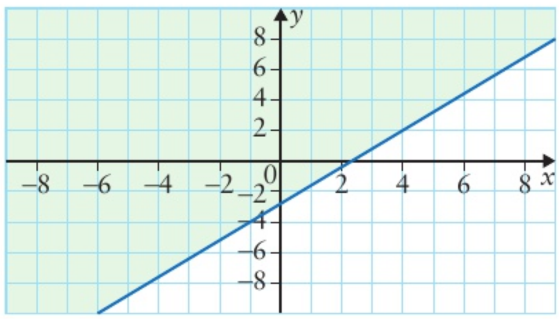
b



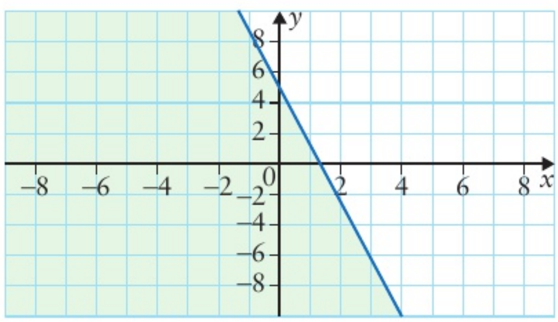
c



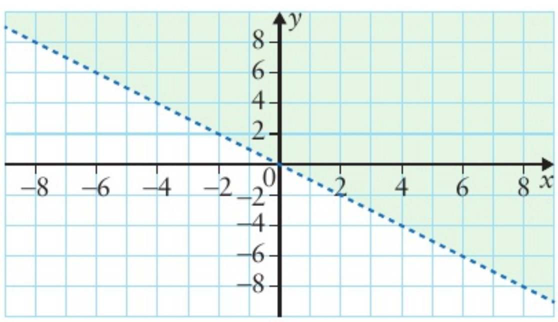
d



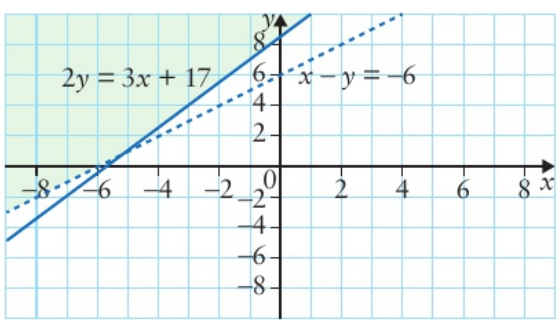
e



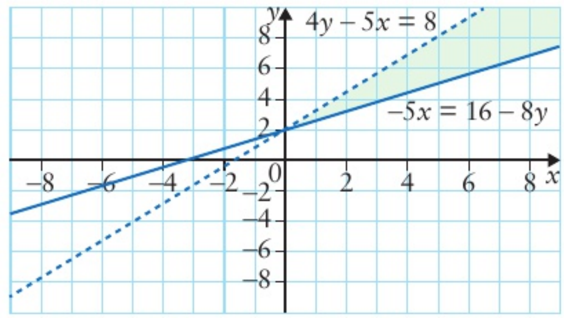
f



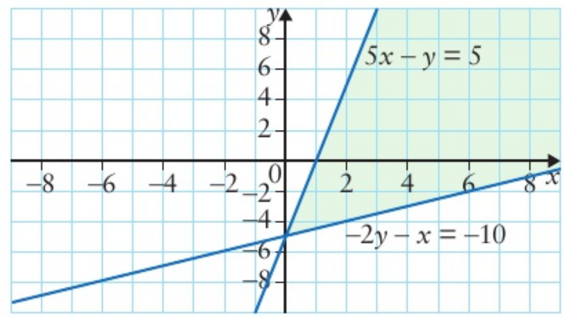
2 a



b



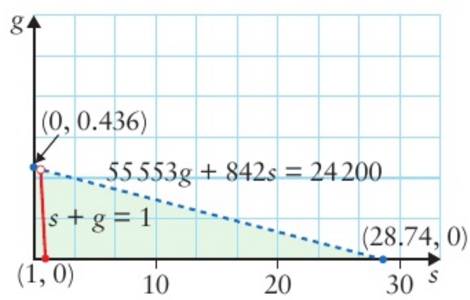
c



3 Let g be weight of gold in kg and s be weight of silver in kg. Two conditions must be satisfied:

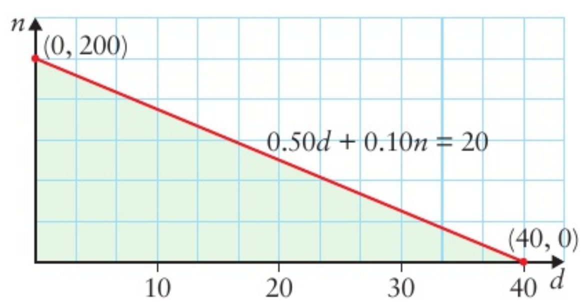
$55\,553\,g + 842s < 24\,200$

$g + s = 1$



4 Let d be the number of minutes during the day and n the number of minutes at night.

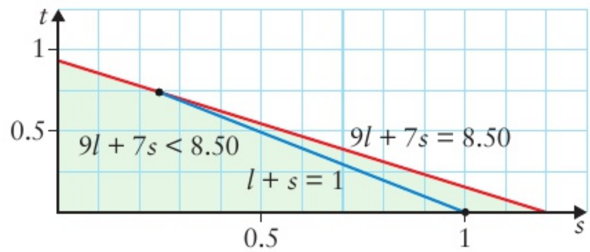
Then $0.50d + 0.10n \leqslant 20$.



5 Let l be weight of the large beans in kg and s weight of small beans in kg. Two conditions

$9l + 7s \leqslant 8.50$

$l + s = 1$



6 a $x + 4y \leqslant 8$

$3y - 2x + 21 \geqslant 0$

$3x + 4y \geqslant -8$

b $3x - 2y \leqslant 30$

$2x + y \geqslant 10$

$x + 8 \geqslant 4y$

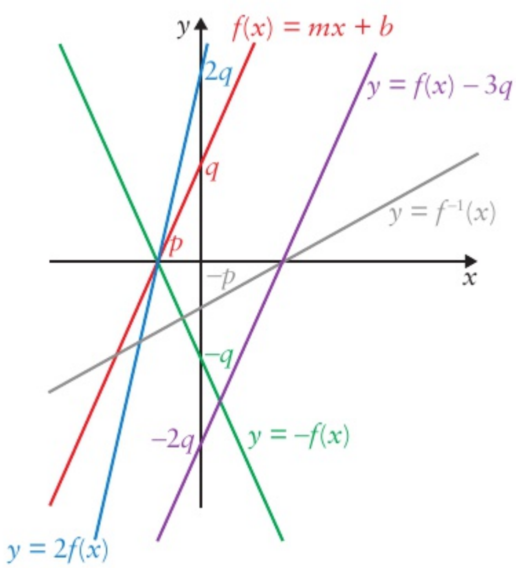
Check your knowledge questions

1 a $-m$ b $3m$ c m d $2m$

2 $f(x) = -\frac{q}{p}x + q$. All graphs are below.

a $y = -\frac{2q}{p}x + 2q$ b $y = \frac{q}{p}x - q$

c $y = -\frac{q}{p}x - 2q$ d $y = -\frac{p}{q}x + p$



3 a HG

b CD

c GF

d FE

e FE

f BC: $y = 2x + 6$

CD: $y = -2x + 6$

DE: $2y + x = 6$

GF: $y = 2x - 6$

HG: $y = -2x - 6$

AB: $2y - x = 6$

AH: $2y + x = -6$

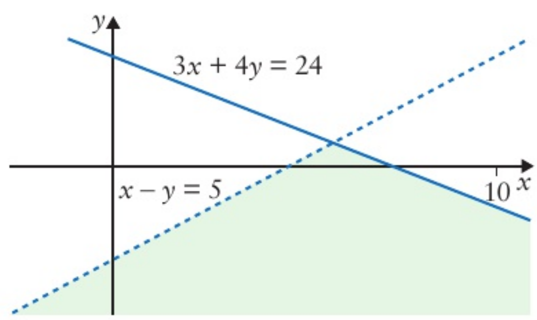
4 a $2x - 9y = -17$

b $9x + 2y = -34$

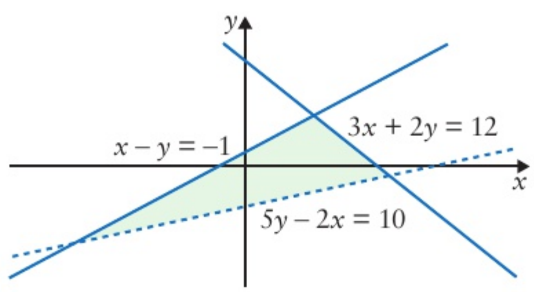
c $y = \frac{6}{5}x - 6$

d $x = -13$

5 a



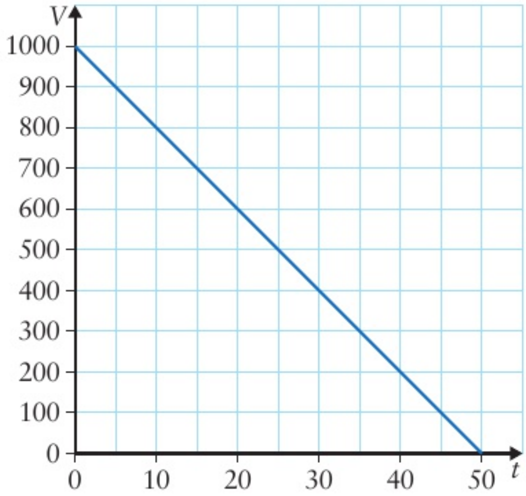
b



6 a $V = 1000 - 20t$

b Gradient = rate of drain per minute

c Domain [0, 50]



d 400

e Volume when container is full

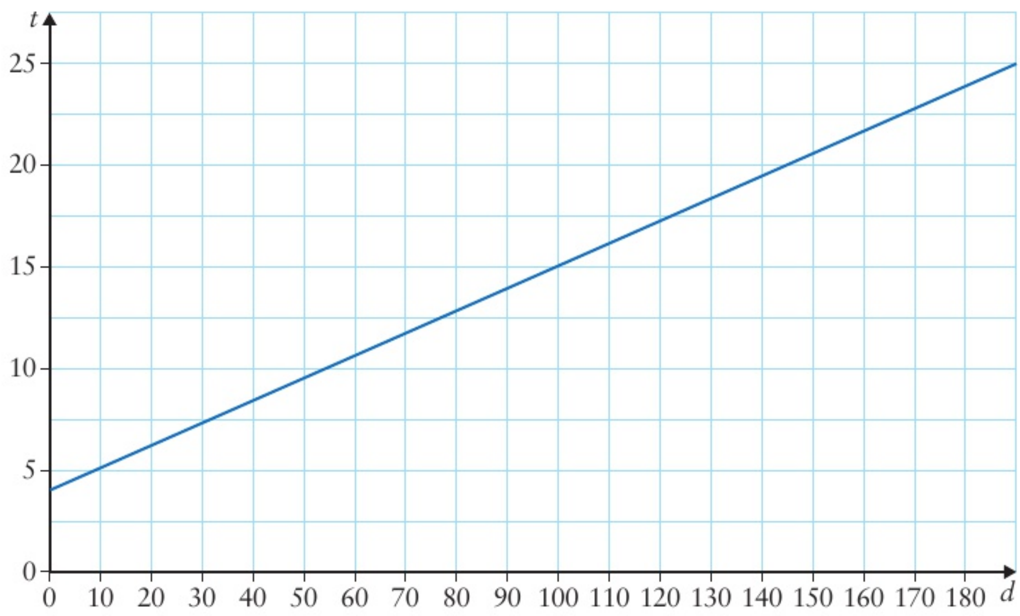
f after 50 minutes

7 a $t = 0.11d + 4.2$

b 5.3 milliseconds

c The time to say ouch when pricked very close to brain.

d



e The speed with which signals travel in the nerve.

8 a $y = 4$; $y = -4$; $x = 4$; $x = -4$; $y = x + 6$; $y = x - 6$; $y = -x + 6$; $y = -x - 6$

b $16 + 8\sqrt{2}$ units

c 56 units^2

Chapter 5 answers

Do you recall?

1 $y = \left(-\frac{1}{3}\right)x + \left(\frac{10}{3}\right)$

2 First plot the y-intercept. Then, starting at the y-intercept use the gradient to plot a second point by moving down 2 units and right 1 unit. Lastly, connect the two points to form a line.

3 No, it is not a solution.

4 The gradient is $-\frac{2}{3}$.

5 The lines are not parallel.

Practice questions 5.1

1 (2, 3)

2 (2, 2)

3 (1, 2)

4 No solution

5 (−5, −3)

6 (−3, 3)

7 No solution

8 (−5, 0)

9 (6, −3)

10 Infinitely many solutions

11 No solution

12 (5, −5)

13 No solution

14 After 10 weeks, the plants will be at the same height of 17 cm.

15 a Yes, Tony will pass Kate

b It will take Tony 106.67 seconds to finish the track. It will take Kate 145.45 seconds to finish the track.

16 Sample answers given:

a $y = -x + 4$

b $y = \frac{2}{3}x + 3$

c $3y = 2x + 12$

17 They will meet at (−3, −7), which doesn't make sense, as time can't be negative.

18 16 bottles of apple juice and 12 bottles of orange juice

19 There were 105 625 people in each city when they had the same population.

20 They need to produce 25 packages each week to break even.

21 a i Gary's is cheaper for 2 hours of work.

ii Frank's is cheaper for 5 hours of work.

b After 2.5 hours of work, they both cost the same £215.

22 They sold 26 mixed media cards and 78 stamped cards.

Practice questions 5.2

1 (1, 3)

2 (−10, 5.5)

3 (−1.25, −3.125)

4 (5, −6)

5 (3.6, 3.8)

6 No solution

7 (−4.5, −1)

8 Infinitely many solutions

9 (−2.4, 0.8)

10 (−3, −3)

11 Infinitely many solutions

12 (−4.5, −2)

13 $x = 11$, $y = 4.5$; Perimeter = 120 m; Area = 756 m^2

14 (3.5, 2)

15 Stephanie is 42 and her father is 81.

16 25 pounds of seafood flavour and 50 pounds of poultry flavour

17 They will be the same cost for 100 people, so for fewer than 100 people, the Red Hall will be a better deal.

18 Florian earns £13.25 per hour, and Simon earns £16.75 per hour.

19 37 and −14

20 The length of the frame is 18 cm and the length of the hole is 15 cm.

Practice questions 5.3

1 (−1, −1)

2 (5.2, 4)

3 (0, −7)

4 (4, 0.5)

5 Infinitely many solutions

6 (−8, 1)

7 (4, 2)

8 (3, −3.5)

9 No solution

10 (3.5, 1.75)

11 No solution

12 (−1, −3)

13 (−8, −1)

14 No solution

15 (−2, −4)

16 €1400 in the account earning 4.5% and €10,600 in the account earning 6%

17 5 kilos of candies and 5 kilos of nuts

18 Sarah drove for 8 hours and Matt drove for 13 hours.

19 1.6 litres of the 70% solution and 2.4 litres of the 20% solution

20 The boat travelled at 32 km/h and the current had a speed of 8 km/h.

21 €275.00

22 €210.00

23 Length = 48 units; width = 36 units

24 Solution is (10, 36), which means that 10 items need to be sold in order to break even.

Check your knowledge questions

1 (5, −8)

2 (3, 2)

3 No solution

4 (−5, 4)

5 Infinitely many solutions

6 (−3, −12)

7 (8, 7)

8 (3, −6)

9 (−4, 11)

10 (9, −9)

11 (−11, 7)

12 No solution

13 (−3, −10)

14 (−15, 20)

15 (3, 6)

16 (11, 10)

17 No solution

18 (19, 13)

19 (20, −17)

20 Infinitely many solutions

21 (−4, 16)

22 (0, −6)

23 No solution

24 (−9, 17)

25 (1, 13)

26 (−17, 16)

27 €3000 was invested at 3% and €2000 was invested at 5%.

28 20 minutes

29 $13.75

30 The acute angles measure 74.5° and the obtuse angles measure 105.5°.

31 He needs to sell 146.3 pounds of beans in order to break even.

32 $x = 3.6$ m, $y = 20.4$ m. The overall dimensions are 16 m × 24 m.

33 4.2 litres are needed, but he only has 3.5 litres, so no, he doesn't have enough.

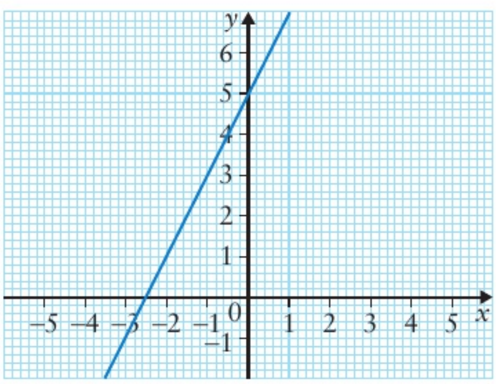
Chapter 6 answers

Chapter 6 is an enrichment chapter at Standard level. Please access the answers for this chapter via the link on this page of your eBook.

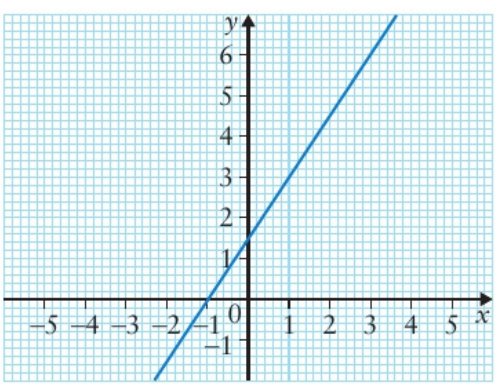
Chapter 7 answers

Do you recall?

1



x-intercept $(-2.5, 0)$ y-intercept $(0, 5)$



x-intercept $(-1, 0)$ y-intercept $\left(0, \frac{3}{2}\right)$

2 $(-1, 2)$

3 $(1, 2)$ does, because its coordinates satisfy the equation of the curve.

$(2, 4)$ does not, because its coordinates do not satisfy the equation of the curve.

Practice questions 7.1

1 a

x	−3	−2	−1	0	1	2	3
y	−31.5	−14	−3.5	0	3.5	14	31.5

b

x	−3	−2	−1	0	1	2	3
y	−10.8	−4.8	−1.2	0	1.2	4.8	10.8

c

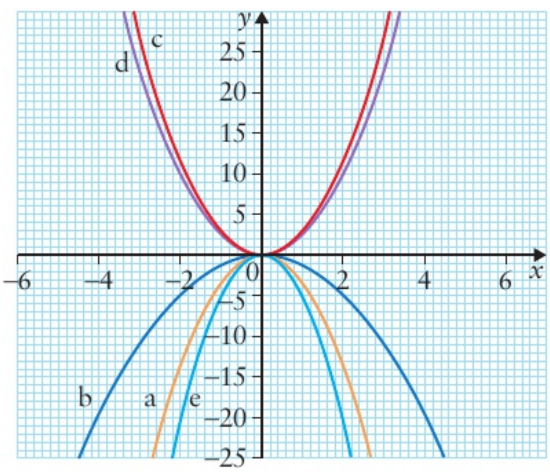
x	−3	−2	−1	0	1	2	3
y	27	12	3	0	3	12	27

d

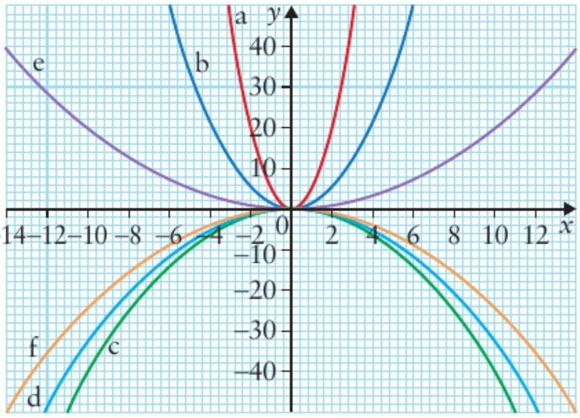
x	−3	−2	−1	0	1	2	3
y	22.5	10	2.5	0	2.5	10	22.5

e

x	−3	−2	−1	0	1	2	3
y	−45	−20	−5	0	5	20	−45



2 a, b and e are concave up.
c, d and f are concave down.



3 A–e, B–a, C–b, D–c

Practice questions 7.2

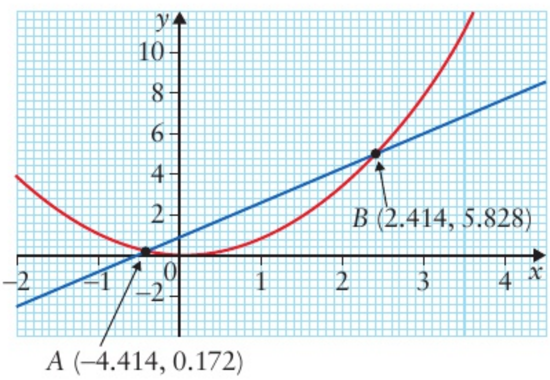
1 **a** $(2, 4)$, $(-2, 4)$

b $\left(-\sqrt{\frac{5}{3}}, 5\right)$, $\left(\sqrt{\frac{5}{3}}, 5\right)$ or $(-1.29, 5)$, $(1.29, 5)$

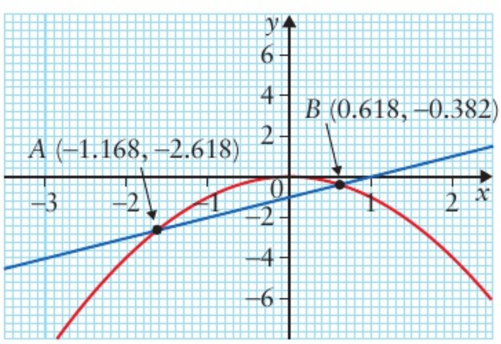
c no intersections

d $\left(-\frac{1}{\sqrt{2}}, 1\right)$, $\left(\frac{1}{\sqrt{2}}, 1\right)$ or $(-0.707, 1)$, $(0.707, 1)$

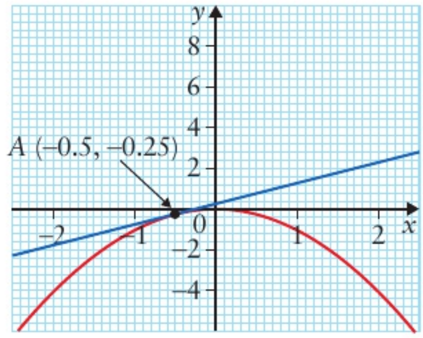
2 **a**



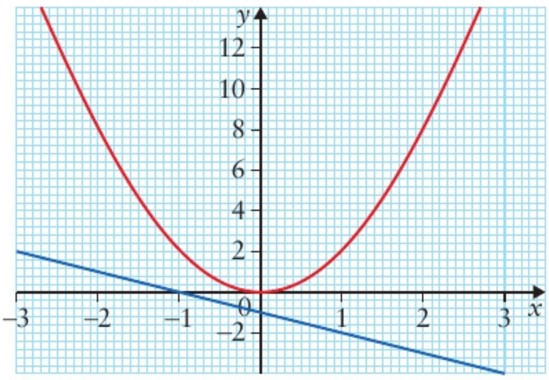
b



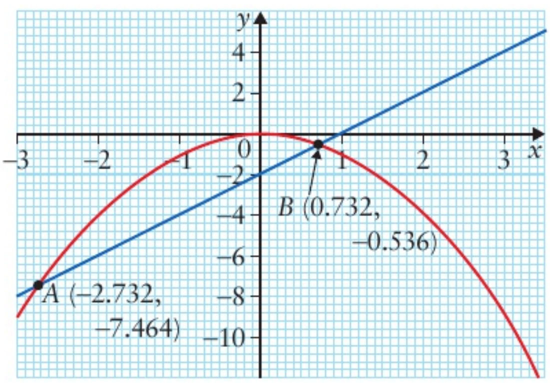
c



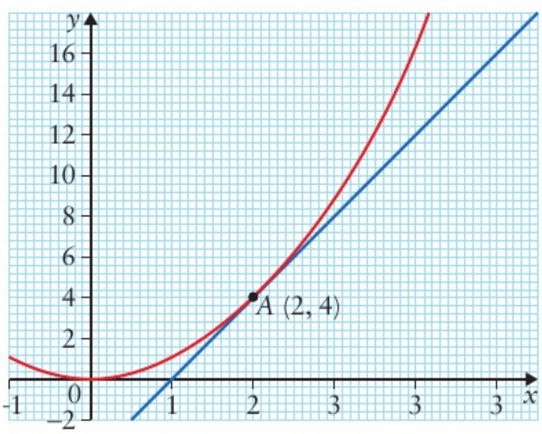
d



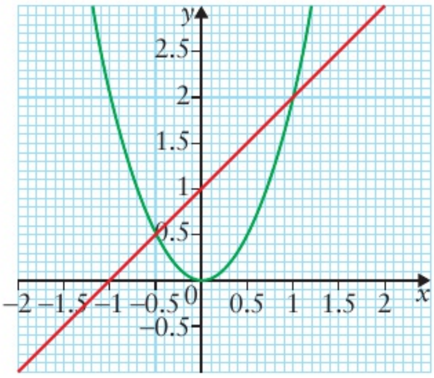
e



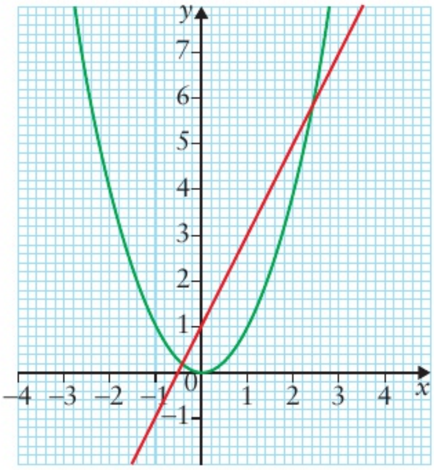
f



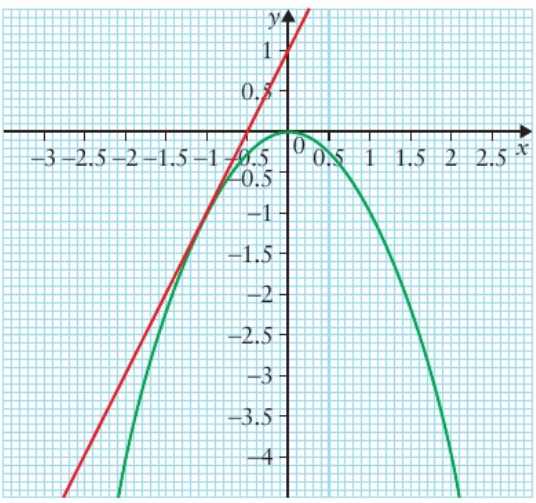
3 **a** two distinct points



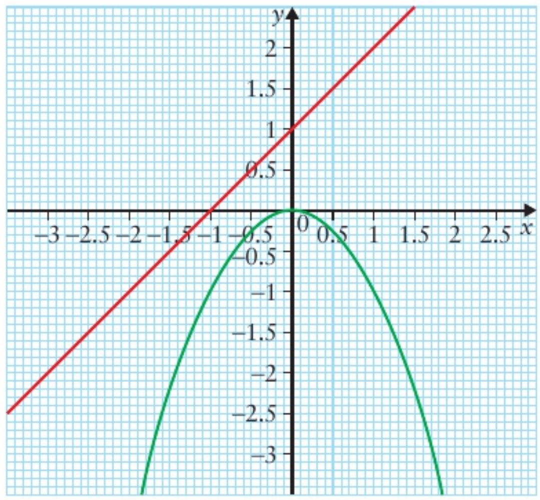
b two distinct points



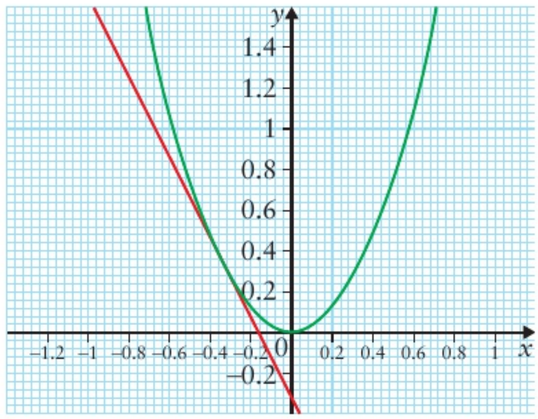
c two coinciding points



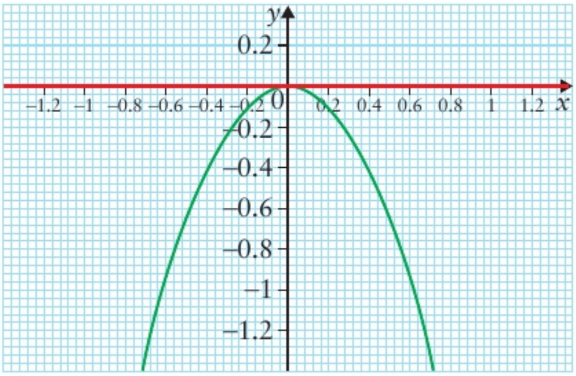
d no points



e two coinciding points



f two coinciding points



4 1.63 cm

Practice questions 7.3

1 a $y = (x + 2)^2$ b $y = x^2 + 1$
c $y = (x - 3)^2$ d $y = x^2 - 4$
e $y = (x + 1)^2$ f $y = x^2 - 2$
g $y = 2x^2 + 3$ h $y = 2(x - 2)^2$
i $y = \frac{1}{2}x^2 + 3$ j $y = 2(x + 1)^2$
k $y = -2(x + 1)^2$ l $y = 3(x - 2)^2 - 1$

2 a reflection in the x-axis
b vertical stretch factor 2
c vertical compression factor 3
d vertical stretch away from the x-axis factor $\frac{1}{2}$ and reflection in the x-axis
e horizontal shift 1 unit to the right
f reflection in the x-axis *then* vertical shift 3 units up
g vertical shift 3 units down
h horizontal shift 2 units to the left
i horizontal shift 2 units to the right *and* vertical shift 3 units up
j horizontal shift 1 units to the left *and* vertical shift 2 units down
k horizontal shift 2 units to the right and vertical stretch away from the x-axis factor 2, *then* vertical shift 2 units down
l horizontal shift 2 units to the right and reflection in the x-axis, *then* vertical shift 4 units up
m horizontal shift 2 units to the left, reflection in the x-axis, and vertical stretch away from the x-axis factor $\frac{1}{3}$, *then* vertical shift 3 units up

3 a (0, 0) b (0, 0) c (0, 0)
d (0, 0) e (1, 0) f (0, 3)
g (0, −3) h (−2, 0) i (2, 3)
j (−1, −2) k (2, −2) l (2, 4)
m (−2, 3)

4 A–d, B–b, C–a, D–e

5 a $y = (x - 1)^2 + 1$ b $y = (x - 3)^2 - 1$
c $y = -(x - 2)^2 - 2$ d $y = -(x + 3)^2 + 1$

Practice questions 7.4.1

1 a $x = 1$ or $x = 2$ b $x = 0$ or $x = -3$
c $x = 1$ or $x = -2$ d $x = -\frac{1}{2}$
e $x = 4$ f $x = 0$
g $x = -1$ or $x = 1$ h $x = \frac{1}{2}$ or $x = -1$
i $x = \frac{1}{6}$ or $x = \frac{10}{3}$

2 a $x = 0$ or $x = \frac{1}{3}$ b $x = -2$ or $x = 2$
c $x = -2$ or $x = -3$ d $x = 2$ or $x = 3$
e $x = -2$ or $x = -5$ f $x = 3$ or $x = 7$
g $x = 5$ o r $x = 7$ h $x = -7$ or $x = 8$
i $x = -5$ or $x = -\frac{2}{3}$

3 a $x^2 - 5x + 6 = 0$ b $x^2 - x = 0$
c $x^2 + x - 2 = 0$ d $x^2 + 5x + 6 = 0$
e $x^2 + 4x + 4 = 0$ f $x^2 - 2x + 1 = 0$

4 a $x = -\frac{1}{2}$ or $x = 1$ b $x = \frac{1}{2}$ or $x = -1$
c $x = -\frac{1}{3}$ or $x = \frac{1}{2}$ d $x = \frac{2}{5}$ or $x = 3$
e $x = \frac{1}{2}$ or $x = 5$ f $x = -1$ or $x = \frac{7}{4}$
g $x = -5$ or $x = -\frac{2}{3}$ h $x = 2$ or $x = \frac{7}{3}$

5 a $(0, 0)$ and $(3, 9)$

b $(1, 1)$ and $(4, 16)$

c $(-3, -9)$ and $(5, -25)$

d $(2, -4)$ and $(1, -1)$

e $(-5, 100)$ and $\left(-\frac{1}{4}, \frac{1}{4}\right)$

f $(5, 100)$ and $\left(-\frac{1}{4}, \frac{1}{4}\right)$

g $(-3, 45)$ and $\left(-\frac{1}{5}, \frac{1}{5}\right)$

h $(2.5, 12.5)$ and $(-2, 8)$

Practice questions 7.4.2

1 a $x^2 + 6x + 9 = (x + 3)^2$

b $x^2 + 8x + 16 = (x + 4)^2$

c $x^2 - 2x + 1 = (x - 1)^2$

d $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

e $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$

f $2x^2 - 8x + 8 = 2(x - 2)^2$

2 a $x = 3 \pm \sqrt{5}$

b $x = -3 \pm 2\sqrt{2}$

c $x = 2$ or $x = -1$

d $x = 1 \pm \frac{\sqrt{5}}{3}$

e $x = -\frac{3}{2} \pm \sqrt{3}$

3 a two b two

c one d one

e no real solutions f no real solutions

4 a $x = -1.24$ or $x = 3.24$

b $x = -12.1$ or $x = 0.0828$

c $x = 0$ or $x = 12$

d $x = -0.414$ or $x = 2.41$

e $x = -5.46$ or $x = 1.46$

f no real solutions

g $x = -0.541$ or $x = 5.54$

h $x = -0.140$ or $x = 7.14$

i $x = -0.414$ or $x = 2.41$

j $x = \frac{1}{2}$ or $x = 1$

5 a $x = -3$ or $x = 5$ b $x = 2$ or $x = 4$

c $x = \frac{1}{3}$ or $x = 2$ d $x = \frac{1}{2}$ or $x = 5$

6 a $x = -2$ or $x = -\frac{2}{3}$

b $x = 1 \pm \sqrt{3}$

c no real solutions

d $x = \frac{3 \pm \sqrt{7}}{2}$

7 a $(0.356, -0.161)$ and $(-2.11, -0.777)$

b $(3.29 ,-2.62)$ and $(0.709, -0.042)$

c $(-2.50, -0.750)$

d $(-2, 12)$ and $(-2, 12)$ (two coinciding intersections)

e no intersections

Practice questions 7.4.3

1 a x-intercepts $(1, 0)$ and $(5, 0)$, y-intercept $(0, 5)$, vertex $(3, -4)$

b x-intercepts $(1, 0)$ and $(-5, 0)$, y-intercept $(0, -10)$, vertex $(-2, -18)$

c x-intercepts $(-3, 0)$ and $(-3, 0)$ (two coinciding intersections), y-intercept $(0, 9)$, vertex $(-3, 0)$

d x-intercepts $(-6, 0)$ and $(0, 0)$, y-intercept $(0, 0)$, vertex $(-3, -9)$

e x-intercepts $(-2, 0)$ and $(0.5, 0)$, y-intercept $(0, -2)$, vertex $(-0.75, -3.125)$

f x-intercepts $(1, 0)$ and $(1, 0)$ (two coinciding intersections), y-intercept $(0, 2)$, vertex $(1, 0)$

g no x intercepts, y-intercept $(0, 1)$, vertex $(-0.25, 0.875)$

h x-intercepts $(0, 0)$ and $(2, 0)$, y-intercept $(0, 0)$, vertex $(1, 1)$

2 A: no x-intercepts, y-intercept not visible, vertex $(-3, 1)$, axis of symmetry $x = -3$

B: no x-intercepts, y-intercept $(0, 3)$, vertex $(1, 2)$, axis of symmetry $x = 1$

C: no x-intercepts, y-intercept not visible, vertex $(-3, -1)$, axis of symmetry $x = -3$

D: *x*-intercepts (0.6, 0) and (3.4, 0), *y*-intercept (0, −2), vertex (2, 2), axis of symmetry $x = 2$

3 **a** (−1, 0), (4, 0)

b (1, 0), (−3, 0)

c $\left(\frac{1}{2}, 0\right), (-1, 0)$

d $\left(\frac{1}{3}, 0\right), \left(\frac{1}{2}, 0\right)$

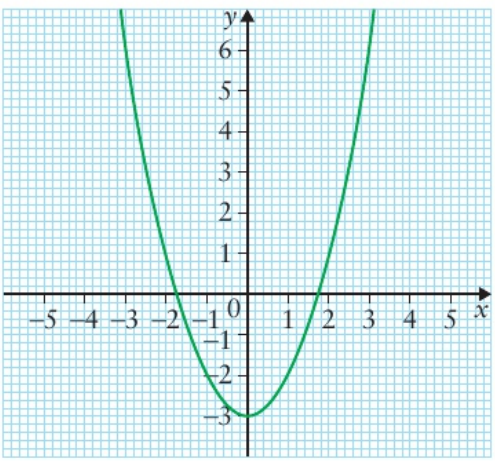
e (−1, 0), (4, 0)

f $\left(-\frac{1}{3}, 0\right), \left(\frac{2}{3}, 0\right)$

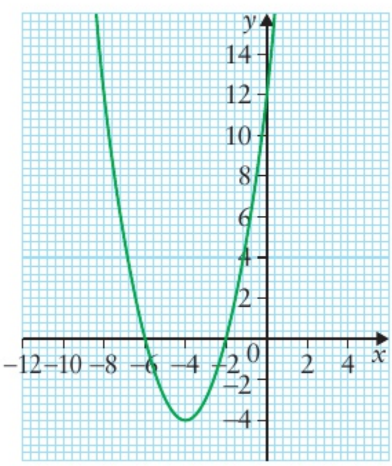
g (1, 0), (1, 0)

h (−1, 0), (1, 0)

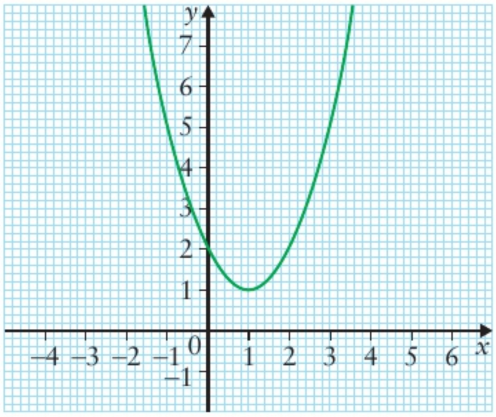
4 **a** *x*-intercepts $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$; *y*-intercept (0, −3); vertex (0, −3)



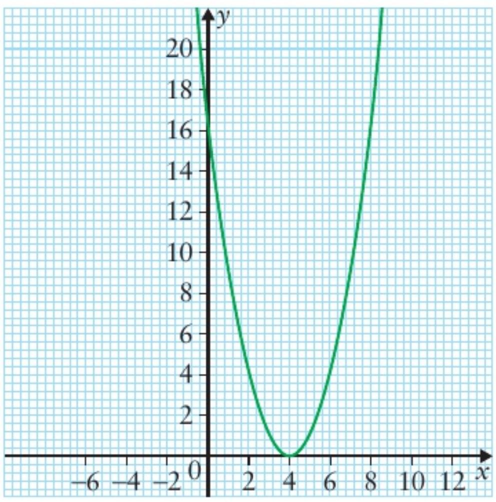
b *x*-intercepts (−6, 0), (−2, 0); *y*-intercept (0, 12); vertex (−4, −4)



c *x*-intercepts none; *y*-intercept (0, 2); vertex (1, 1)



d *x*-intercepts (4, 0), (4, 0); *y*-intercept (0, 16) ; vertex (4, 0)



5 **a** 7 **b** 7 **c** 3

6 **a** −7 **b** 0 **c** 11

7 **a** $y(0) = c = -2$

b $x = -\frac{b}{2a} = -1$

c Substitute values of *b* and *c* into equation gives $y = ax^2 + 2ax - 2$

d $a = 3$

e $y = 3x^2 + 6x - 2$

8 **a** $y = -x^2 + 2x$ **b** $y = x^2 - 6x + 9$

Practice questions 7.4.4

1 **a** two distinct solutions

b no real solutions

c two coinciding solutions

d two coinciding solutions

e no real solutions

f two distinct solutions

2 Students' own answers

3 $m = 1$

4 $m = 2, -2$

5 $m = -\frac{1}{4}$

6 $m = 2, -2$

Check your knowledge questions

1 a i 4 m ii 2 m iii 4 m

b $h(x) = -x(x - 4) = -x^2 + 4x$

c i 1.75 m ii 1 m and 3 m

2 $4\sqrt{2} + 2 \approx 7.7$, length 7.7 cm
$4\sqrt{2} - 2 \approx 3.7$, height 3.7 cm

3 a 18.4 dollars b 20 litres

c 20 dollars d 5 litres

4 a i 75 m ii 60 m iii 35 m

b 80 m c 4 s

5 a x-intercepts at $(-10, 0)$ and $(10, 0)$
so $y = a(x + 10)(x - 10)$
y-intercept at $(0,13)$, so $a = -0.13$

b i $(-6.2, 8)$ and $(6.2, 8)$

ii 12.4 m

6 a i 12.4 m

ii 21.2 m

b 32.4 m

c i $A(17.6, 6.97)$

ii $C(28, 11.1)$

iii 30.1 m

Chapter 8 answers

Do you recall?

1 a $x = \frac{19}{4}$ b $x = \pm 5$

2 a $72\ cm^2$ b $40\ cm^2$

3 Surface Area = $108\ cm^2$; Volume = $72\ cm^3$

4 $a = 60°$ (supplementary angle)
$b = 120°$ (vertical angles are congruent)
$c = 60°$ (vertical angles are congruent)
$d = 120°$ (corresponding angles are congruent)
$e = 60°$ (supplementary angle)
$f = 120°$ (vertical angles are congruent)
$g = 60°$ (vertical angles are congruent)

5 $x = 1, y = -2$

6 $y - 3 = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x + \frac{7}{2}$

Practice questions 8.1

1 a 80° b 60° c Yes

d 3 : 2 e 3 : 2

2 a A and C

b A and C

c A and B; C and D

d A and D

e B and C

3 a W b L c N d M

e P f U g R h Q

i V j T

4 a B: corresponding angles are not congruent

b C: not all sides are congruent

c B: not all sides are congruent

d A: no pair of congruent sides

5 A and C have the same length to width ratio of 3 : 1
B and D have the same length to width ratio of 3 : 2
E and F have the same length to width ratio of 5 : 3

6 a M b P c N d K

e L f Q g R h S

i T j U

7 a P b F c T

d W e L

8 3564 km

9 a $11.7\text{ m} \times 20\text{ m}$ b 234 m^2
c 0.342

Practice questions 8.2

1 a A and C; B and D
b A and D
c B and C

2 a M b P c N d K
e L f Q g R h S
i T j W

3 a B and C by SAS Similarity
b A and B by SAS Similarity
c A and B by SAS Similarity
d A and B by SAS Similarity

4 a $ADE \sim ABC$ by AA Similarity
b $ABC \sim ECD$ by AA Similarity
c $EAB \sim DAC$ by SAS Similarity
d $DCE \sim BCA$ by SSS Similarity
e $AED \sim ABC$ by AA Similarity
f $ACB \sim ECD$ by AA Similarity

Practice questions 8.3

1 a $\sphericalangle A$ is congruent to $\sphericalangle D$ (given in the diagram); $\sphericalangle ABC$ is congruent to $\sphericalangle DBE$ as they are vertically opposite angles; triangle $ABC \sim$ triangle DBE by AA Similarity. $x =$ 4.8

b $\sphericalangle ACB$ is congruent to $\sphericalangle DCE$ as they are vertically opposite angles; $\frac{AC}{DC} = \frac{CB}{CE} = \frac{26}{16}$; triangle $ACB \sim$ triangle DCE by SAS Similarity. $x = 27.1$ (3 s.f.)

c $\frac{GF}{GH} = \frac{FE}{HJ} = \frac{GE}{GJ} = \frac{20}{28} = \frac{25}{35} = \frac{30}{42} = \frac{5}{7}$ triangle $GFE \sim$ triangle GHJ by SSS Similarity. $x = 83°$.

d $\frac{AB}{AE} = \frac{BC}{ED} = \frac{AC}{AD} = \frac{18}{24} = \frac{24}{32} = \frac{30}{40} = \frac{3}{4}$ triangle $ABC \sim$ triangle AED by SSS Similarity. $a = 53°$.

e $\sphericalangle E$ is congruent to $\sphericalangle C$ (given in the diagram); $\sphericalangle EBA$ is congruent to $\sphericalangle CBD$ as they are vertically opposite angles; triangle $EBA \sim$ triangle CBD by AA Similarity. $y = 29.75$

2 a AA Similarity; $e = 3.14$, $f = 1.57$
b AA Similarity; $x = 23.3$
c SAS Similarity; $x = 28.8$
d SAS Similarity; $h = 42°$
e AA Similarity; $a = 150$, $b = 282$
f AA Similarity; $x = 28.3$, $y = 11.7$

3 a $x = 19$
b $x = 21$
c $x = 15$, $y = 24$

4 a i $\sphericalangle BED$ is congruent to $\sphericalangle BAC$ and $\sphericalangle BDE$ is congruent to $\sphericalangle BCE$, as they are corresponding angles on parallel lines; triangle $BED \sim$ triangle BAC.
ii $4:9$
iii $BE = 15\frac{5}{9}\text{ m}$; $EA = 19\frac{4}{9}\text{ m}$
iv $4:5$
v given that $BD/DC = 4:5$; part iv shows $BE/EA = 4:5$

b i $3:5$
ii $3:5$
iii $\sphericalangle EDH = \sphericalangle FDG$ common angle; $DE:DF = DH:DG$; triangle $EDH \sim$ triangle FDG by SAS Similarity
iv Yes, corresponding angles of similar triangles are congruent.
v Yes, corresponding angles of similar triangles are congruent.
vi Yes, corresponding angles are congruent.
vii You have shown that if a line divides *two sides* of a triangle in the *same ratio* it must be *parallel* to the third side.

5 a $a = 18.6$ m

b $b = 20.4$ m

c $c = 25.2$ m

6 a Since triangle $ABC \sim$ triangle LMN, $\measuredangle ACB$ is congruent to $\measuredangle LNM$ and $\frac{AC}{LN} = \frac{BC}{MN}$. Since AD is a median of triangle ABC, $BD = DC$, and since LP is a median of triangle LMN, $MP = PN$. Since D is between B and C, $BC = BD + DC$, and since P is between M and N, $MN = MP + PN$. Using substitution, we get $BC = 2DC$ and $MN = 2PN$. Again, using substitution in $\frac{AC}{LN} = \frac{BC}{MN}$, we get $\frac{AC}{LN} = \frac{2DC}{2PN}$, which simplifies to $\frac{AC}{LN} = \frac{DC}{PN}$. Therefore, triangle $ADC \sim$ triangle LPN by SAS Similarity. Hence, $\frac{AD}{LP} = \frac{AC}{LN}$.

b Since triangle $ABC \sim$ triangle LMN, $\measuredangle ACB$ is congruent to $\measuredangle LNM$. Since AE is an altitude of triangle ABC and LQ is an altitude of triangle LMN, $\measuredangle AEC$ is congruent to $\measuredangle LQN$ as all right angles are congruent. Hence, triangle $AEC \sim$ triangle LQN by AA Similarity. Therefore, $\frac{AE}{LQ} = \frac{AC}{LN}$.

7 5.25 m

8 138.6 m

9 a It is given that $AD \parallel CE$ and $\measuredangle DAC$ is congruent to $\measuredangle DAB$. Hence, $\measuredangle DAC$ is congruent to $\measuredangle ECA$, as they are alternate interior angles, and $\measuredangle BAD$ is congruent to $\measuredangle BEC$ as they are corresponding angles. Hence, using substitution, $\measuredangle BEC$ is congruent to $\measuredangle ECA$, and triangle EAC is isosceles. Therefore, AE is congruent to AC.

b Since $AD \parallel EC$, $\measuredangle BAD$ is congruent to $\measuredangle BEC$ and $\measuredangle BDA$ is congruent to $\measuredangle BCE$. Then, by AA Similarity, triangle $BEC \sim$ triangle BAD. Hence, $\frac{BE}{BA} = \frac{BC}{BD}$. Since A is between B and E, $BE = BA + AE$, and since D is between B and C, $BC = BD + DC$. Substituting in, we get:

$$\frac{BA + AE}{BA} = \frac{BD + DC}{BD}$$

$$1 + \frac{AE}{BA} = 1 + \frac{DC}{BD}$$

$$\frac{AE}{BA} = \frac{DC}{BD}$$

$$\frac{BA}{AE} = \frac{BD}{DC}.$$

c From part b, we know $\frac{BA}{AE} = \frac{BD}{DC}$, and from part a, we know that triangle EAC is an isosceles triangle. Therefore, $AE = AC$, and using substitution, we get $\frac{BA}{AC} = \frac{BD}{DC}$.

Practice questions 8.4

Section 8.4 is enrichment material at Standard level. Please access the answers for this section via the link on this page of your eBook.

Check your knowledge questions

1 a Yes, AA Similarity

b No, no angles known or 3rd side

c Yes, AA Similarity

2 a No, sides not in equal ratios

b No, sides not in equal ratios

c No, sides not in equal ratios

3 a Yes, SAS Similarity

b No, corresponding sides not in equal ratios

c Yes, SAS Similarity

4 a $a = 21$

b $a = 25$

c $a = 8.8$

5 $x = 19.5, y = 12.49$

6 **a** $x = 18$

b $x = 26.1$

c $a = 50$

7 **a** $AB \parallel ED$ and $BC \parallel AE$ (given); $\sphericalangle EDA$ is congruent to $\sphericalangle BAC$ (alternate interior angles are congruent); $\sphericalangle BCA$ is congruent to $\sphericalangle EAD$ (alternate interior angles are congruent): triangle $ABC \sim$ triangle DEA by AA Similarity.

b $\dfrac{AB}{BD} = \dfrac{BD}{DC} = \dfrac{AD}{BC}$

$\dfrac{18}{12} = \dfrac{12}{8} = \dfrac{15}{10}$

$\dfrac{3}{2} = \dfrac{3}{2} = \dfrac{3}{2}$

Triangle $ABD \sim$ triangle BDC by SSS Similarity

8 17.6 cm × 22 cm

9 $x = 5\frac{1}{3}, y = 6\frac{2}{3}$

10 $x = 26.1$

11 $h = 8.281\text{m}$

Chapter 9 answers

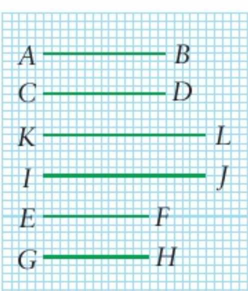
Do you recall?

1 $A\hat{B}E = 42.51°$, $C\hat{D}E = 34.78°$, $A\hat{E}B = 102.71°$, $C\hat{E}D = 102.71°$

2 **a** (5, 5) **b** $4\sqrt{5} \approx 8.94$

Practice questions 9.1

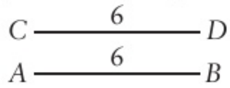
1



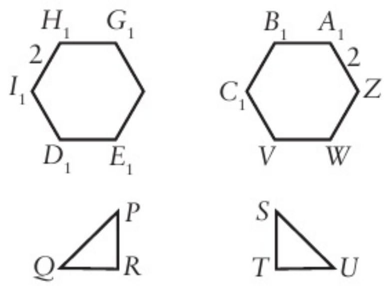
$AB \cong CD$, $KL \cong IJ$ EF and GH are not congruent with any line

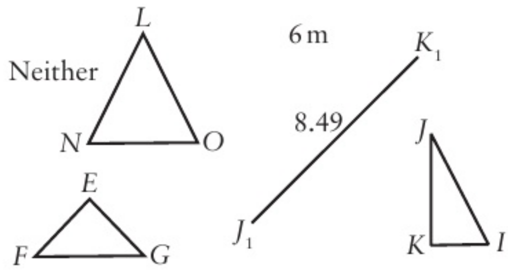
2 Equal quantities: 300 cm and 3 m; 60 mm and 6 cm

Congruent line segments:



Congruent shapes:





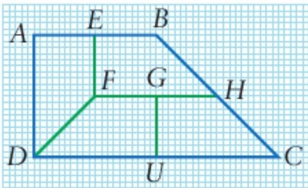
3 Students' own answers. All answers will be different here, all side lengths should be equal, all angles should be equal.

4 Students' own answers. All answers will be different here, corresponding side lengths should be the same length, corresponding angles should be the same size.

5 **a** Students' answers will vary, depending on the shapes chosen from the detailed photo.

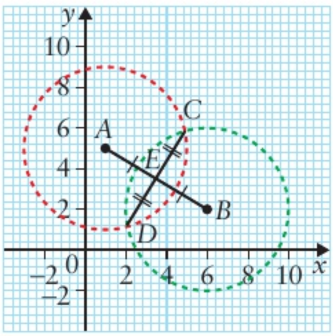
b Students' answers will vary, depending on the shapes chosen from the detailed photo.

6 The shape can be divided into 4 congruent trapezia regions



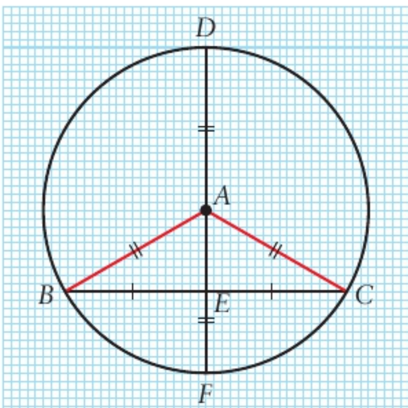
Practice questions 9.2

1

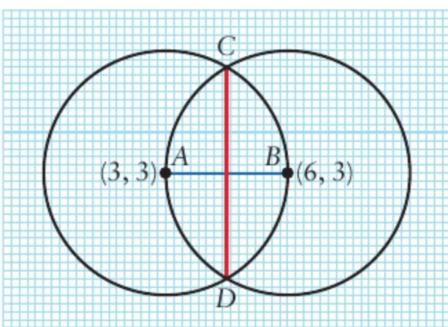


$AE \cong EB$, and $CE \cong ED$

2

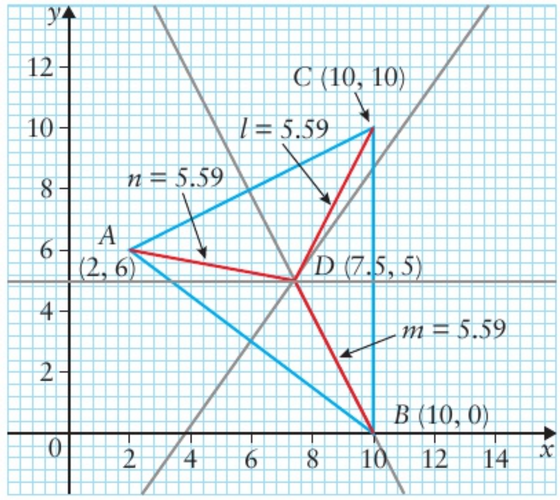


3



$|CD| = 5.2\,\text{cm}$

4



a The perpendicular bisectors are drawn in grey in the diagram above.

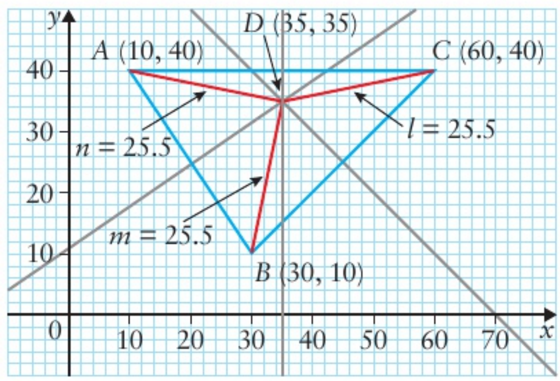
b D is (7.5, 5)

c The distance from D to the three vertices is 5.59

d The lines AD, DC and DB are all equal lengths, hence $AD \cong DC \cong DB$

5 Students' own answers. All answers will be different here, but they should match the description in the question.

6



The point of intersection of the three perpendicular bisectors marks the point where the swimming pool would be an equal distance from each of the three sites. This is at the point (35, 35). This would be a good position for the swimming pool as it is an equal distance from all three sites.

Practice questions 9.3.1

1 a Given that $AB \cong DF$, $AC \cong DE$, and $BC \cong EF$ then $\triangle ABC \cong \triangle DFE$

b Given that $GH \cong JK$, $GI \cong JL$, and $H\hat{G}I \cong K\hat{J}L$ then $\triangle HGI \cong \triangle KJL$

c Given that $N\hat{M}O \cong R\hat{P}Q$ and $M\hat{O}N \cong P\hat{R}Q$, $MO \cong PR$, then $\triangle MNO \cong \triangle PQR$

d Given that $C\hat{A}B \cong E\hat{D}F$ and are 90°, and $CB \cong EF$, $AB \cong DF$ then $\triangle CAB \cong \triangle EDF$

2 Given that $AE \cong EB$, $AD \cong BC$, and $D\hat{A}E \cong E\hat{B}C$ then $\triangle AED \cong \triangle BEC$

3 As $\triangle ABC$ is equilateral, then $AB \cong BC \cong AC$. As point D is the circumcentre, then $DA \cong DB \cong DC$. Therefore, $\triangle ABD \cong \triangle ADC \cong \triangle BDC$.

4 a As $EFGHI$ is a regular pentagon, then $I\hat{H}G \cong G\hat{F}E$ and $IH \cong HG \cong GF \cong FE$, therefore $\triangle IHG \cong \triangle GFE$.

b No, $\triangle IGE \ncong \triangle GFE$ because $I\hat{G}E \ncong G\hat{F}E$, and $IG \ncong GF$.

5 a As AB and CD are parallel lines, then $A\hat{B}C \cong B\hat{C}D$,and $B\hat{A}D \cong A\hat{D}C$, therefore $AE \cong EB$ and $EC \cong ED$. Hence as $AC \cong BD$, $AE \cong BE$, and $EC \cong ED$ then $\triangle AEC \cong \triangle BED$.

b $A\hat{E}C = 67.38°$

Practice questions 9.3.2

1 As the octagon is regular, then all sides are of equal lengths and hence all the perimeter lines are congruent, and all the internal angles are equal, hence $DCBA \cong EFGH \cong BAHG \cong CDEF$. This means that $AD \cong HE \cong CF \cong BG$, given that $AH \cong DE \cong BC \cong GF$ then $AHED \cong BGFC$.

2 a $B\hat{E}D = 58.2°$ b $A\hat{C}E = 40.11°$

3 a $E(1, 2)$ b $A\hat{D}C = 135°$

4 a If an additional line segment is drawn drawing point B to A then two triangles are formed. As $BE \cong BC$ and $AE \cong AC$ and BA is common to both triangles, then $\triangle EBA \cong \triangle BAC$ and hence $B\hat{E}A \cong B\hat{C}A$.

b $B\hat{E}A = 23.29°$

5 Drawing the line segment AC generates two triangles $\triangle BAC$ and $\triangle DAC$. As $AB \cong AD$ and $BC \cong DC$ and AC is a common side in both triangles, then $\triangle BAC \cong \triangle DAC$. Hence $B\hat{A}C \cong D\hat{A}C$, $B\hat{C}A \cong D\hat{C}A$, and $A\hat{B}C \cong A\hat{D}C$.
therefore $A\hat{B}C = A\hat{D}C$
$A\hat{B}C = 180 - \left(\frac{141.4}{2} + \frac{38.6}{2}\right)$
$A\hat{B}C = 90°$
$A\hat{B}C = A\hat{D}C = 90°$

6 Students' own answers. All answers will be different here, but they should identify that it is always the case.

7 18.9

8 Solution is given via the online links in the question.

Practice questions 9.3.3

1 $|EF| = 3.5$ cm (1 d.p.)

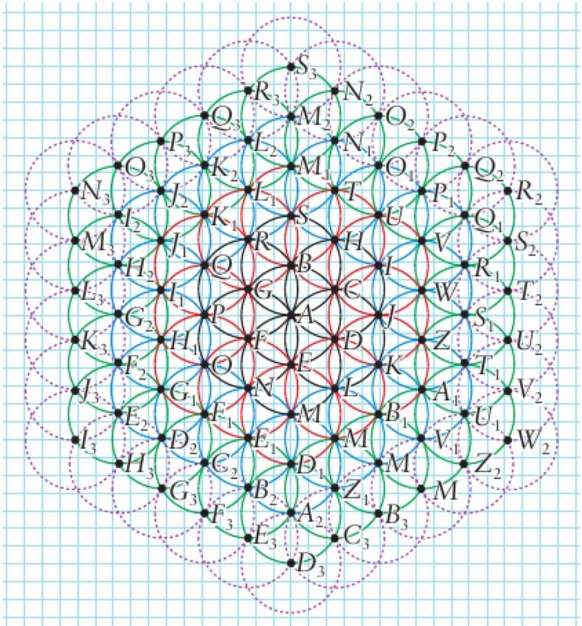
2 $|ED| = |EC| = 3.5$ cm

3 16.6 cm

4 a, b, c

Pattern number	1	2	3	4	5	n
Number of circles in the outer ring	1	6	12	18	24	$6(n-1)$

d Expect to see $6 \times 5 = 30$ in the outer ring of pattern 6. This is confirmed by extending the pattern:



5

Pattern number	1	2	3	4	5	n
Total number of circles	1	7	19	37	61	$3n^2 - 3n + 1$

The pattern is a quadratic sequence $T_n = 3n^2 - 3n + 1$

Check your knowledge questions

1 $AB \cong CD$, $EF \cong GH$, $OP \cong IJ$, $QR \cong KL$, $ST \cong NM$

2 16 small colour squares, 16 small right-angled colour triangles, 16 large right-angled colour triangles. There is no distinction between the different colours as the congruence relates to abstract shapes and their size.

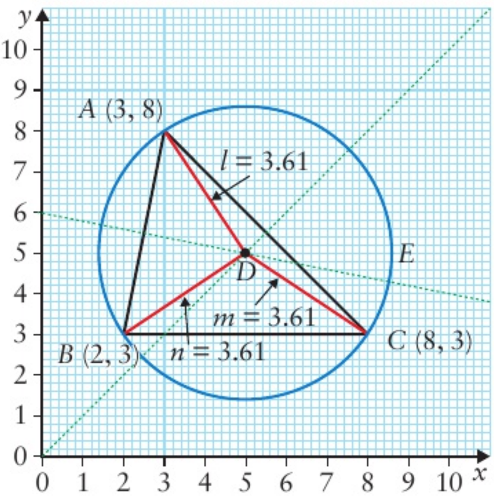
3 As DE is the perpendicular bisector of AB then $BF \cong FC$, the lines AB and AC are radii of the circle so $AB \cong AC$, and as AF is common to both triangles, then all three sides of the triangles are congruent, $\Delta ABF \cong \Delta ACF$.

4 a As $AB \cong CD$, $B\hat{A}E \cong E\hat{C}D$ and $A\hat{E}B \cong C\hat{E}D$ (intersecting lines) then $\Delta AEB \cong \Delta CED$

b $AE = 2.235$ c $C\hat{E}D = 126.86°$

5 a b Circumcentre point D (5,5)

c d $AD = DC = DB = 3.61$



6 a 3 b 5 c 2

d $\sqrt{13} \approx 3.61$ e 53.13°

7 $r = 3$

8 a $FG = 3.5$ b $DF = 6.76$

9 a 10 b 14 c $4n + 2$ d $4n - 6$

Chapter 10 answers

Do you recall?

1 1

2 $\frac{1}{4}$

3 $\frac{1}{3}$

4 $96\ \text{cm}^2$

5 $h = 7$ units; Area = 84 units^2

Practice questions 10.1.1

1 a i 20.2 ii $(-10.5, 6.5)$

b i 5.4 ii $(-8, -7.5)$

c i 27.3 ii $(3, 6.5)$

d i 7.1 ii $(7.5, 5.5)$

e i 14.8 ii $(2.5, -9.5)$

f i 16.8 ii $(-4, -11.5)$

g i 30 ii $(-1, -6)$

2 a $B = (9, -4)$ b $B = (-8, -18)$
c $B = (18, -13)$ d $A = (9, -6)$
e $A = (1, -9)$ f $A = (-11, -22)$

3 a 11.2 km
b 12.3 km
c 8.4 km
d 8.2 km
e 12.1 km

4 13.7 units

5 a The fourth vertex is 2 left and 6 down from (8, 3) which is (6, −3).
b Perimeter: 31.6
Area: 60.0
c (2.5, 1.5)

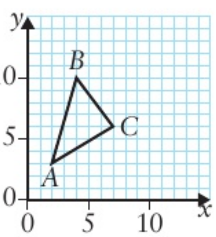
6 $AB = \sqrt{80}$, $AC = \sqrt{145}$, $BC = \sqrt{145}$
Two sides are the same length so the triangle is isosceles.

7 a If S is the midpoint S is $(-1, 3)$
If G is the midpoint S is $(14, 0)$
If C is the midpoint S is $(-16, 6)$
b The distance between S and G is $\sqrt{416}$ when $S = (-16, 6)$ and C is the midpoint.

8 a $A\ (4, -2, 0)$ $B\ (0, -2, 0)$
$C\ (0, 0, 0)$ $D\ (4, 0, 0)$
$Q\ (0, -2, 8)$ $R\ (0, 0, 8)$
$S\ (4, 0, 8)$
b $CP = \sqrt{84} = 9.17$
c $DQ = \sqrt{84} = 9.17$
d $AS = \sqrt{68} = 8.25$

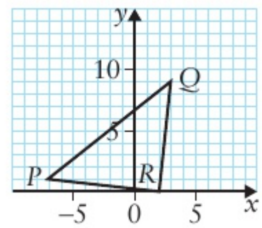
Practice questions 10.1.2

1 a



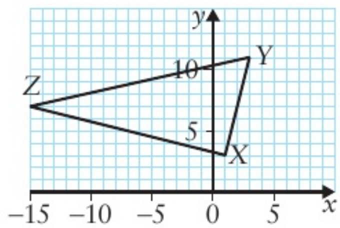
Scalene, not right-angled

b



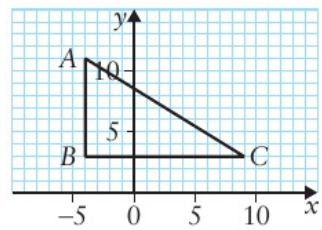
Isosceles, right-angled

c



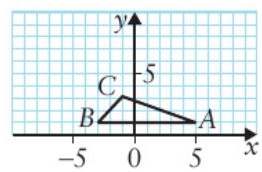
Scalene, right-angled

2 a



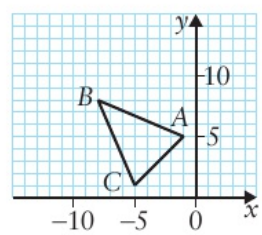
Area = 52
Perimeter = 36.26

b



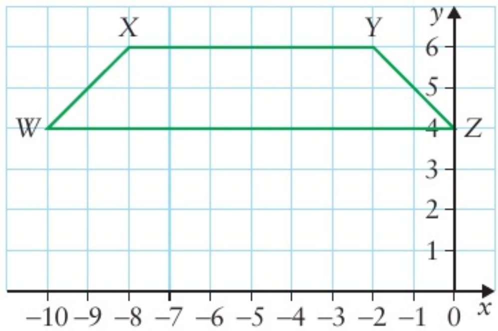
Area = 8
Perimeter = 17.15

c

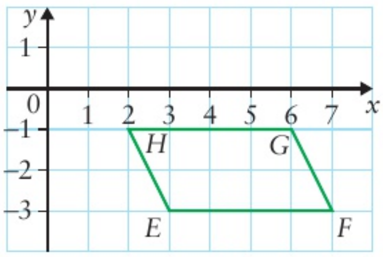


Area = 20
Perimeter = 20.89

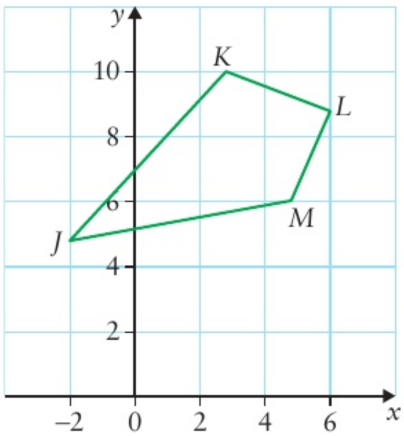
3 **a** Isosceles trapezium



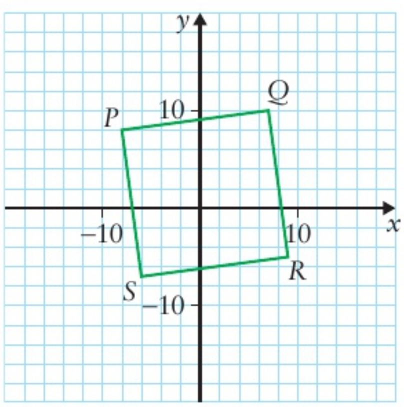
b parallelogram



c kite

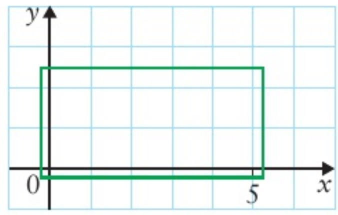


d square



4 **a** (5.5, 2.5), (5.5, −0.5), (−0.5, −0.5), (−0.5, 2.5)

b



c The midpoints form a rectangle.

5 **a** To be an isosceles trapezium, there must be one pair of parallel sides and the two non-parallel sides must have the same length.

$AB = CD = \sqrt{20} = 4.47$. These two sides have the same length.

Slope of AD is -1 Slope of BC is -1

Therefore AD and BC are parallel.

Therefore it is an isosceles trapezium.

b midpoint of AB is (2, 7), midpoint of BC is (6, 6), midpoint of CD is (7, 2), midpoint of DA is (3, 3).

c The quadrilateral is a rhombus as all four sides are the same length ($\sqrt{17} = 4.12$)

6 **a** One plane is $\sqrt{34} = 5.83$ km and the other is $\sqrt{89} = 9.43$ km

b $\sqrt{173} = 13.15$ km

c scalene triangle

Practice questions 10.2

1 (2, 0)

2 (−3, −6)

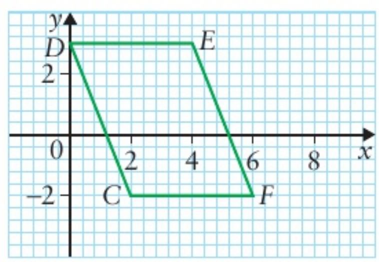
3 (−4, 3)

4 (6, 0)

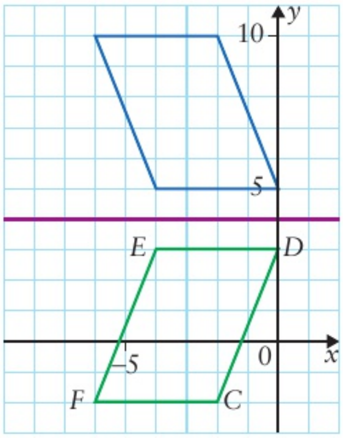
5 (−3, 10)

6 (−8, 2)

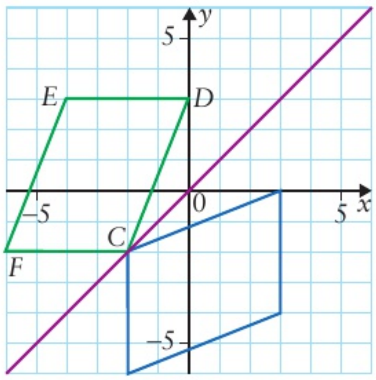
7 **a**



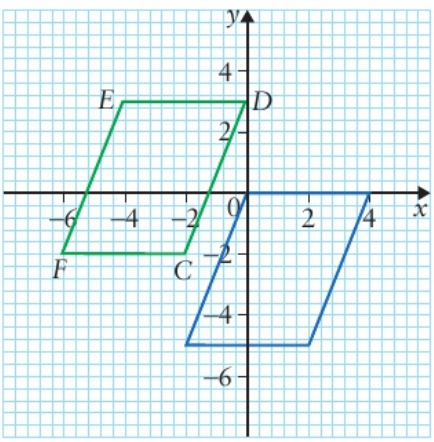
b



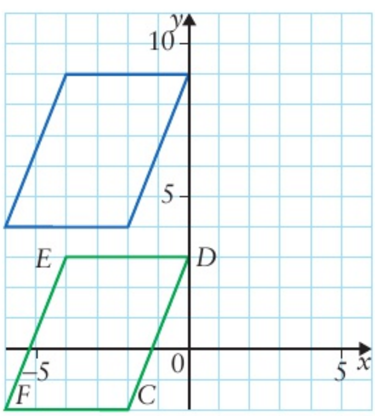
c



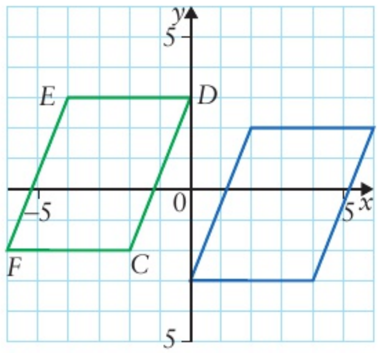
d



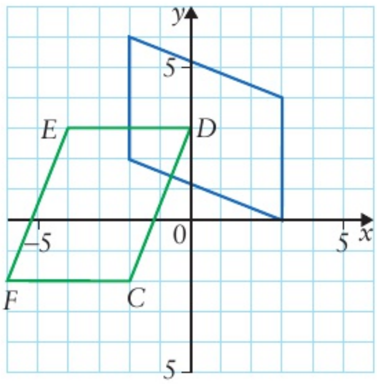
e



f



g



8 A to B: reflection in the line $x = 0$.
A to C: reflection in the line $y = -x$.

9 **a** F to E: translate left 5 units and up 1 unit
F to B: translate down 4 units
F to A: reflection in the x-axis ($y = 0$)
F to C: 180° about (0, 0)
F to D: first rotate 90° about (0, 0) and then translate 3 units left

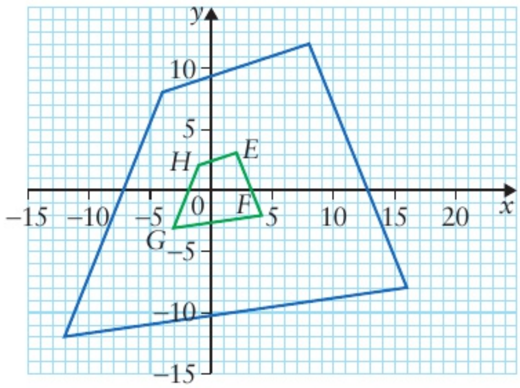
b A to B reflect in $y = -2$
A to F reflect in the x-axis ($y = 0$)
A to C reflect in the y-axis ($x = 0$)

10 Rotation: for example F to C. Both are right-angled triangles. The legs of each triangle are 2. Therefore congruent by SAS.
Reflection: for example F to A. Both are right-angled triangles. The legs of each triangle are 2. Therefore congruent by SAS.
Translation: for example F to E. Both are right-angled triangles. The legs of each triangle are 2. Therefore congruent by SAS.

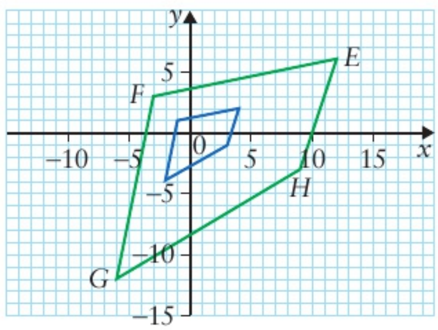
11 **a** A' (9, 1), B' (1, −1), C' (4, −2)
b midpoint of AB is (1, 6), midpoint of AC is (2.5, 5.5), midpoint of BC is (−1.5, 4.5)

c image of midpoint of AB is $(5, 0)$, image of midpoint of AC is $(6.5, -0.5)$, image of midpoint of BC is $(2.5, -1.5)$
Midpoint of $A'B'$ is $(5, 0)$, midpoint of $A'C'$ is $(6.5, -0.5)$, midpoint of $B'C'$ is $(2.5, -1.5)$. They are the same.

12 a



b



13 $x = 6.25$
$y = 14$

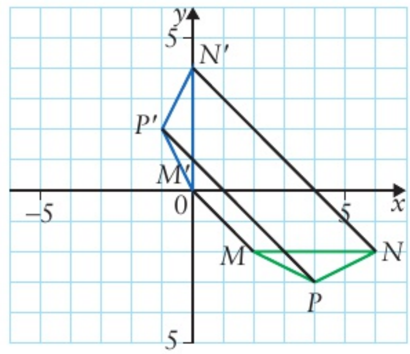
14 a (red axis is the x-axis, green axis is the y-axis, blue axis is the z-axis)
$A\ (0, 0, 0)$, $B\ (0, -2, 0)$, $C\ (2, -2, 0)$, $D\ (2, 0, 0)$
$E\ (0, 0, 2)$, $F(0, -2, 2)$, $G\ (2, -2, 2)$, $H\ (2, 0, 2)$

b $A'\ (0, 0, 0)$, $B'\ (0, -6, 0)$, $C'\ (6, -6, 0)$, $D'\ (6, 0, 0)$
$E'\ (0, 0, 6)$, $F'\ (0, -6, 6)$, $G'\ (6, -6, 6)$, $H'\ (6, 0, 6)$

c Preimage: Total surface area = 24 units^2.
Image: Total surface area = 216 units^2.
The surface area of the image is 9 times larger.

d The volume of the preimage is 8 units^3.
The volume of the image is 216 units^3.
The volume of the image is 27 times larger.

15 a



b Slope of $MM' = -1$, slope of $NN' = -1$, slope of $PP' = -1$

c The perpendicular slope is 1.

d The midpoint of MM' is $(1, -1)$, NN' is $(3, 1)$, PP' is $(1.5, -0.5)$

e $y = x - 2$

16 Find the midpoint of the line segment joining a vertex to its image. The midpoints will all be the same as this is the centre of rotation. The midpoint of XX' is $(1, 0)$, so the centre of rotation is $(1, 0)$. (Note that this method works for a 180° rotation, but for other angles of rotation the coincidence of the perpendicular bisectors of the linking line segments is required).

Practice questions 10.3

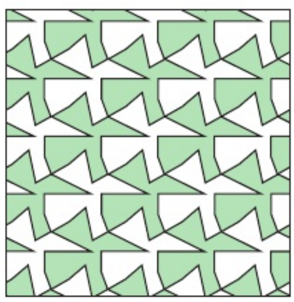
1 a translation

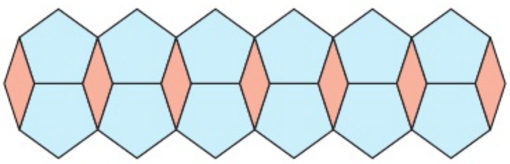
b rotation

c rotation

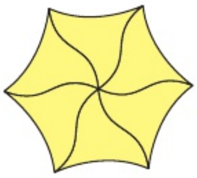
d reflection and translation

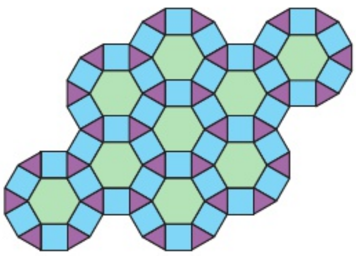
2



3 

4 

5 

6 

Check your knowledge questions

1 $AB = \sqrt{(3-1)^2 + (8-4)^2} = \sqrt{4+16} = \sqrt{20}$
$GH = \sqrt{(12-8)^2 + (1-3)^2} = \sqrt{16+4} = \sqrt{20}$
Yes, $AB = GH$.

2 $EF = \sqrt{(5-1)^2 + (2-1)^2} = \sqrt{16+1} = \sqrt{17}$
$CD = \sqrt{(9-6)^2 + (5-2)^2} = \sqrt{9+9} = \sqrt{18}$
No, $AB \neq CD$.

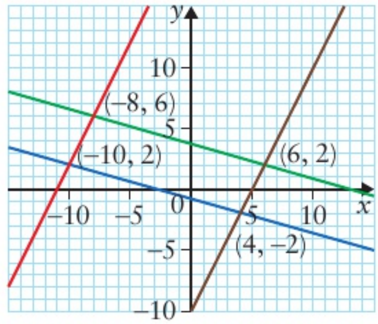
3 Midpoint of $XY = \left(\frac{7+(-3)}{2}, \frac{-4+(-2)}{2}\right) =$ $(2, -3)$
Midpoint of $HJ = \left(\frac{5+(-1)}{2}, \frac{7+(-13)}{2}\right) =$ $(2, -3)$
Yes, they have the same midpoint.

4 $(31, -4)$

5 a The paths intersect at $(0.5, 0.75)$.
b Total Distance: 10 units, which is 1000 m.

6 a $(3, 1)$, $(-1, -11)$, $(5, -13)$ or $(9, -1)$
b any point on circles with A or B as centre of radius AB to form isosceles triangles

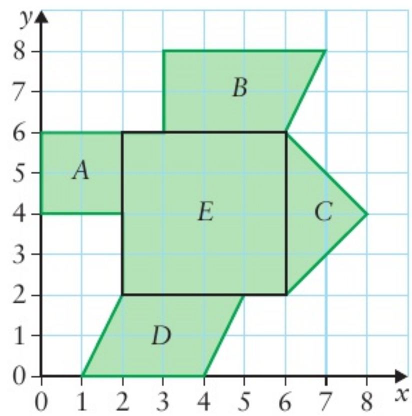
7 a $(-12, 2)$ or $(-4, -10)$ or $(6, 8)$
b any point on the line $y = x + 2$

8 

Parallelogram

9 This shape can be reconfigured as a parallelogram with a base of $CB = \sqrt{(9-0)^2 + (3-(-3))^2} = \sqrt{81+36} = \sqrt{117}$ and a height of $CE = \sqrt{(-4-0)^2 + (3-(-3))^2} = \sqrt{16+36} = \sqrt{52}$, so the area $= \sqrt{117} \times \sqrt{52} = \sqrt{6084} = 78$

10 The garden can be subdivided into different polygons:



Perimeter: $19 + \sqrt{5} + \sqrt{8} + \sqrt{8} + \sqrt{5} + \sqrt{5} \approx$ 31.365 units
Perimeter = 31.365 units which is 62.73 metres
A: Square with area of 4
B: Trapezoid with area of $\frac{1}{2} \times 4(6+8) = 28$
C: Triangle with area of $\frac{1}{2} \times 8 \times 4 = 16$
D: Parallelogram with area of $8 \times 4 = 32$
E: Square with area of 16
$A + B + C + D + E = 39$ square units, with each square unit being $4\,\text{m}^2$

Total Area = $156\,\text{m}^2$

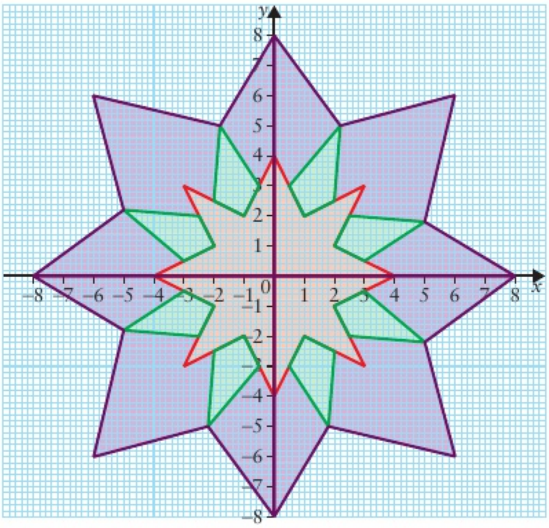
Total Volume with depth $0.2\,\text{m} = 31.2\,\text{m}^3$

The topsoil will cost €1014.00, and the fencing will cost €219.56.

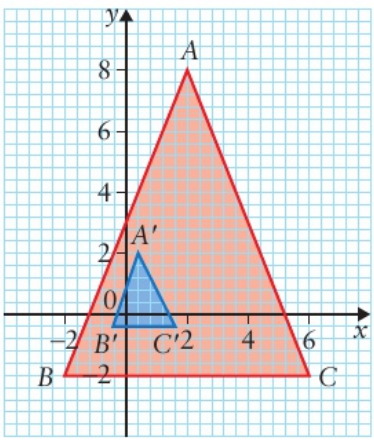
11 The coordinates of the new triangle are A' $(4, -3)$, B' $(-11, 1)$, and C' $(-6, -2)$.

12 a Kite

b



13

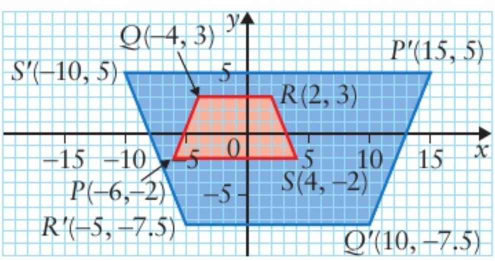


Area of pre-image = 40 units^2

Area of image = 2.5 units^2

$\frac{2.5}{40} = \frac{1}{16} = \frac{1}{4^2}$

14 a



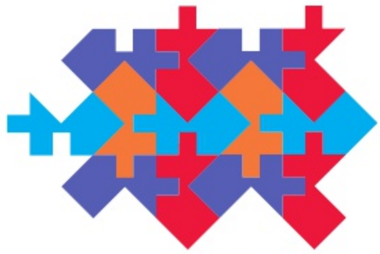
b Sides: $\frac{10}{25} = \frac{6}{15} = \frac{2}{5}$

Area: $\frac{5(6 + 10)}{2} = 40$ and

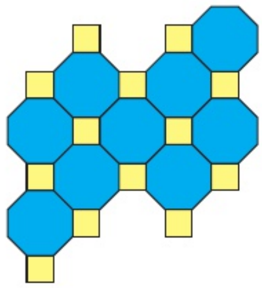
$\frac{12.5(15 + 25)}{2} = 250$;

ratio of areas: $\frac{40}{250} = \frac{4}{25}$

15



16



Chapter 11 answers

Do you recall?

1 a 16π **b** $144 - 36\pi$

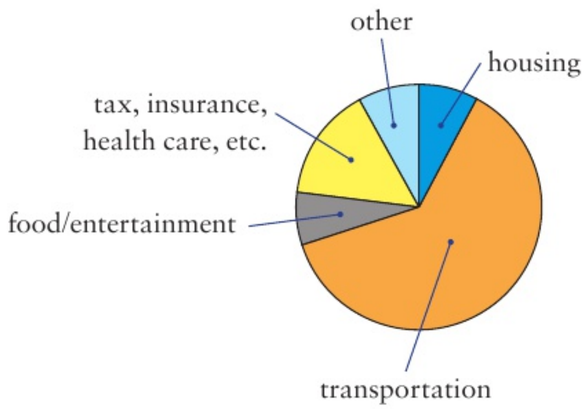
2 Area $= 3.74 + 0.605\pi\,\text{m}^2$

perimeter $= 5.6 + 1.1\pi\,\text{m}$

3 a $3.6 + 0.8\pi\,\text{km}$ **b** $73\,\text{km}$

4 $\frac{27}{25}\pi\,\text{m}^2$ and $\frac{9}{25}\pi\,\text{m}^2$

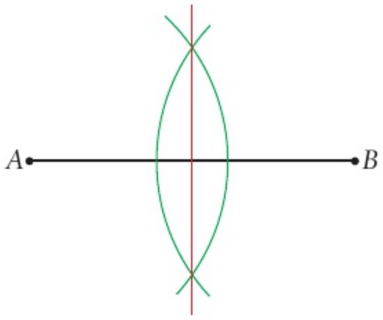
5



6 a $x = 62°$ **b** $y = 124°$ **c** $z = 56°$

7 $n = 50°$

8



Practice questions 11.1

1 a $\measuredangle COA$ b $\measuredangle AOD$
c $\measuredangle BDC$ d $\measuredangle ACD$

2 a i $\measuredangle AYX$
ii $\measuredangle BXY$
b i $\measuredangle XAY$ and $\measuredangle XBY$
ii $\measuredangle AXB$ and $\measuredangle AYB$

3 a i $\measuredangle BEC$ and $\measuredangle BAC$
ii $\measuredangle BDC$ and $\measuredangle BFC$
b i $\measuredangle BAC$ and $\measuredangle BEC$
ii $\measuredangle BDC$ and $\measuredangle BFC$

Practice questions 11.2

1 a 44° b 25° c 34°
d 39° e 39° f 39°
g 96° h 118° i 150°

2 a $a = b = 33°$ b $d = c = 62°$
c $e = f = 62°$ d $e = 39°$
e $i = h = 58°$ f $k = 98°$
g $m = n = 90°$ h $p = 180°$
i $t = 90°$

3 a i 30° ii 60° iii 35°
iv 70° v 65° vi 130°
b i 30° ii 60° iii 35°
iv 70° v 65° vi 130°
vii Yes

4 a 120° b 125°

5 a $d = 33°$, complementary angles
b $f = 18°$, complementary angles
c $g = 104°$, supplementary angles
d $i = 102°$, $h = 89°$, supplementary angles
e $a = 53°$, $b = 127°$, angle at circumference is half angle at centre.
f $c = 70°$, $d° = 110$, angle at circumference is half angle at centre and supplementary angles
g $m = 50°$, $n = 40°$, exterior angle in a triangle is the sum of remote interior angles
h $m = 46°$, $n = 44°$, exterior angle in a triangle is the sum of remote interior angles
i $a = 110°$, $b = 70°$, opposite angles of cyclic quadrilateral are supplementary, angles on a straight line
j $e = 85°$, $f = 95°$, opposite angles of cyclic quadrilateral are supplementary, angles on a straight line

6 i size of $\measuredangle ACB = 36°$ since $\measuredangle ACB \cong \measuredangle ADB$ and $\measuredangle ACB \cong \measuredangle OBC$
ii size of $\measuredangle AOB = 72°$ and thus, size of $\measuredangle ACB = 36°$ and $\measuredangle ACB \cong \measuredangle OBC$

7 46°

8 Connect O to A, C, D and E.
$\measuredangle COA \cong \measuredangle DOE$ since they are angles at the centre subtended by arcs of equal size angles at B and F have sizes half the central angles and thus will have equal measures.

9 35°

10 72°

11 64°

12 An arc of a circle centre O and radius $\frac{l}{2}$.

13 a angles subtended by the same arc.
b $\measuredangle ABD = \gamma$, $\measuredangle BDC = \beta$, $\measuredangle BDA = \delta$
c $\alpha + \beta = 90°$, Angle in a semi-circle.
Similarly, $\delta + \gamma = 90°$ and thus,
$\alpha + \beta + \delta + \gamma = 180°$

d Since $\measuredangle DBC = \alpha$, $\measuredangle ABD = \gamma$, $\measuredangle BDC = \beta$, and $\measuredangle BDA = \delta$, the result follows.

14 $\measuredangle A$ and $\measuredangle BEF$ are supplementary, and $\measuredangle BEF \cong \measuredangle C$, and thus $\measuredangle A$ and $\measuredangle C$ are supplementary.
Therefore $AF \parallel DC$ because interior angles on one side of the transversal AC are supplementary.

Practice questions 11.3

1 **a** **i** Perpendicular bisector of chord AB.

ii Perpendicular bisector of chord CD.

iii P, because it is equidistant from A, B, C, and D.

b Choose any 4 points on the circle and draw 2 chords (make sure the chords are not parallel), then construct the perpendicular bisector for each chord. The centre is where the perpendicular bisectors intersect.

c See Worked example 11.6.

2 **a** 6 cm **b** $4\sqrt{7}$ m **c** 10 mm

3 **a** $2\sqrt{7}$ cm **b** $\sqrt{41}$ cm

4 **a** $\triangle POC \cong \triangle QOC$ by SSS theorem

b $\measuredangle POC \cong \measuredangle QOC$, corresponding angles in congruent triangles.

c $\triangle PON \cong \triangle QON$ by SAS theorem

d $\overline{NP} \cong \overline{NQ}$, corresponding sides in congruent triangles. Thus ON is the perpendicular bisector of segment/chord QP

5 **a** 120° **b** 90° **c** 60°

d 51.4° **e** 45° **f** 36°

6 **a** With radius equal to the radius of the circle mark 6 points on the circle, then join every other point with a segment.

b Draw a diameter. Then construct the perpendicular bisector of that diameter and join the points of intersection of the circle with the diameter and perpendicular bisector.

c Follow same instructions as part a, but join every 2 adjacent points.

d Follow same instructions as part b, and then bisect each angle at the centre and join the points of intersection with the circle.

Practice questions 11.4

1 **a** $c = 37°$

b $d = e = 57°$

c $k = m = 90°$, $n = 132°$

d $p = 8.2$ cm, $q = 64°$

2 **a** $x = 39°$ **b** $y = 120°$

c $a = 48°$ **d** $e = f = 105°$

3 **a** $a = 90°$, $b = c = 25°$, $d = 65°$

b $e = f = 57°$, $g = 123°$

c $g = h = 35°$, $k = 110°$

d $k = 90°$, $m = 65°$

e $n = 90°$, $r = 91°$, $v = 62°$

f $y = 112°$, $x = w = 34°$, $z = 56°$

g $c = 90°$, $d = \sqrt{51}$ m, $e = \sqrt{89}$ m

h $f = 8$ cm

i $t = 5$ m

4 $x = 128°$, $y = 52°$, $z = 26°$

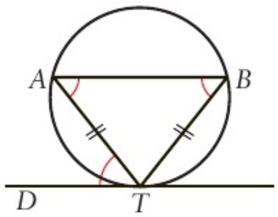
5 Connect BT

a $\measuredangle BOT = 2\measuredangle$A, but, $\measuredangle BTP \cong \measuredangle$A, and result follows.

b $\measuredangle ATQ \cong \measuredangle ABT$, Thus,
$\measuredangle ATQ + \measuredangle RBA + \measuredangle PBT = \measuredangle ABT + \measuredangle RBA + \measuredangle PBT = 180°$ since this is a straight angle.

c Quadrilateral $OTPB$ is cyclic because the opposite angles at T and B are right angles
$\measuredangle BPT = 180° - \measuredangle \text{TOB} = 180° - 2\measuredangle BAT$

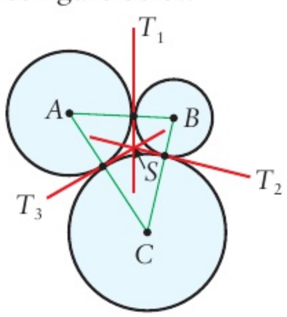
6 Angles at A and B are congruent since they are subtended by equal chords.



$\sphericalangle DTA \cong \sphericalangle B$, and thus $\sphericalangle DTA \cong \sphericalangle A$ and so the lines AB and DT are parallel because the alternate interior angles are congruent.

7 Take the bisector of angle P. This bisector is the locus of points equidistant from PQ and PR. Similarly, the bisector of angle R, is the locus of points equidistant from PR and QR. So, where they intersect is equidistant from all sides. Let the point of intersection be O, and take the distance of O from say PR as a radius and draw a circle. Since O is equidistant from all sides, then it is tangent to all sides.

8 See figure below



If we consider the intersection between tangents T_1 and T_2. Knowing that two tangents from the same point outside the circle to the circle are congruent, then this point is equidistant from AB and BC. Thus, it is on the bisector of angle B. Similarly, the intersection between T_1 and T_3 is on the bisector of angle A. Similarly, the inter section between T_2 and T_3 is on the bisector of angle C. But the angle bisectors are concurrent (Q7 above), and so, these points are the same.

9 a $a = 105°$, $m = 25°$, $n = 50°$
b 50°
c 210°

Practice questions 11.5

1 a $b = 18$ b $c = 12$ c $d = 16$
d $e = 6$ e $f = 1$ f $h = 4$
g $r = 12$ h $t = 12$
i $a = 15, b = \dfrac{32}{3}$
j $c = 15, d = \dfrac{40}{3}$ k $x = 4\sqrt{2}$

2 a $PT = 6\,\text{m}$ b $AB = 6.9\,\text{cm}$
c $CD = \dfrac{59}{5}\,\text{m}$ d $EG = \dfrac{75}{2}\,\text{m}$
e $CD = \dfrac{511}{15}\,\text{m}$ f $CD = \dfrac{56}{5}\,\text{cm}$
g $PT = 13.4\,\text{cm}$

3 $CE = 3\sqrt{3}\,\text{m}$

4 8 cm

5 $AB = 4\sqrt{2}\,\text{cm}$

6 a $\sphericalangle PTS = 90°$ as STN are collinear. Since $OS \parallel PT$ $\sphericalangle OSN = 90°$ (complementary angles)
b $\sphericalangle OPT = 90°$ too, so $OPTS$ is a rectangle.
c $PT \approx 10.2\,\text{cm}$

7 $AB = 8\,\text{cm}$

Check your knowledge questions

1 a $n = 50°$ b $p = 20°$ c $x = 115°$
d $k = 220°$ e $a = 68°, b = 22°, x = 68°$
f $a = 40°, b = 100°, x = 50°$

2 a $e = 45°$, $\sphericalangle THG = 85°$ and $\sphericalangle FHG = 40°$
b $BE = 4\,\text{cm}$, $OE = 3\,\text{cm}$. $x \times x = 8 \times 2$
c $x = 95°$. $\sphericalangle BAT = \sphericalangle BAC + \sphericalangle CAT$

3 $\sphericalangle QTR = 70°$, $\sphericalangle RQT = 70°$, $\sphericalangle QRT = 40°$. Opposite angles of a parallelogram, exterior angle to a cyclic quadrilateral, sum of angles in a triangle.

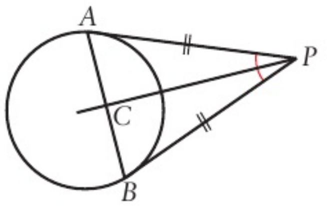
4 Draw perpendicular from O to $ABCD$ to meet it at M.
In the large circle: $AM = DM$
In the small circle: $BM = CM$
Subtract the two equations and the result will follow.

5 90°

6 a $y = 4$. Intercepts of chords products.
b $f = 19$. Intercepts of secants products.
c $r = 5$ cm. Tangent and Intercepts of secants.

7 a $a = 31°, b = 59°$ b $c = 142°, d = 109°$
c $f = 53°, e = 65°$ d $x = \sqrt{85}$ m
e $y = \frac{36}{5}$ m
f $r = 6.89$ m, $t = 6.6$ m
g $v = 45°, w = 35°, x = 100°$
h $a = 0.9$ m, $b = 110°$
i $c = d = 65°, e = 50°$
j $f = 45°, g = 90°, i = j = 45°$
k $k = 152°$ l $n = 41°, m = 68°$
m $x = 4\sqrt{2}$ n $m = 154°, n = 77°$

8 a i $WP^2 = PB \times PA = PT^2$
ii $AB = 11.2$ cm
b $\triangle PAB$ is isosceles and hence the angle bisector of the vertex angle is the perpendicular bisector of the base AB



Now, AB is a chord and thus, the perpendicular bisector passes through the centre.
c 48° and 132°
d $OP = 12$ cm

Chapter 12 answers

Do you recall?

1 12 cm
2 $\frac{1}{6}, \frac{6}{7}$
3 $\frac{3\sqrt{7}}{7}, \frac{\sqrt{2}}{2}, 2\sqrt{3}$
4 a 4 b 8
5 The definition of similarity says that two polygons are similar if corresponding angles are congruent and corresponding sides are proportional. For triangles there are special shortcuts:
(AA) two triangles are similar if two pairs of corresponding angles are congruent.
(SAS) two triangles are similar if two pairs of sides are in proportion and the pair of corresponding included angles is congruent.
(SSS) two triangles are similar if all the three pairs of corresponding sides are in proportion.
6 180°
7 … their measures add to 90°
8 1, 3, 7 and 5
4, 8, 2 and 6

Practice questions 12.1

1	a	c, A	b	e, D	c	h, G	d	k, J
2	a	c	b	f	c	g	d	j
3	a	a	b	d	c	i	d	l
4	a	b	b	e	c	h	d	k

Practice questions 12.2

1 a $\tan(A) = \frac{5}{4}, \tan(C) = \frac{4}{5}$
b Side $BC = 5\sqrt{7}$, $\tan(A) = \frac{\sqrt{7}}{\sqrt{2}}, \tan(C) = \frac{\sqrt{2}}{\sqrt{7}}$
c Side $BC = \sqrt{2}$, $\tan(A) = \frac{\sqrt{2}}{\sqrt{15}}, \tan(C) = \frac{\sqrt{15}}{\sqrt{2}}$
d Side $BC = 2\sqrt{11}$, $\tan(A) = \frac{2\sqrt{11}}{\sqrt{5}}$,
$\tan(C) = \frac{\sqrt{5}}{2\sqrt{11}}$

e Angle A and angle C add to 90°. They are complementary angles.

f $\tan A$ is the reciprocal of $\tan C$.

$\tan A = \dfrac{1}{\tan C}$

2 a $\tan 60° = \dfrac{x}{y}$ b $\tan 35° = \dfrac{x}{h}$

3 a $x = 22.9$

b $x = 8.14$

4 a i $x = 3.91$, ii $\theta = 52°$, iii Hypotenuse ≈ 6.35

b i $x = 12.87$, ii $\theta = 43°$, iii Hypotenuse ≈ 17.60

c i $x = 3.48$ ii $\theta = 75°$, iii Hypotenuse ≈ 13.46

d i $x = 3.40$, ii $\theta = 67°$, iii Hypotenuse ≈ 8.69

5 a 0.268 b 1.072 c 2.246

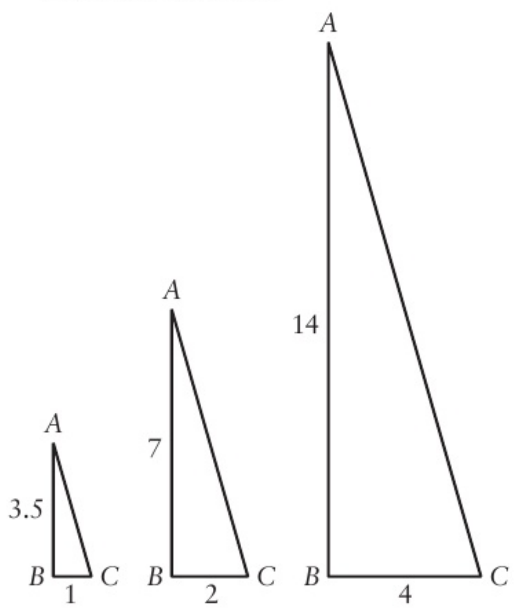
d 57.290 e 477.464

f 5 729 577.951

g The tangent increases very quickly.

h There is no limit as it is undefined.

6 a Draw three similar triangles, for example as the ones shown



b The triangles are all similar since the ratios of their sides are equal.

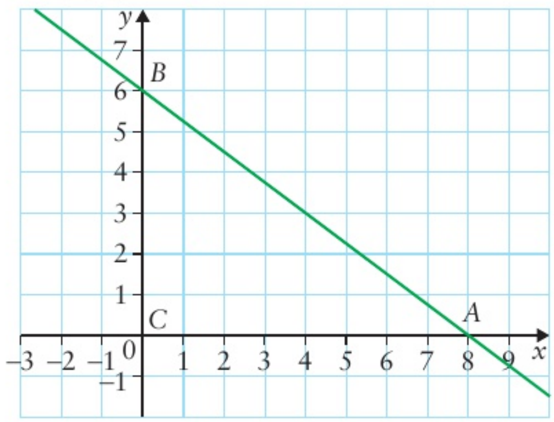
7 a $A = 20.56°$

b $A = 64.98°$

c $A = 2.58°$

d $A = 72.31°$

8 a, b The line $y = -0.75x + 6$ creates a triangle with the x- and y-axis ABC.
The x-intercept is $A(8,0)$ and y-intercept is $B(0, 6)$.



c Then $\tan A = \dfrac{6}{8}$, and using a calculator we have that $A = 36.87°$. The acute angle is 36.87° and the angle with the positive x-axis is therefore 143.13°.

9 a The legs run along the axes, forming a right-angle at the origin, and take their lengths from the coordinates. The absolute values are used as it is unknown whether m and b are positive or negative

b $\tan B = \dfrac{|b|}{\left|\dfrac{-b}{m}\right|} = |m|$

where B is taken to be the acute angle, measured positively.

c The tangent of the angle made by a line with the positive x-axis is equal to the gradient of the line. This holds true, even when the angle is obtuse, or measured clockwise (as negative).

10 a The angle, $\alpha°$, that line $y = 0.5x + 4$ makes with the positive axis is given by $\tan \alpha° = 0.5 \Rightarrow 26.57$. The angle, β, that line $y = 0.25x + 8$ makes with the positive x-axis is given by $\tan \beta° = 0.5 \Rightarrow \beta \Rightarrow 14.04$

b The angle between the two lines is the difference of the two angles: 12.53°

c Yes, because the resulting angles are the same.

d False. e.g. take $A = 30°$ and $B = 45°$. $\tan(A+B) = 2+\sqrt{3} = 3.732$, $\tan 30 + \tan 45 = \frac{1}{\sqrt{3}} + 1 = 1.577$

11 a $\frac{360°}{n}$ b $90° - \frac{180°}{n}$

c $\frac{x\tan\left(90° - \frac{180°}{n}\right)}{2}$

d $\frac{nx^2\tan\left(90° - \frac{180°}{n}\right)}{4}$

e $36\ \text{cm}^2$

f $43.01\ \text{cm}^2$

Practice questions 12.3

1 a $\sin(A) = \frac{8}{17}$, $\cos(A) = \frac{15}{17}$, $\sin(C) = \frac{15}{17}$, $\cos(C) = \frac{8}{17}$

b $\sin(A) = \frac{7}{25}$, $\cos(A) = \frac{24}{25}$, $\sin(C) = \frac{24}{25}$, $\cos(C) = \frac{7}{25}$

c $\sin(A) = \frac{1}{\sqrt{5}}$, $\cos(A) = \frac{2}{\sqrt{5}}$, $\sin(C) = \frac{2}{\sqrt{5}}$, $\cos(C) = \frac{1}{\sqrt{5}}$

2 a 0.61153 b 0.99985

c 0.99998 d 0.70711

3 $\sin(57°) = \frac{x}{y}$, $\cos(32°) = \frac{x}{y}$, $\tan(16°) = \frac{y}{x}$

4 a $\theta = 54°$, $x = 6.180$

b $\theta = 48°$, $x = 16.148$

c $\theta = 85°$, $x = 1.137$

d $\theta = 61°$, $x = 3.878$

5 a $x = 10.1$ b $x = 12.8$

c $x = 5.6$ d $x = 3.2$

6 a $\theta = 57.796°$ b $\theta = 42.205°$

c $\theta = 42.826°$ d $\theta = 32.635°$

7 28°

8 a 36.9° b 3 m

9 a 4604 m

b actual length 4618 m so estimate is close

10 11.45 cm

11 69.08°

12 $20.18\ \text{cm}^2$

13 16.96 cm

14 33.75°

15 a 2.121

b 0.832

c Student's own answers

16 a 130.2 b 191.7 c 16 278

17 70.53°

Practice questions 12.4

1 a $x = y = \sqrt{2}$

b $x = 4\sqrt{3}$ cm, $y = 8\sqrt{3}$ cm

c $x = \frac{21}{10}\sqrt{3}$ m, $y = \frac{21}{10}$ m

d $x = \sqrt{15}$ cm, $y = 2\sqrt{5}$ cm

2 a $a = c = \frac{\sqrt{6}}{2}$ b $e = 6, f = 3\sqrt{3}$

c $g = \frac{3}{2}\ i = \frac{3\sqrt{3}}{2}$ d $j = \frac{4\sqrt{3}}{3}, k = \frac{8\sqrt{3}}{3}$

3 $\frac{5\sqrt{2}}{2}$ cm

4 $6\sqrt{3}$ cm

5 a 60° b Equilateral

c $\frac{5\sqrt{3}}{2}$ cm d $\frac{75\sqrt{3}}{2}\ \text{cm}^2$

Practice questions 12.5

1 23.6 m

2 99.7 m

3 1400 m

4 3.84 m

5 50.3°

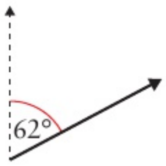
6 2072 m

7 The height of the shortest skyscraper is 63.8 m. The highest skyscraper is 73 m tall.

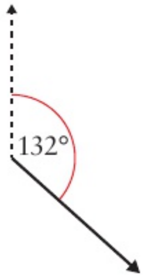
8 1440 m

Practice questions 12.6

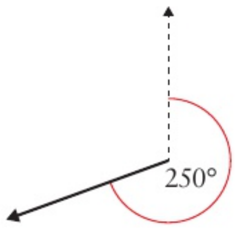
1 a



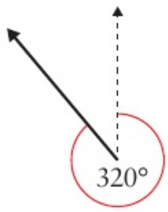
b



c



d



2 a 047° b 120° c 323°
d 059° e 236° f 212°

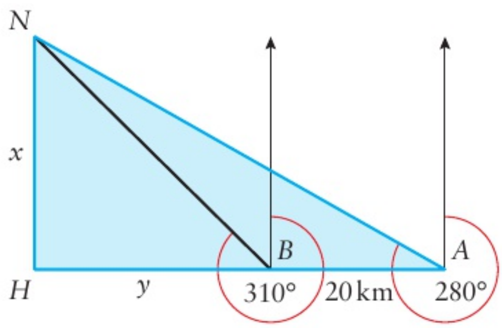
3 a 220° b 040° c 270°
d 090° e 299° f 119°

4 47.3 km

5 109.5 km

6 039°

7 a



b $x = 4.465$ km c 6.946 km

8 a 28.23 km b 211°

Check your knowledge questions

1 a $4\sqrt{6}$, $\sin(A) = \frac{5}{7} = 0.714$,

$\sin(C) = \frac{2\sqrt{6}}{7} = 0.700$,

$\cos(A) = \frac{2\sqrt{6}}{7} = 0.700$, $\cos(C) = \frac{5}{7} = 0.714$,

$\tan(A) = \frac{5}{2\sqrt{6}} = 1.021$,

$\tan(C) = \frac{2\sqrt{6}}{5} = 0.978$

b $\sqrt{97}$, $\sin(A) = \frac{4}{\sqrt{97}} = 0.406$,

$\sin(C) = \frac{9}{\sqrt{97}} = 0.914$,

$\cos(A) = \frac{9}{\sqrt{97}} = 0.914$,

$\cos(C) = \frac{4}{\sqrt{97}} = 0.406$,

$\tan(A) = \frac{4}{9} = 0.444$, $\tan(C) = \frac{9}{4} = 2.250$

c 9.086, $\sin(A) = \frac{9.086}{11} = 0.826$,

$\sin(C) = \frac{6.2}{11} = 0.564$, $\cos(A) = \frac{6.2}{11} = 0.564$,

$\cos(C) = \frac{9.086}{11} = 0.826$,

$\tan a = \frac{9.086}{6.2} = 1.466$,

$\tan c = \frac{6.2}{9.086} = 0.682$

d $2\sqrt{2}$, $\sin(A) = \frac{1}{\sqrt{2}} = 0.707$,

$\sin(C) = \frac{1}{\sqrt{2}} = 0.707$, $\cos(A) = \frac{1}{\sqrt{2}} = 0.707$,

$\cos(C) = \frac{1}{\sqrt{2}} = 0.707$, $\tan(A) = 1$, $\tan(C) = 1$

2 a 6.36 b 2.59 c 18.32 d 21.12

3 a 18.2° b 66.6° c 23.8° d 124.8°

4 a 230° b 050° c 270°
d 90° e 298° f 118°

5 11.1 m

6 247.4 cm

7 Two possible bearings: 135° or 315°

8 9 and $9\sqrt{3}$

9 1950 m

10 1730 m

11 a 179.9 km b 92.7 km c 202.3 km

Chapter 13 answers

Do you recall?

1 A vertical line test: if every possible vertical line crosses the graph at most once, then the relation is either many-to-one or one-to-one, and so it represents a function.

2 a 8

b $x = \pm 2$

c i vertical shift by three units *up*

ii horizontal shift by three units *to the right*

iii reflection in the y-axis (the graph does not change as it was symmetric about the y-axis)

iv vertical *stretch* by a factor of three

v horizontal *compression* by a factor of three

Practice questions 13.1.1

1 a A mapping diagram for which it never occurs that two arrows reach the same point in the range. Since the question asks for the mapping diagram of a *function*, no two arrows can leave the same point in the domain.

b A mapping diagram for which at least one point in the range is reached by at least two arrows. Since the question asks for the mapping diagram of a *function*, no two arrows can leave the same point in the domain.

2 a either *a* or *b*

b *b* (otherwise the diagram does not represent a function)

c either *b* and *c* or *b* and *d*

d none

Practice questions 13.1.2

1 a has an inverse

x	-1	-2	-3	-4
$f^{-1}(x)$	0	1	2	3

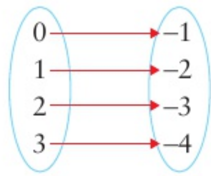
b does not have an inverse, because $f(1) = f(3)$ and $f(2) = f(4)$.

c has an inverse, and $f^{-1}(x) = f(x)$

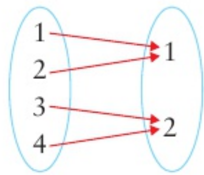
x	0	2	4	6
$f^{-1}(x)$	0	2	4	6

d does not have an inverse, because $f(-2) = f(2)$.

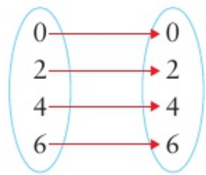
2 a



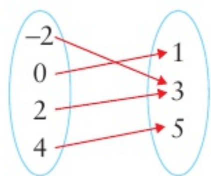
b



c



d

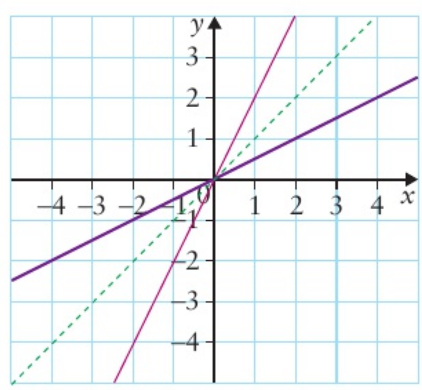


b a and c are one-to-one because each point in the range is reached by only one arrow. b and d are many-to-one because there are points in the range reached by more than one arrow.

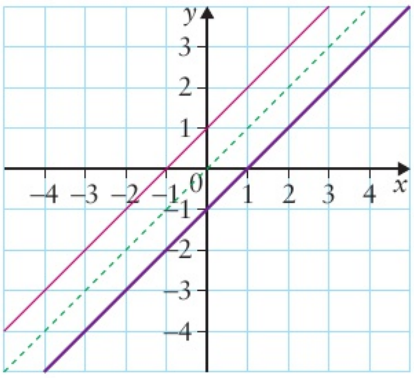
3 Have an inverse: a, b, e, h, i

Do not have an inverse: c, d, f, g, j, k

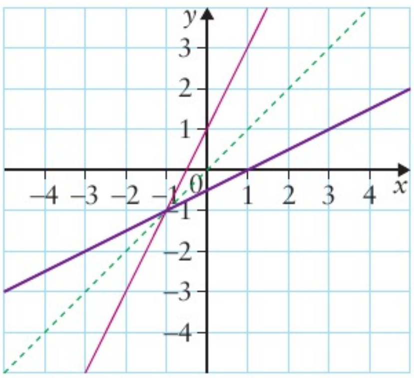
4 a



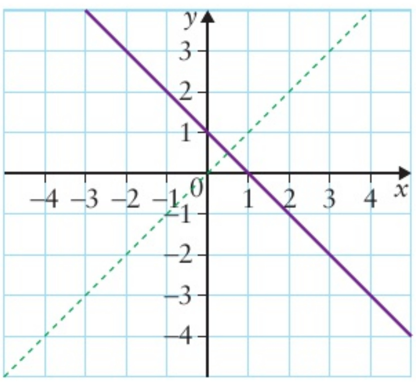
b



c



d



5 **a** $f(x) = 2x, f^{-1}(x) = \frac{1}{2}x$

b $f(x) = x + 1, f^{-1}(x) = x - 1$

c $f(x) = 2x + 1, f^{-1}(x) = \frac{1}{2}(x - 1)$

d $f(x) = f^{-1}(x) = 1 - x$

Practice questions 13.1.3

1 **a** $f^{-1}(x) = \frac{1}{3}(x - 1)$

b $f^{-1}(x) = 2x - 1$

c $f^{-1}(x) = \sqrt[3]{x - 1}$

d $f^{-1}(x) = -x - \frac{2}{3}$

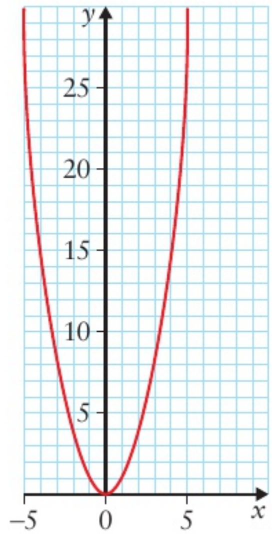
e $f^{-1}(x) = x$

f $f^{-1}(x) = \frac{1}{x} - 1$

g $f^{-1}(x) = \frac{1}{x}$

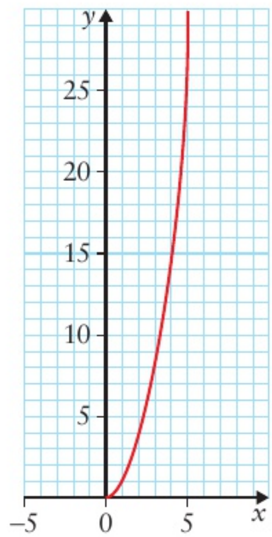
h $f^{-1}(x) = \frac{x - 1}{x + 1}$

2 **a** **i**



ii No, because the graph does not pass the horizontal line test.

b **i**



ii Yes, because the graph now passes the horizontal line test.

iii $g^{-1}(x) = \sqrt{x}$.

c For instance, on the interval from −90 to 90 the graph of the function passes the horizontal line test. Other choices are possible, e.g. from 90 to 270, etc.

3 Possible solutions are:

a $(-90, 90)$

b $[0, +\infty)$

c $[1.5, +\infty)$

4 a $p(V) = \frac{k}{V}, V(p) = \frac{k}{p}$

b $k = 12$

c $V(4) = 3$

5 a No, since it does not pass the horizontal line test. There are two different values for the run-up that result in each value of the performance. For instance, a run-up of either 10 or 50 metres both result in a performance of about 6 metres. When the run-up exceeds 30 metres, it is too long and performance in the long jump decreases!

b It would make sense to focus on the region $r < 30$, because it leads to the maximum performance and not away from it.

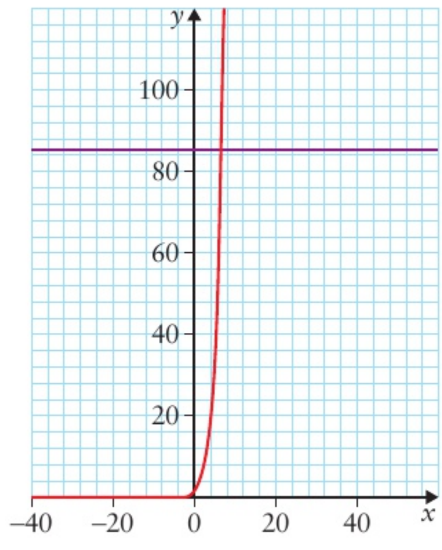
c $L(10)$ = performance with a run-up of 10 metres. $L(10) \approx 6$ metres.
$r(5)$ = run up needed to jump 5 metres. There are two possible values, about 7.5 or 52.5 metres, which is why the function $L(r)$ does not have an inverse over the whole interval $0 < r < 60$.
$L(0)$ = performance with no run-up.
$L(0) \approx 2.5$ metres.
$r(10)$ = run up needed to jump 10 metres. It is impossible for this athlete to jump this long, so there is no such run-up value!
$r(0)$ = run up needed to jump 0 metres. It is impossible for this athlete to jump this short, so there is no such run-up value!

6 a No, because this function is many-to-one (many students having one history teacher).

b This relation is not even a function, because of one word having possibly many translations (synonyms in the target language). It is not invertible since there are synonyms in the English language too, i.e. different points in the domain that are sent to the same point in the range.

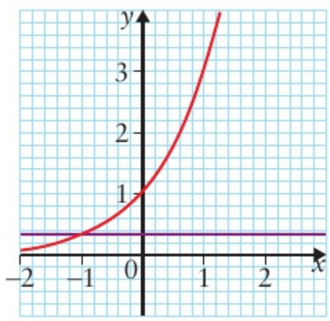
Practice questions 13.2.2

1 a



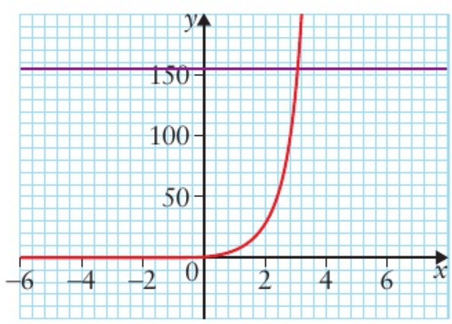
$x \approx 6.44$

b



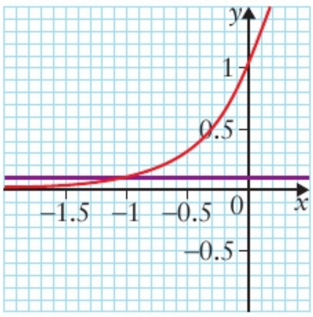
$x \approx -0.982$

c



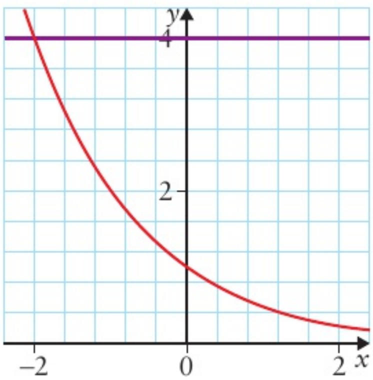
$x \approx 3.14$

d



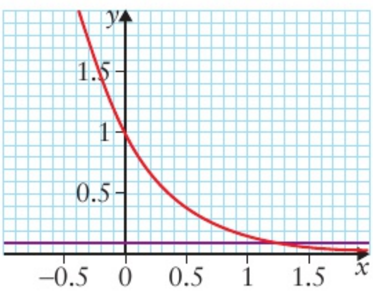
$x \approx -1.05$

e



$x = -2$

f

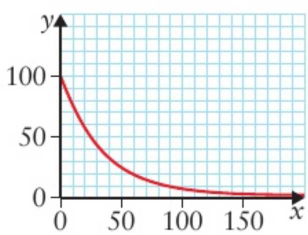


$x \approx 1.18$

2 a i 25 grams

ii $Th(t) = 100 \times \left(\frac{1}{2}\right)^{\frac{t}{24}}$, so after 100 seconds there are $100 \times \left(\frac{1}{2}\right)^{\frac{100}{24}} \approx$ 5.57 grams left

b



c i 72 seconds

ii 79.7 seconds

iii Never

3 a i $\frac{8}{100}$

ii $\frac{32}{100}$

iii 80.6%

b 16.9 days

c exactly three more days, so after 19.9 days

4 a i 1.77 cm

ii 0.0276 cm

iii 0.000153 cm

b i $H(t) = 12 + \frac{W(t)}{20} = 12 + 0.25 \times 2^{-1.5t}$

ii $T(t) = H(t) + W(t) = 12 + 5.25 \times 2^{-1.5t}$

Practice questions 13.3

1 a $4 = \log_2 16$ b $2 = \log_5 25$

c $3 = \log_3 27$ d $1 = \log_5 5$

e $-1 = \log_2 \frac{1}{2}$ f $-2 = \log_3 \frac{1}{9}$

g $0 = \log_7 1$ h $\frac{1}{2} = \log_{25} 5$

i $\frac{1}{2} = \log_9 3$ j $\frac{4}{3} = \log_{27} 81$

k $-\frac{1}{2} = \log_{100} 0.1$ l $2 = \log_{\sqrt{3}} 3$

2 a $2^3 = 8$ b $10^2 = 100$

c $7^{-2} = \frac{1}{49}$ d $6^1 = 6$

e $6^0 = 1$ f $16^{\frac{1}{2}} = 4$

g $27^{\frac{1}{3}} = 3$ h $27^{-\frac{1}{3}} = \frac{1}{3}$

i $2^{-3} = \frac{1}{8}$ j $3^{\frac{1}{2}} = \sqrt{3}$

k $\sqrt{5}^2 = 5$ l $4^{\frac{3}{2}} = 8$

3 a 2 b 5 c −1

d −2 e $\frac{1}{2}$ f 0

g $-\frac{1}{2}$ h 2 i $\frac{1}{2}$

j $\frac{3}{2}$ k $-\frac{2}{3}$

4 a 1 b 2 c 2 d 2

e 0 f 1 g 2 h 2

5 a 2 b 1 c 0 d 5

e 0 f −1 g 1 h 6

6 a $\log_b x + \log_b x - \log_b z$

b $\log_b 3 + \log_b b - \log_b 3(=1)$

c $\frac{1}{2}\log_3 x - 2\log_3 y - \log_3 z$

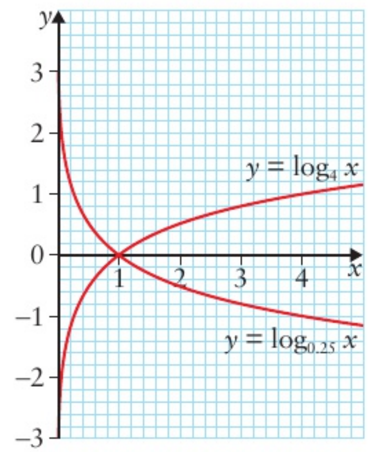
d $\frac{\log_b x^5}{\log_b x} = \frac{5\log_b x}{\log_b x} = 5$

e $\log_a x + \log_a (x+1) + \log_1 (x+2)$

f $\log_b y + \frac{1}{2}\log_b (y^2 + 1)$

Practice questions 13.4.1

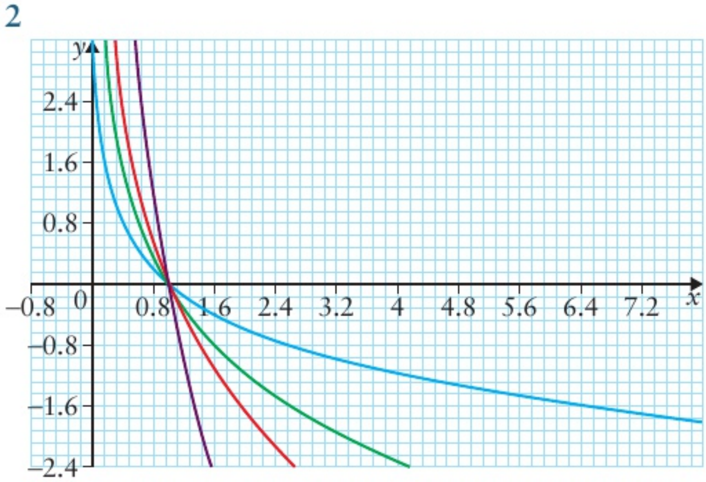
1 a



b If $w = \log_{\frac{1}{a}} x$, then $\left(\frac{1}{a}\right)^w = x$. It follows that $a^{-w} = x$.

On the other hand, if $y = \log_a x$, then $a^y = x$. So $y = -w$.

2



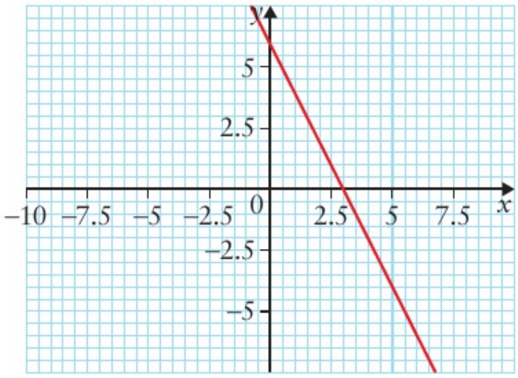
Practice questions 13.4.2

1	a	$x = 3$		b	$x = 3$
	c	$x = 4$		d	$x = 3$
	e	$x = 4$		f	$x = 2$
	g	$x = 2$		h	$x = 2$
	i	$x = -2$		j	$x = -3$
	k	$x = -2$		l	$x = -3$
	m	$x = -5$		n	$x = -4$
	o	$x = -3$		p	$x = -2$
2	a	$3 < x < 4$		b	$4 < x < 5$
	c	$1 < x < 2$		d	$2 < x < 3$
	e	$-3 < x < -2$		f	$-3 < x < -2$
	g	$-3 < x < -2$		h	$-2 < x < -1$
3	a	$x = 3.91$		b	$x = 4.19$
	c	$x = 1.54$		d	$x = 2.66$
	e	$x = -2.32$		f	$x = -2.10$
	g	$x = -2.86$		h	$x = -1.91$
4	a	$x = 10\,000$		b	$x = 2$
	c	$x = 3$		d	$x = 1$
	e	$x = 27$		f	$x = \frac{1}{27}$
	g	$x = 0.01$		h	$x = 2$
	i	$x = \frac{\sqrt{5}}{5}$		j	$x = 8$
5	a	$x = 8$		b	$x^3 = 8, x = 2$
	c	$x^3 = 27, x = 3$		d	$x^{\frac{1}{2}} = 4, x = 16$
	e	$x^{\frac{1}{2}} = \sqrt{3}, x = 3$		f	$x^{-3} = \frac{1}{8}, x = 2$
6	a	$x = \frac{25}{2}$		b	$x = 54$
	c	$x = 8$		d	$x = \sqrt[3]{2}$
	e	$x = 25$		f	$x = 1$
7	a	$T = 2$		b	$V(2) = U(2) = 5$

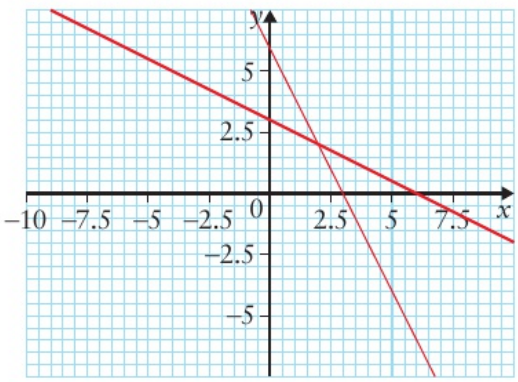
Check your knowledge questions

1 **a** 7 **b** 2 **c** $\dfrac{(x-1)}{3}$

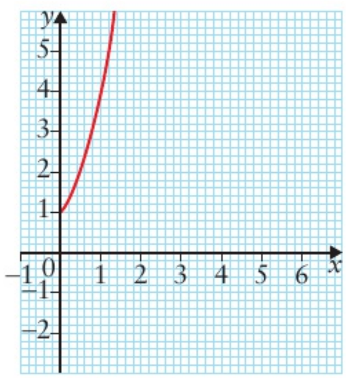
2 **a** $f(x)$



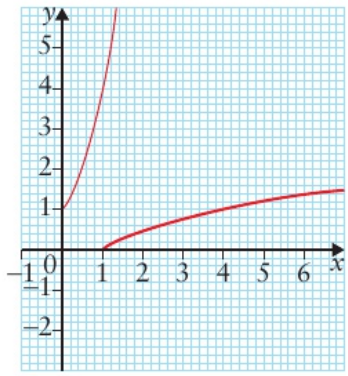
has an inverse, $f^{-1}(x) = \dfrac{6-x}{2}$ with graph



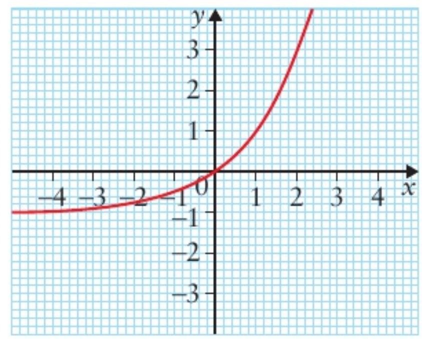
b $g(x)$



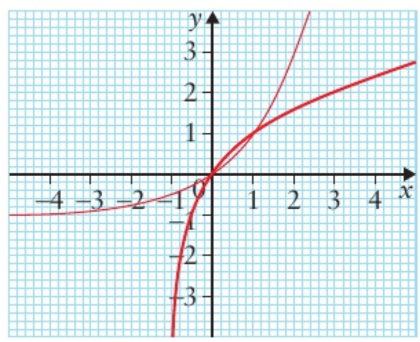
has an inverse, $g^{-1}(x) = \sqrt{x} - 1$ for $x \geqslant 1$ with graph



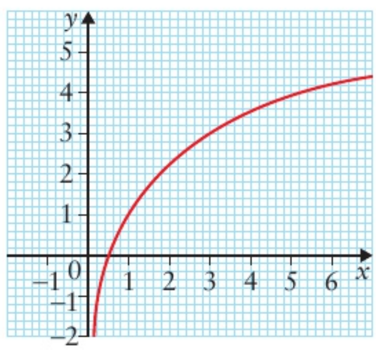
c $h(x)$



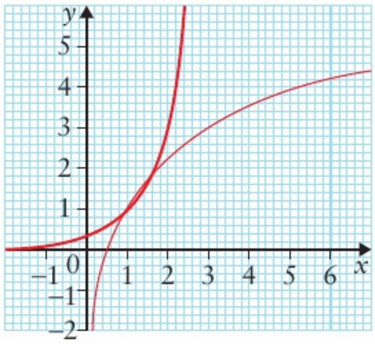
has inverse $h^{-1}(x) = \log_2(x+1)$ with graph



d $s(x)$



has inverse, $s^{-1}(x) = 3^{x-1}$ with graph



3 **a** **i** 4.91 m **ii** 19.6 m **iii** 78.5 m

b $t(h) = \sqrt{\frac{2h}{g}}$ **c** 123 m

4 **a** $x = 7$ **b** $x = 2$ **c** $x = 2$

d $x = 2$ **e** $x = 2$ **f** $x = 1$

g $x = 2.40$ **h** $x = -\frac{1}{3}$

5 **a** $x = \frac{1}{16}$ **b** $x = 2.64$

c $x = \pm\frac{\sqrt{3}}{3}$ **d** no (real) solutions

6 **a** $p + q + r$ **b** $p + q - r$

c $2p - q - r$ **d** $\frac{1}{2}p - 3q + 2r$

e $-2p + 3q + r$ **f** 0

g 0 **h** $p + q + r$

i $-2p + q + 3r$

7 **a** 6.95

b **i** 1.20×10^{30} W **ii** 1.51×10^{25} W

8 **a** **i** \$525 **ii** \$551.25

b 14.2 years **c** 7.18%

Chapter 14 answers

Do you recall?

1 $\sqrt{80} = 4\sqrt{5}$ cm

2 27 cm^2

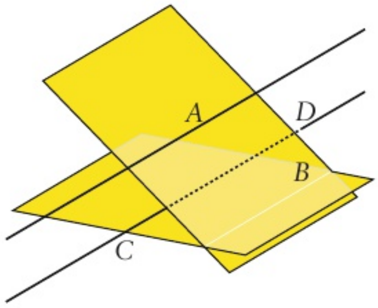
3 5.0 cm

Practice questions 14.1

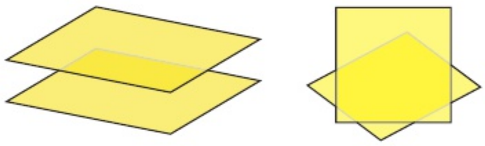
1 **a** True

b True

c False

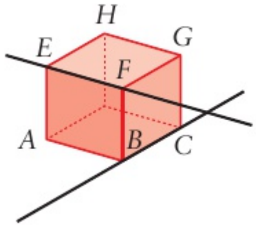


d False: either the two planes are parallel or they are intersecting in a line

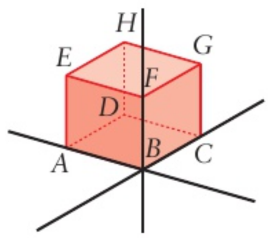


e True

f False. Skew lines by definition do not intersect neither are they parallel

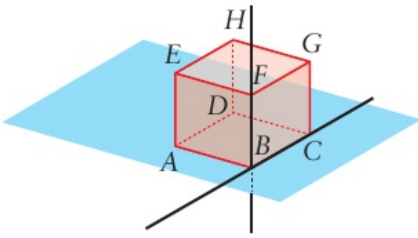


g False. Two intersecting lines are always coplanar. The third however can be non-coplanar



h True

i False. Take for example plane EHD in the figure below. It is not perpendicular to the line FB which instead is perpendicular to line BC.



j True

2 First mark some points on the staircase as shown

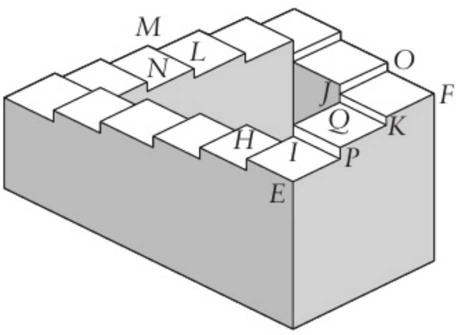
a Two parallel planes are, for example, two steps such as HIE and KJF or two parallel walls as CDE and FOG

b HIE and JQK or CDE and DGF

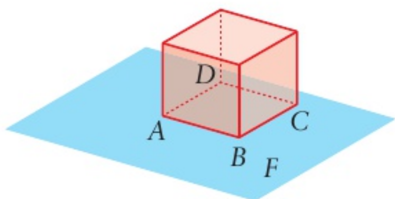
c ED and CDG or CDE and HI

d HI and KF or DE and GF

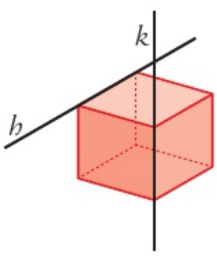
e CD and GF or KF and ML



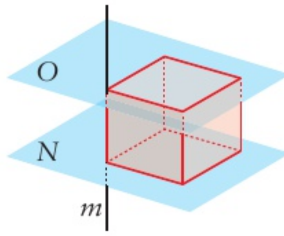
3 a



b



c



4 a DEA

b GC, HD, IE, FG, HG, IH, IJ

c FB, GC, HD, IE

d All combinations of 4 points of I, J, F, G, H, or of A, B, C, D, E, or of A, B, F, J, or of B, C, G, F, or of D, C, G, H, or of I, H, D, E, or of A, E, I, J

e Three points are always coplanar.

f B

g Line IH

h Point H

Practice questions 14.2

1 a i 3 ii 2

iii The answer is not unique: for example AB and EI

iv The answer is not unique: CB and GF

v There are 7 faces, 15 edges, and 10 vertices. The formula $F + V - E = 2$ is verified: $7 + 10 - 15 = 2$

b i 3 ii 2

iii The answer is not unique: for example AB and CG

iv The answer is not unique: for example AB and EF

v There are 6 faces, 12 edges, 8 vertices. The formula $F + V - E = 2$ is verified: $6 + 8 - 12 = 2$

c i 3 ii 2

iii The answer is not unique: for example AB and FD

iv There are no parallel edges.

v There are 6 faces, 10 edges, 6 vertices. The formula $F + V - E = 2$ is verified: $6 + 6 - 10 = 2$

2 a Octahedron

b Icosahedron

c Tetrahedron

d Dodecahedron

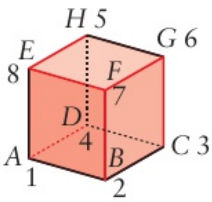
3 $5 + 11 - 9$ is not equal to 2. Therefore, it is not a convex polyhedron.

4 a 7

b Hexagonal pyramid

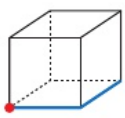
5 a 24 cm

b Number the vertices as shown below

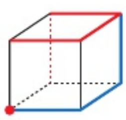


Then one possible path is 1, 2, 3, 4, 5, 6, 7, 8, 1

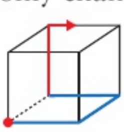
c To count all these possible paths, notice that there are 3 faces containing the starting vertex and each face has two pairs of edges, as shown below. So there are 6 options at the beginning



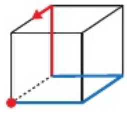
Then, you can change face and get this only chance



Or remain on the same face and have this only chance



Since this other path does not go back to the starting point



In total there are $6 \times 2 = 12$ possible paths that the ant can take.

Practice questions 14.3

1 a 10π cm^2 b 84 cm^2

c 171 cm^2 d $(48 + 36\sqrt{13})$ cm^2

e $(260 + 36\sqrt{5})$ cm^2

2 £3165.75

3 226.2 cm^2

4 £0.0115

5 279 cm^2. First, find the radius from the lateral surface area, then find the areas of the two bases and add it to the lateral area.

6 a 45° b 2.77 cm

c 2.3 cm d 25.46 cm^2

e 18.37 cm f 271.34 cm^2

7 a 154π or 483.8 cm^2

b $132\pi + 56$ or 470.7 cm^2. The surface area has decreased.

8 a A triangular prism

b 138 150 ft^2

c 14.6 feet

d 145 675.5 ft^2

Practice questions 14.4

1 a 147 cm^2

b 69.7 cm^2

2 a 208 cm^2

b $4 + \sqrt{13} + 2\sqrt{37}$ cm^2

3 a 42.5 cm^2

b 266.7 cm^2

4 $1945.96\ m^2$

5 a 1.573. The Golden ratio is 1.618. This is an approximation of the Golden Ratio accurate to about 97.2%

b $85\,878.69\ m^2$

6 a $144\pi\ cm^2$ b $75\pi\ cm^2$

c $160\pi\ cm^2$ d $1200\pi\ cm^2$

e $6400\pi\ cm^2$

7 a 17 cm b $136\pi\ cm^2$ c $200\pi\ cm^2$

8 a 12.7 cm (rounded to 1 d.p.)

b $28.4\pi\ cm^2$

c $33.4\pi\ cm^2$

9 $224\pi\ cm^2$

10 a $1618.70\ cm^2$ b $2391.54\ cm^2$

11 $5534.86\ ft^2$

12 $8.88\ m^2$

13 4.3 cm

14 a 5 cm b $75\pi\ cm^2$

15 5 cm

Practice questions 14.5

1 a $100\pi\ cm^2$ b $144\pi\ cm^2$

2 a $628\ cm^2$ b $157\ cm^2$

3 $231\,200\ cm^2$

4 4.0 cm

5 a $855\ cm^2$ b $102\ m^2$

6 $510\,064\,472\ km^2$

7 $345.6\ cm^2$

Practice questions 14.6

1 $1320\ cm^3$

2 $108\ cm^3$

3 $588\sqrt{2}\ cm^3$

4 $44\ cm^3$

5 a $1050\ cm^3$ b $360\ cm^3$ c $648\ cm^3$

6 $202.5\ m^3$

7 a $971.9\ m^3$ b $19.1\ m^3$

8 $10\,108\,074\ m^3$ (to nearest whole number)

9 $513\ cm^2$

10 84.8 litres (for 100 people)

11 a $5.775\ m^3$ in each tank

b 1.28 m

c 141 mm

12 3.93 bales (three significant figures)

Practice questions 14.7

1 a $50.3\ cm^3$

b $1620\ cm^3$

c $314\ cm^3$

d $33.3\ cm^3$

2 a 3.5 cm

b $57.17\ cm^3$ (four significant figures)

3 a 4.28 cm b $40.38\ cm^3$

4 a Each wooden stick should be longer than 13 feet (the slant height of the tepee).

b $314\ ft^3$

5 a $1\,078\,000\ m^3$ (four significant figures)

b 2.4 : 1

6 $640.9\ cm^3$ (four significant figures)

7 radius: 5 cm; height: 12 cm; slant height: 13 cm.

8 17.32 cm (four significant figures)

9 a $125\ cm^3$ b $51.0\ cm^3$

c $524\,000\,000\ m^3$ d $758\ cm^3$

10 $288\ \pi\ cm^3$

11 volume: $60.75\ \pi\ cm^3$; surface area: $60.75\pi\ cm^2$

12 a $1571.092\ cm^3$

b $1452.739\ cm^3$

13 9.65 cm

14 $1767\ m^3$ (four significant figures)

15 a 6371 km (four significant figures)

b $5.101 \times 10^8\ km^2$ (four significant figures)

c $3.570 \times 10^8\ km^2$ (four significant figures)

16 Students' own answers

17 a volume = 6044 cm³ surface area = 1605 cm²

b Students' own answers

18 34.64 cm (four significant figures)

19 64 : 27

20 a $r = \sqrt{144 - |x|^2}$ cm

b $(144 - |x|^2)\pi$ cm²

c $|x| = \sqrt{72}$ cm = 8.485 cm (four significant figures)

Check your knowledge questions

1 a Line through *A* and line *CE*

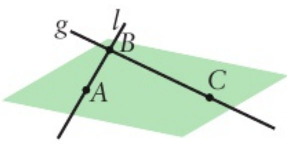
b Line through *A* and line through *B*

c Any 4 of *A*, *B*, *C*, *D*, *E*

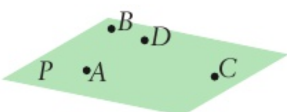
d *C*, *B*, *E*

e Line *CE* and line through *B*

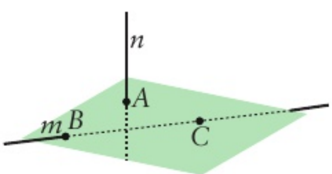
2 a



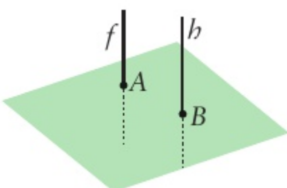
b



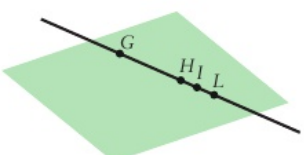
c



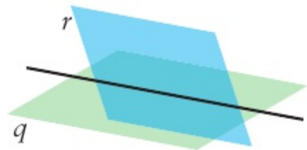
d



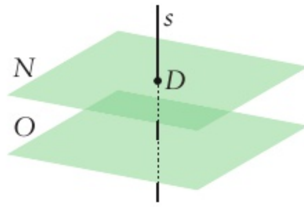
e



f



g



3 a Yes, it is a polyhedron with flat polygonal faces, straight edges and sharp corners as vertices.

b *ABD*, *ACD*, *BCD*, *ABC*

c *AC*, *AB*, *BC*, *AD*, *BD*, *CD*

d *A*, *B*, *C*, *D*

e $F = 4$, $E = 6$, $V = 4$, $F + V - E = 2$, $4 + 4 - 6 = 2$, yes

4 It is not a polyhedron: $6 + 11 - 8 = 9$

5 a Surface: 219 cm²
Volume: 165 cm³

b Surface: 669 cm²
Volume: 941 cm³

c Surface: 191 cm²
Volume: 145 cm³

d Surface: 88 cm²
Volume: 63 cm³

6 27.713 cm²

7 Surface: 105.4 cm²
Volume: 58.7 cm³

8 15 minutes

9 12.0 m² (three significant figures)

10 Surface: 144 π cm²
Volume: 288 π cm³

11 Surface: 273.37 cm²
Volume: 340.34 cm³

Chapter 15 answers

Do you recall?

1 $\sin(A) = \frac{5}{\sqrt{61}}$ $\cos(A) = \frac{6}{\sqrt{61}}$

$\tan(A) = \frac{5}{6}$

$A = 39.8°$

2 $x^2 + y^2 = 1$

3 No, if the known angle was between the two known sides they would be congruent.

4 α. Angles of elevation and depression are equal.

Practice questions 15.1

1 a 0.50 b 0.87 c 0.58 d 0.87

e 0.50 f 1.73 g 0.50 h −0.87

$\cos(180 - x) = -\cos(x)$,

$\sin(x) = \cos(90 - x)$, $\tan(x) = \frac{\sin(x)}{\cos(x)}$

2 a 50° b 120° c 60° d 139°

e 90° f 180° g 0° h 107°

i 160° j 66° k 80°

3 a 45° and 135° b 70° and 110°

c 10° and 170° d 20° and 160°

e 40° and 140° f 51° and 129°

g 79° and 101°

4 a 4th b 3rd c 2nd d 1st

5 a

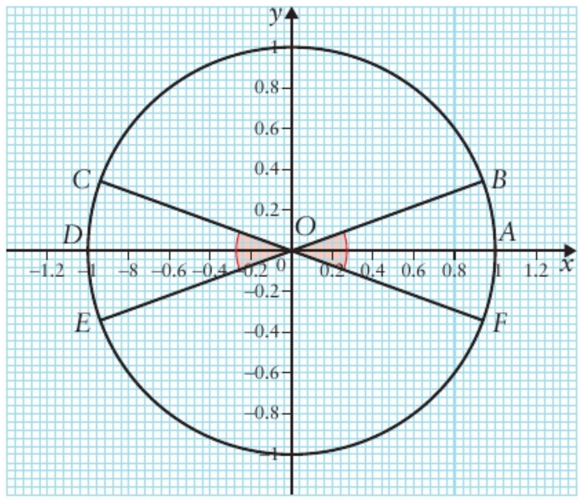
θ	32°	45°	107°	164°
$\sin(\theta)$	0.53	0.71	0.96	0.28
$\sin(180° - \theta)$	0.53	0.71	0.96	0.28
$\cos(\theta)$	0.85	0.71	−0.29	−0.96
$\cos(360° - \theta)$	0.85	0.71	−0.29	−0.96

b $\sin(\theta) = \sin(180° - \theta)$

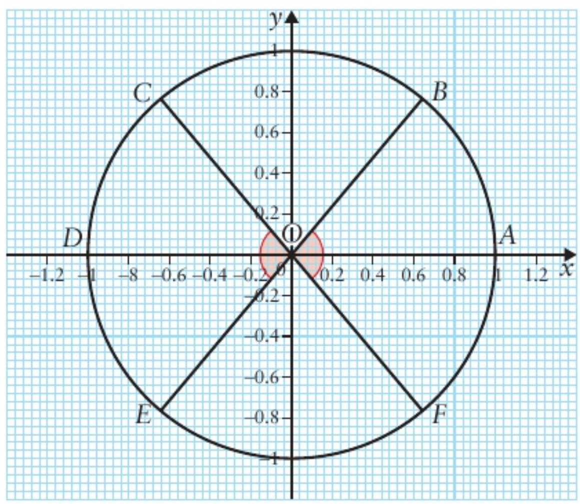
c $\cos(\theta) = \cos(360° - \theta)$

Practice questions 15.2

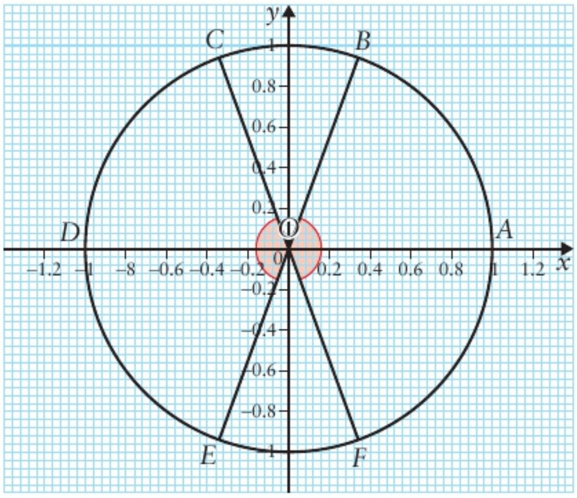
1 a $20 = AOB$, $180 - 20 = AOC$,
$180 + 20 = AOE$, $360 - 20 = AOF$



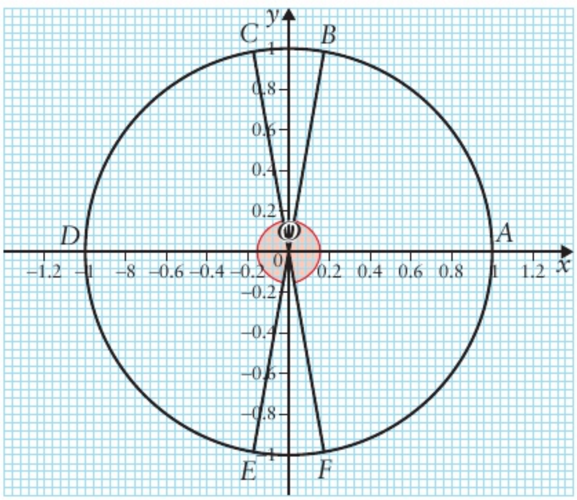
b $50 = AOB$, $180 - 50 = AOC$,
$180 + 50 = AOE$, $360 - 50 = AOF$



c $70 = AOB$, $180 - 70 = AOC$,
$180 + 70 = AOE$, $360 - 70 = AOF$



d $80 = AOB$, $180 - 80 = AOC$,
$180 + 80 = AOE$, $360 - 80 = AOF$



2

	$\theta = 34°$	$\theta = 117°$
$\sin\theta$	0.56	0.89
$\sin(180° - \theta)$	0.56	0.89
$\sin(180° + \theta)$	−0.56	−0.89
$\sin(360° - \theta)$	−0.56	−0.89
$\cos\theta$	0.83	−0.45
$\cos(180° - \theta)$	−0.83	0.45
$\cos(180° + \theta)$	−0.83	0.45
$\cos(360° - \theta)$	0.83	−0.45

3 **a**

	$\theta = 30°$
$\sin\theta$	0.5
$\sin(180° - \theta) = \sin 150° = \sin 30°$	0.5
$\sin(180° + \theta) = \sin 210° = -\sin 30°$	−0.5
$\sin(360° - \theta) = \sin 330° = -\sin 30°$	−0.5
$\cos\theta$	$\frac{\sqrt{3}}{2}$
$\cos(180° - \theta) = \cos 150° = -\cos 30°$	$-\frac{\sqrt{3}}{2}$
$\cos(180° + \theta) = \cos 210° = -\cos 30°$	$-\frac{\sqrt{3}}{2}$
$\cos(360° - \theta) = \cos 330° = \cos 30°$	$\frac{\sqrt{3}}{2}$

	$\theta = 45°$
$\sin\theta$	$\frac{\sqrt{2}}{2}$
$\sin(180° - \theta) = \sin 135° = \sin 45°$	$\frac{\sqrt{2}}{2}$
$\sin(180° + \theta) = \sin 225° = -\sin 45°$	$-\frac{\sqrt{2}}{2}$
$\sin(360° - \theta) = \sin 315° = -\sin 45°$	$-\frac{\sqrt{2}}{2}$
$\cos\theta$	$\frac{\sqrt{2}}{2}$
$\cos(180° - \theta) = \cos 135° = -\cos 45°$	$-\frac{\sqrt{2}}{2}$
$\cos(180° + \theta) = \cos 225° = -\cos 45°$	$-\frac{\sqrt{2}}{2}$
$\cos(360° - \theta) = \cos 315° = \cos 45°$	$\frac{\sqrt{2}}{2}$

	$\theta = 60°$
$\sin\theta$	$\frac{\sqrt{3}}{2}$
$\sin(180° - \theta) = \sin 120° = \sin 60°$	$\frac{\sqrt{3}}{2}$
$\sin(180° + \theta) = \sin 240° = -\sin 60°$	$-\frac{\sqrt{3}}{2}$
$\sin(360° - \theta) = \sin 300° = -\sin 60°$	$-\frac{\sqrt{3}}{2}$
$\cos\theta$	0.5
$\cos(180° - \theta) = \cos 120° = -\cos 60°$	−0.5
$\cos(180° + \theta) = \cos 240° = -\cos 60°$	−0.5
$\cos(360° - \theta) = \cos 300° = \cos 60°$	0.5

b

	$\theta = 30°$	$\theta = 45°$	$\theta = 60°$
$\tan(\theta)$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\tan(180° - \theta)$	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$
$\tan(180° + \theta)$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\tan(360° - \theta)$	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$

4 a 153° b 138° c 110° d 94°

5 a 83° b 56° c 32° d 8°

6 a 106° b 74° c 29° d 151°
e 23° f −23° g 88° h −88°

7 a 15.7°, 164.3° b 24.2°, 155.8°
c 34.3°, 145.7° d 38°, 142°
e 55.1°, 124.9° f 64.2°, 115.8°

8 a 74.3°, 285.7° b 105.7°, 254.3°
c 114.2°, 245.8° d 124.3°, 235.7°
e 88.7°, 268.7° f 89.4°, 269.4°

Practice questions 15.3

1 $\sin(ACD) = h/b$, then $h = b \times \sin(ACD)$, but supplementary angles have the same sine ratio therefore, $h = b \times \sin(ACD) = b \times \sin(ACB)$.

The area of the triangle is: Area $= \frac{1}{2}ab \times \sin(C)$

2 a 28.0 cm^2
b 985.5 m^2
c 6.7 cm^2
d 2465.7 cm^2
e 166.9 km^2

3 a $22u^2$ b $11u^2$ c $37u^2$

4 28.0 cm^2

5 76.5 cm

6 14.3 cm

7 $\frac{\sqrt{3}}{4}a^2$

8 1 137 533.3 km^2

9 35.7 m^2

10 61.1 m^2

11 37.2 km^2

12 a 25.6 cm^2 b 56 cm^2

Practice questions 15.4

1 a $\frac{\sin 45°}{7.21} = \frac{\sin 56.31°}{8.49} = \frac{\sin 78.69°}{10} = 0.10$

b $\frac{\sin 33.22°}{6.88} = \frac{\sin 44.73°}{8.84} = \frac{\sin 102.05°}{12.29} = 0.08$

c $\frac{\sin 40.74°}{29.9} = \frac{\sin 67.56°}{42.35} = \frac{\sin 71.7°}{43.5} = 0.02$

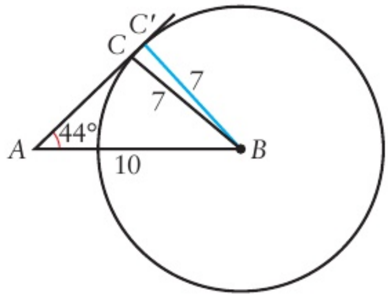
2 a 4.29 m
b 5.07 m
c 2.62 m
d 3.02 m
e 2.63 m

3 5.275 cm^2

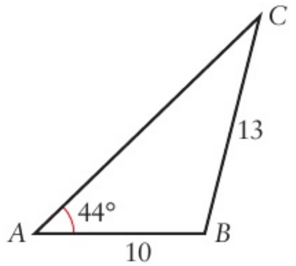
4 28.1 km

5 a 82.92°, 97.08°

b The two possible triangles are ABC and ABC', since the circle with radius 7 and centre B intersects the ray AC in two points: C and C'.

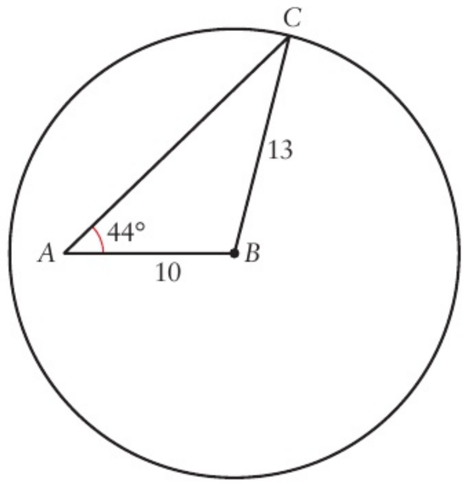


6 a



b 32.3°

c The circle with centre B and radius 13 intersects ray AC in only 1 point. Therefore, there can be only one triangle



Another reason is the following: The other possible angle with the same sine value is $180° - 32.3° = 147.7°$, but $147.7° + 44° > 180°$, so there would not exist angle B.

7 Call the yacht Y. $TY = 4\,\text{km}$, $SY = 5\,\text{km}$

8 $3176.2\,\text{m}^2$

9 5.95 m

10 a 37.775°

b 151.51 pm

c 165.61 pm

11 a Poor House to C = 35 732.38 ft, King's Arbour to C = 21 654.23 ft

b Poor House to E = 38 577.68 ft, CE = 30 159.28 ft

c Same distance: 35 732.38 ft

d Poor House to D = 27 646.09 ft, DE = 32 411.69 ft

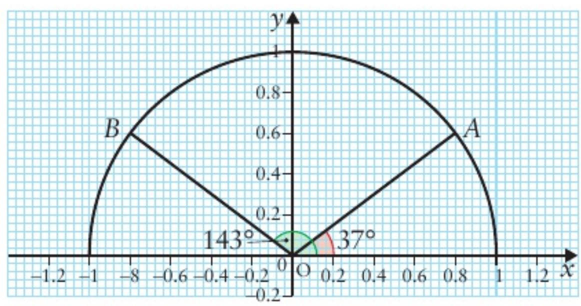
e Assuming that the triangle is isosceles with length of sides equal to the average of 27404 and 27646.09. Then, we get that the required distance is 54 646 km.

Check your knowledge questions

1 $A(0.6, 0.8)$, $B(-0.91, 0.4)$

2 a 0.64 b −0.34 c 0.82 d −0.94

3 a



$\theta = 143°$, $\sin\theta = 0.60$

b $\cos(143°) = -0.7986$, $\cos(37°) = 0.7986$

c $\tan(37°) = \dfrac{\sin(37°)}{\cos(37°)} \approx \dfrac{0.60}{0.80} \approx 0.75$, with calculator, $\tan(37°) = 0.7536$

d $\cos^2(37) + \sin^2(37) = 0.6378 + 0.3622 = 1$

4 a 133° b 100° c 141°

5 25.5° and 154.5°

6 144.1°, 215.9°

7 33.4°, 213.4°

8 a 145.9° b 102.1°

9 **a** $BC = 5.49$ cm, $AC = 9.82$ cm, $C = 44°$

b $AC = 4.87$ cm, $BC = 5.14$ cm, $A = 70°$

c $AC = 13.03$ cm, $BC = 6.18$ cm, $A = 25°$

10 **a** First solution: $C = 50.32°$, $B = 91.68°$, $AC = 12.99$ cm. Second solution: $C = 129.7°$, $B = 12.3°$, AC = 2.77 cm

b First Solution $C = 74.2°$, $B = 64.8°$, $AC = 20.7$ cm. Second Solution: $C = 105.8°$, $B = 33.2°$, $AC = 12.5$ cm

11 **a** 15 cm² **b** 34.9 cm²

12 **a** 11.3 cm **b** 33.7 cm²

13 **a** 3.97° **b** 40° **c** 65.4 m

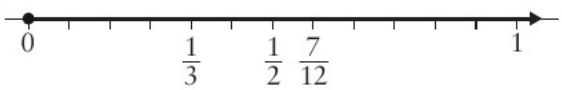
14 12.4 m

15 $AD = 9.74$ cm

Chapter 16 answers

Do you recall?

1



2 Possibilities include $\sqrt{2}$, $\sqrt{3}$, π

3 Only $\frac{2}{3} + \frac{3}{4}$

4 The largest square (on the hypotenuse) has area equal to the sum of the areas of the other squares (on the legs)

5 n

6 $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$

7 $(a + b)(a - b)$

8 $\sqrt{2}$, all the others are equal.

Practice questions 16.1.1

1 **a** 4 **b** 5 **c** 7 **d** 1
e 3 **f** 37 **g** $\frac{1}{6}$ **h** $\frac{1}{24}$
i $\frac{3}{16}$ **j** $\frac{5}{4}$ **k** $\frac{11}{8}$ **l** a
m $\frac{3xy}{20}$

2 **a** $\frac{2}{3}$ **b** $\frac{1}{2}$ **c** $\frac{2}{3}$ **d** $\frac{101}{103}$
e $\frac{17}{22}$ **f** $\frac{2}{3}$ **g** $\frac{3}{2}$ **h** $\frac{16}{21}$
i $\frac{4}{3}$ **j** $\frac{2}{3}$ **k** $\frac{3}{10}$ **l** $\frac{x}{y}$
m $\frac{8x}{3y}$

3 **a** smaller gear 9 rotations, larger gear 7 rotations

b 9 : 7, 18 : 14, 27 : 21, …

c They all simplify to 9:7

4 **a** 8 mm: $\frac{5}{16}$, 24 mm: $\frac{15}{16}$
b $\frac{3}{4}$
c 27 mm

Practice questions 16.1.2

1 **a** $\frac{19}{4}$ **b** $\frac{31}{1}$ **c** $\frac{9}{4}$ **d** $\frac{3}{2}$
e $\frac{6}{5}$ **f** $\frac{157}{50}$ **g** $\frac{707}{500}$ **h** $\frac{1}{10}$
i not possible **j** $\frac{-2}{1}$

2 **a** $0.\dot{0}2325581395348837209\dot{3}$
b $0.\dot{0}4651162790697674418\dot{6}$

3 8: 63414, 73170, 85365, 87804, 90243, 92682, 95121, 97560

Practice questions 16.1.3

1 **a** **i** $\frac{1}{4}$ **ii** $\frac{4}{25}$ **iii** $\frac{10}{121}$
b $\frac{1}{x}, \frac{1}{x^2}$

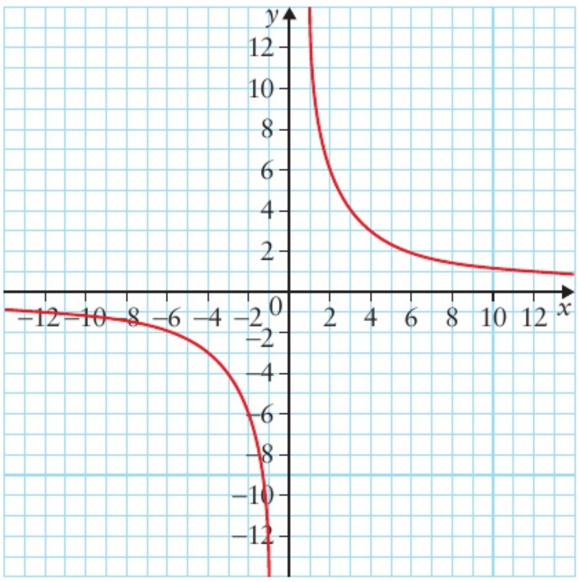
2 **a** $\frac{x-1}{x}$ **b** $\frac{1}{6x}$ **c** $\frac{x^2+3}{x}$
d $\frac{x^2+16}{4x}$ **e** $\frac{3x-1}{x-x^2}$ **f** $\frac{2x^2}{x^2-4}$
g $\frac{1}{x^2-1}$ **h** $\frac{2}{x^3-x}$ **i** $\frac{2}{4x-x^3}$

3 $R = \frac{R_1^3 - R_1}{3R_1^2 - 1}$

Practice questions 16.1.4

1 a

x	−12	−6	−4	−3	−2	−1	0	1	2	3	4	6	12
$f(x)$	−1	−2	−3	−4	−6	−12	N/A	12	6	4	3	2	1



b $x = 0$

c It is not possible.

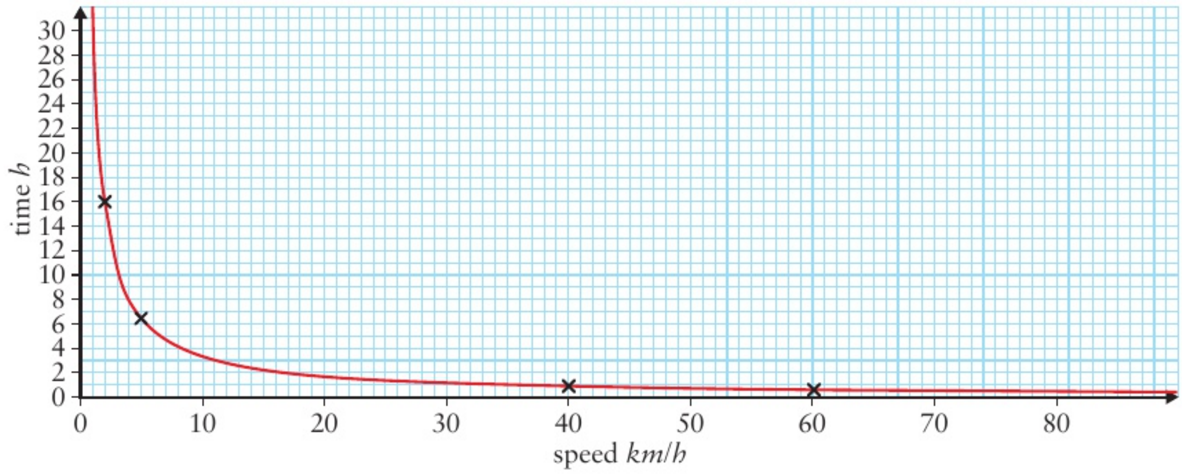
2 a 16 hours

b 6 hours 24 minutes

c 48 minutes

d 24 minutes

e



f 32 minutes

3 a $y = 1$ and $x = 3$

b $f(x) = \dfrac{x-2}{x-3} = \dfrac{x-3}{x-3} + \dfrac{1}{x-3} = 1 + \dfrac{1}{x-3}$, so as $|x|$ becomes large, y approaches 1 and as x approaches 3, $|y|$ becomes large.

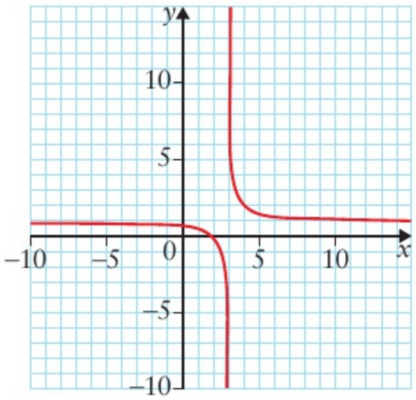
c $y = g(x)$ has the same asymptotes

d $y = h(x)$ has $y = 2$ and $x = 3$, $y = j(x)$ has $y = 0$ and $x = 3$ as asymptotes.

Practice questions 16.1.5

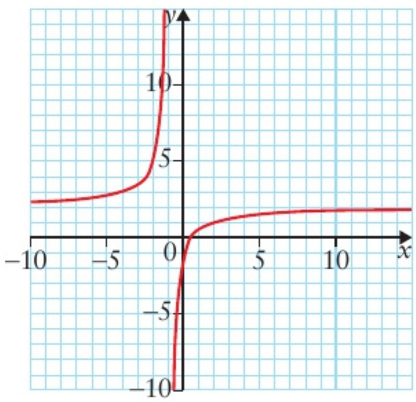
1 **a** $x \neq 3$, **b** $f(x) \neq 1$, **c** $x = 3$, $y = 1$

d



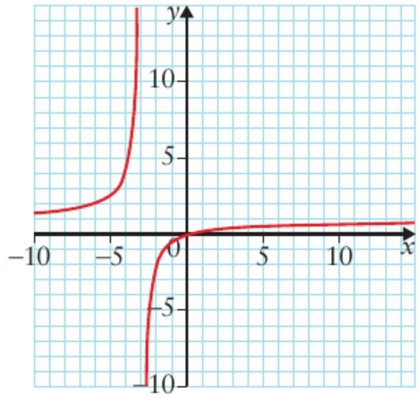
2 **a** $x \neq -1$, **b** $f(x) \neq 2$, **c** $x = -1$, $y = 2$

d



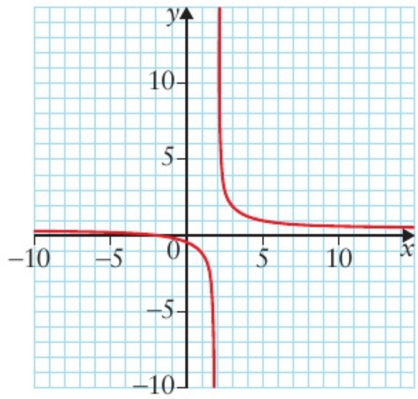
3 **a** $x \neq -3$, **b** $f(x) \neq 1$, **c** $x = -3$, $y = 1$

d



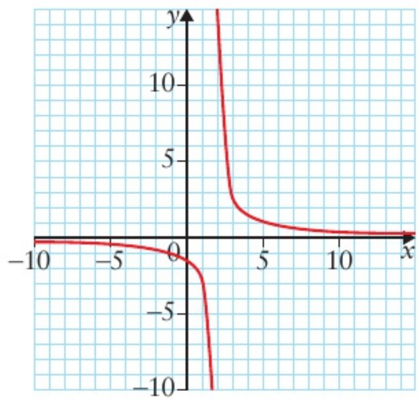
4 **a** $x \neq 2$, **b** $f(x) \neq \frac{1}{2}$, **c** $x = 2$, $y = \frac{1}{2}$

d



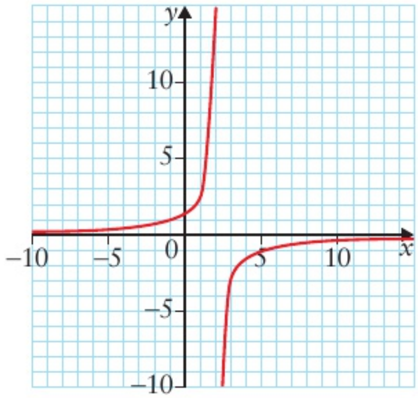
5 **a** $x \neq 2$, **b** $f(x) \neq 0$, **c** $x = 2$, $y = 0$

d



6 **a** $x \neq 2$, **b** $f(x) \neq 0$, **c** $x = 2, y = 0$

d



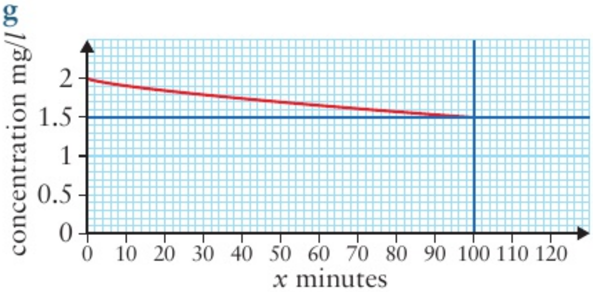
7 **a** 600 000 mg **b** 1600 mg

c 696 000 mg **d** 420 000 litres

e 1.66 mg/litre

f flouride: $600\,000 + 1600x$ mg,
water: $300\,000 + 2000x$ litres,
concentration: $\dfrac{600\,000 + 1600x}{300\,000 + 2000x}$ mg/litre,
simplified: $\dfrac{1500 + 4x}{750 + 5x}$ mg/litre

g



h The concentration will remain slightly above 1.5 mg/litre when the tank is full.

i Setting the concentration in the added water to n mg/l we use the graph of $y = \dfrac{300 + nx}{150 + x}$, where y mg/l is the total concentration and x minutes is the time. Adjusting n dynamically on the graph we can reach $y = 1.5$ after 100 minutes by choosing $n = 0.75$. So the concentration of fluoride in the added water should be no more than 0.75 mg/l.

Practice questions 16.2.1

1 b, d, e, g are rational, a, c, f are irrational

2 **a** $a = 1 + \sqrt{2}$, $b = 1 - \sqrt{2}$, gives rational $a + b = 2$

b $a = \sqrt{2}$, $b = \sqrt{8}$, gives rational $\dfrac{a}{b} = \dfrac{1}{2}$

c $a = \sqrt{2}$, gives rational $a^2 = 2$

d $\sqrt{a}$ is irrational if a is irrational

3 **a** $c = 0, d = \sqrt{2}, cd = 0$

b irrational

c $c = 0, d = \sqrt{2}, (c + d)^2 = 2$

d $c = 1, d = \sqrt{2}, cd^2 = 2$

Practice questions 16.2.2

1 **a** $3\sqrt{3}$ **b** $6\sqrt{2}$ **c** 3

d $\dfrac{1}{2}$ **e** $\dfrac{1}{3}$ **f** $2\sqrt{2}$

g ab **h** $\dfrac{n}{2}$ **i** 2

j ab **k** 2 **l** 2

m $a\sqrt{b}$ **n** $m\sqrt{m}$ **o** $\sqrt{x}$

2 **a** $4\sqrt{2}$ **b** $4\sqrt{3}$ **c** $3\sqrt{5}$

3 **a** $2\sqrt{3}$ **b** $2\sqrt{5}$ **c** $8\sqrt{2}$ **d** $3\sqrt{7}$

e $7\sqrt{2}$ **f** $9\sqrt{5}$ **g** $4\sqrt{5}$ **h** 0

i $6\sqrt{10}$ **j** $7\sqrt{2}$

4 **a** $6 + 2\sqrt{5}$ **b** $1 + \sqrt{2} + \sqrt{3} + \sqrt{6}$

c $11 - 4\sqrt{6}$ **d** 2

e $1 + n + 2\sqrt{n}$ **f** $n - 2$

Practice questions 16.2.3

1 **a** **i** always rational, $1 + 1 = 2$

b **ii** sometimes rational $(1 + \sqrt{2}) + (1 - \sqrt{2}) = 2$, $(\sqrt{2} + 1) + (\sqrt{2} - 1) = 2\sqrt{2}$

c **iii** sometimes irrational: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, $\sqrt{2} \times \sqrt{2} = 4$

d **i** always irrational $1 + \sqrt{2}$

2 **a** $\dfrac{\sqrt{2}}{2}$ **b** $\sqrt{5}$ **c** $\dfrac{2\sqrt{3}}{3}$

d $2 + \sqrt{3}$ **e** $\dfrac{3 + \sqrt{2}}{7}$ **f** $\sqrt{5} - \sqrt{3}$

3 a $\frac{\sqrt{a}}{a}$ b $\sqrt{n}$ c $\frac{\sqrt{a}+1}{a-1}$

d $\frac{\sqrt{x}-\sqrt{y}}{x-y}$

e $\frac{(\sqrt{p}+\sqrt{q})^2}{p-q} = \frac{p+q+2\sqrt{pq}}{p-q}$

4 No. If it passed through a point with integer coordinates (m, n) then the gradient would give $\sqrt{2} = \frac{n}{m}$, which is not possible since $\sqrt{2}$ is irrational.

5 5, 15, 29, 47

6 a $NC = \sqrt{8}$, $AC = \sqrt{72}$, $BC = 3$

b $NC = 2\sqrt{2}$, $AC = 6\sqrt{2}$

7 $2 + \sqrt{2}$

First picture.

8 The radius of the circle is $\sqrt{2}$ so the curve passes through the points $\left(\frac{\sqrt{2}}{2}, \sqrt{2}\right)$, $(1, 1)$, $\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$. In each case the product of the coordinates is 1, so a possible equation for the curve is $xy = 1$.

9 The material used is in a shape equivalent to a prism with CSA $\sqrt{3} - 1\,\text{m}^2$ and length $\sqrt{3} + 1\,\text{m}$. So the volume is $2\,\text{m}^3$ and the weight (mass) is 4.8 tonnes.

Check your knowledge questions

1 a $\frac{1}{5}$ b $\frac{5}{6}$ c $\sqrt{2}$

2 a $\frac{x+y}{xy}$ b $\frac{2n-3}{n}$ c $\frac{p^2+1}{p}$

d $\frac{1}{x(x-1)}$ e $\frac{1}{x+1}$

3 a i $x \neq 0$, ii $f(x) \neq 0$

b i $x \neq 4$, ii $f(x) \neq 1$

c i $x \neq 0$, ii $f(x) \neq -1$

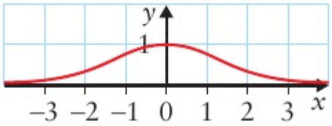
d i $x \neq 2$, ii $f(x) \neq 2$

4 a $x = 1, y = 0$

b $x = 1, y = 2$

c $x = 1, y = 1$

5 Range: $(0, 1]$

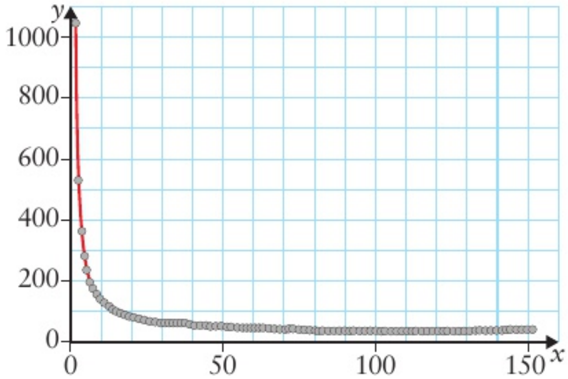


6 a €4000

b $C(x) = \frac{1000}{x} + 30$

c The set of positive integers up to 150.

d



e 200, which exceeds the capacity of the room.

f $(36.67, 1030]$. The lower limit is the minimum possible cost in € per person.

7 a i $\frac{25}{130} \approx 19.2\%$

ii 16 minutes

iii 240 litres

b i 7 litres/minute

ii

8 b and d

9 $x = 3\sqrt{13}$ $y = 6\sqrt{2}$

10 a $\frac{\sqrt{6}}{2}$ b $\frac{\sqrt{6}}{3}$ c $\frac{\sqrt{ab}}{b}$

d $3+\sqrt{7}$ e $12+\sqrt{143}$

f $\frac{a+1-2\sqrt{a}}{a-1}$

Chapter 17 answers

Do you recall?

1 a $5^6 = 15\,625$ b $12^2 = 144$

c $7^6 = 117\,649$ d $2^{12} = 4096$

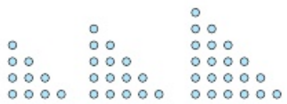
2 $x = 6, y = 1$

3 a -9 b 25 c 500

4 a $n = -2$ b $n = 2$ or $n = 5$

Practice questions 17.1

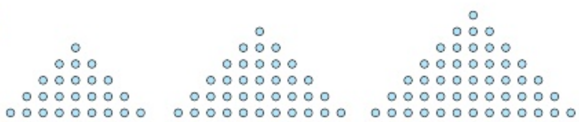
1 a



b 5050

c $\frac{n}{2}(1+n)$

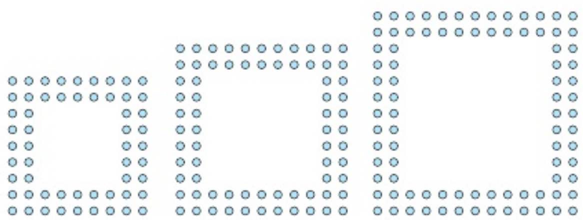
2 a



b 10 201

c $(n+1)^2$

3 a

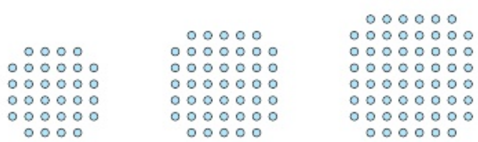


Step 4 Step 5 Step 6

b 1592

c $(2n+1)^2 - (2n-3)^2$ or equivalently $16n - 8$

4 a



Step 4 Step 5 Step 6

b 10 400 c $n^2 + 4n$

5 (answers are not unique)

a 4, 5, 6, 7, 8, 9 or 1, 2, 3, 1, 2, 3

b 8, 10, 12, 14, 16 or 10, 16, 26, 42

c 13, 19, 26 or 32, 64, 256

6 a 2, 6, 10, 14

b $-2, -8, -14, -20$

c 1, 5, 11, 19

d 1, 8, 27, 64

e $4, \sqrt{2}+3, \sqrt{3}+3, 5$

f 1, 4, 27, 256

g $2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}$

h $-\frac{3}{4}, \frac{9}{16}, -\frac{27}{64}, \frac{81}{256}$

i $-1, 1, -1, 1$

7 a $-7, -25, -79, -241$

b 0.5, 2, 5, 11

c 3, 10, 38, 150

d 1, 1, 1, 1

e 0, 0, 0, 0

f $1, 3, -1, 7$

g 1, 2, 2, 4

h 2, 4, 2, 0.5

8 note that answers are not unique

a $u_n = u_{n-1} + 3, u_1 = 4$

b $u_n = u_{n-1} + 2, u_1 = 3$

c $u_n = 2u_{n-1}, u_1 = 4$

d $u_n = 5u_{n-1}, u_1 = 1$

e $u_n = u_{n-1} + u_{n-2}, u_1 = 1, u_2 = 2$

f $u_n = \sqrt{u_{n-1}}, u_1 = 2$

9 note that answers are not unique

a $u_n = 3n + 1$ b $u_n = 2n + 1$

c $u_n = -4n + 3$ d $u_n = n^2 + 1$

e $u_n = 2(-1)^n$ f $u_n = 2\left(-\frac{1}{5}\right)^n$

10 $u_1 = \frac{52}{3}, u_2 = 51$

11 a $u_n = 5n - 3$ b 112

c $u_{71} = 352$

12 a $u_4 = 8, u_6 = 32, u_8 = 128$

b yes, $u_{12} = 2048$

13 a 56

b no, the equation $n^2 + n = 764$ has no integer solution.

Practice questions 17.2

1 a $S_1 = 3, S_2 = 8, S_5 = 35, S_8 = 80$

b $S_1 = 0, S_2 = 2, S_5 = 40, S_8 = 168$

c $S_1 = -2, S_2 = 2, S_5 = -6, S_8 = 8$

d $S_1 = 2, S_2 = 6, S_5 = 62, S_8 = 510$

e $S_1 = 0, S_2 = \frac{1}{3}, S_5 = 2.1, S_8 = 4.34$

2 a 176 b −30 c 32

d 13.1875 e 12 f $\frac{1}{5}$

3 a $\sum_{n=1}^{7} \frac{1}{3n}$ b $\sum_{n=1}^{9} (3n + 1)$

c $\sum_{n=3}^{6} \left(-\frac{1}{2}\right)^n$ d $\sum_{n=1}^{5} (-1)^{n-1} \frac{2}{n^3}$

e $\sum_{n=1}^{13} (2n + 1)$ f $\sum_{n=0}^{4} \frac{3^n}{7}$

4 a false b true

5 665 billion dollars

6 a It is the sequence 1, 2, 4, 8, 16,… In other words, it is the sequence where the terms are powers of 2.

b $a_i = 2^i$ where $i = 0,1,2,3,\ldots$ $(i \in \mathbb{N})$

Practice questions 17.3

1 a arithmetic, $d = -3$

b arithmetic, $d = 2$

c not arithmetic

d not arithmetic

e arithmetic, $d = 1$

f arithmetic, $d = 3$

g arithmetic, $d = 6$

h not arithmetic

i arithmetic, $d = 7$

2 a $u_n = 43 - 3(n - 1)$

b $u_n = -7 + (0.5)(n - 1)$

c $u_n = 12 + 10(n - 1)$

d $u_n = 5 - 5(n - 1)$

e $u_n = 1 + \pi(n - 1)$

f $u_n = 46 + 2(n - 1)$

g $u_n = 49 - 3(n - 1)$

3 u_{20}

4 19

5 $u_n = 32n$

6 a 39, 43 b after 9 years

7 $a = 2, b = 4, d = 3$

8 420.7 ppm

Practice questions 17.4

1 a geometric, $r = -3$

b geometric, $r = -2$

c not geometric

d not geometric

e geometric, $r = 0.3$

f not geometric

g not geometric

h not geometric

i geometric, $r = 0.4$

2 a 5, 15, 45, 135

b $3, 1, \frac{1}{3}, \frac{1}{9}$

c 0.2, −0.04, 0.008, −0.0016

d $2, 2\sqrt{2}, 4, 4\sqrt{2}$

e $1, \pi, \pi^2, \pi^3$

f 1, −1, 1, −1

g $2, 6x, 18x^2, 54x^3$

h a, ax, ax^2, ax^3

3 a 15 b 7 c 13

d 11 e 13

4 1 647 086 or −1 647 086

5 −13 631 488

6 The 17th term

7 £37045.50

8 a $u_n = 360\ (1.02)^n$ where $(2021 + n)$ is the year

b 397

9 Approximately 361

Check your knowledge questions

1 Three possible answers are
1, 3, 5, 7, 9, 11, ... (arithmetic sequence with common difference $d = 2$)
1, 3, 5, 3, 1, 3, 5, 3, 1... (add 2 two times, minus 2 two times)
1, 3, 5, 1, 3, 5, 1, 3, 5, ... (repeat pattern)

2 a −2, 1, 6, 13, 22, 33

b 40

3 a 6, 10, 14, 18, 22

b 1, −2, −5, −8, −11

4 a 3, 18, 78, 318, 1278

b 1, 2, 4, 16, 128

5 a $S_1 = 7, S_2 = 17, S_4 = 46, S_6 = 87$

b $S_1 = -1, S_2 = -6, S_4 = -36, S_6 = -106$

c $S_1 = -4, S_2 = 3, S_4 = 6, S_6 = 9$

d $S_1 = 3, S_2 = 12, S_4 = 120, S_6 = 1092$

e $S_1 = \frac{1}{3}, S_2 = 0.933, S_4 = 2.43, S_6 = 4.09$

6 b 90 b 29 c 0.291

d 1.99 e 63

7 a 11 b −8

8 $k = \frac{7}{5}$

9 $u_n = 1 + 3(n - 1)$

10 a the sequence has a common difference of −4

b $u_n = -3 - 4(n - 1)$

c −403

d 201

11 37

12 a $r = 3$ b $r = -2$ c $r = \frac{1}{\sqrt{2}}$

13 a $k = 5$ b $r = 2$

14 a $u_n = 64(0.5)^{n-1}$

b $n \leqslant 7$

c 127

15 a $u_n = 25 + n$ b 11th row

Chapter 18 answers

Do you recall?

1 a A even chance
B impossible
C certain
D likely
E unlikely
F likely

b B, E, A, D, F, C

2 a 0.5 b 0.9 c 1 d 0

3 0.47

4 a 362 880 b 1 c 1

Practice questions 18.1

1 a $\frac{33}{100}$ b $\frac{67}{100}$

c $\frac{8}{100} = \frac{2}{25}$ d $\frac{92}{100} = \frac{23}{25}$

2 a $\frac{26}{52} = \frac{1}{2}$ b $\frac{26}{52} = \frac{1}{2}$ c 0

d $\frac{4}{52} = \frac{1}{13}$ e $\frac{48}{52} = \frac{12}{13}$ f $\frac{12}{52} = \frac{3}{13}$

g $\frac{13}{52} = \frac{1}{4}$ h $\frac{39}{52} = \frac{3}{4}$ i $\frac{8}{52} = \frac{2}{13}$

3 a $\frac{20}{170} = \frac{2}{17}$ b $\frac{150}{170} = \frac{15}{17}$ c $\frac{85}{170} = \frac{1}{2}$

d $\frac{27}{170}$ e $\frac{40}{170} = \frac{4}{17}$ f $\frac{38}{170} = \frac{19}{85}$

g $\frac{70}{170} = \frac{7}{17}$ h $\frac{3}{170}$

4 The number that is most often encountered is 10 000 repetitions. This exercise really intends to show that the number is bigger than most people think at first.

5 a 0.03

b A sample size of 100 is reasonably large, so it is a fair estimate. However, it would be recommended to take multiple samples to be more certain of the accuracy of the estimate.

6 a 0.16 b 0.17 c 0.19

7 a $\frac{3}{5}$

b The sum is 1, which makes sense since either he does the one or the other. There are no other possibilities.

8 a 0, 1, 2, 3

b

outcome	0	1	2	3
probability	$\frac{6}{50}=0.12$	$\frac{18}{50}=0.36$	$\frac{19}{50}=0.38$	$\frac{7}{50}=0.14$

c 0.74

d The probabilities of the 50 flips are quite close to the theoretical values.

9 $\frac{24+38}{24+38+119+238}=\frac{62}{419}=0.148$. So only about 15% of the cars are electric or hybrid. This means that about 85% of the cars has a fossil fuel engine, which indicates a negative impact on CO_2 emissions and air quality.

10 The experimental probability of flipping heads is 0.57, which is substantially higher than the theoretical probability of 0.50. However, while a sample of 100 flips may seem large, it is much smaller than the 10 000 repeats required. It is recommended to repeat the 100 flips a couple more times to see whether the pattern remains the same, this is quicker than doing 10 000 repeats (but also somewhat less reliable).

11 0.665

12 a $\frac{4}{15}$

b should be a multiple of 15

c 10 Red, 12 Blue, 8 Green

Practice questions 18.2.1

1 a R1, R2, R3, R4, R5, R6, B1, B2, B3,B 4, B5, B6.

b $\frac{1}{3}$

2 By looking at the table we see that 1 outcome out of 8 fits the description, so the probability is $\frac{1}{8}$

	Heads	Tails
Blue	BH	BT
Red	**RH**	RT
Orange	OH	OT
Green	GH	GT

3 BB, BR, BO, BG, RB, RR, RO, RG, OB, OR, OO, OG, GB, GR, GO, GG

a $\frac{2}{16}=\frac{1}{8}$ b $\frac{4}{16}=\frac{1}{4}$

4

	Hearts	Diamonds	Clubs	Spades
Hearts	HH	HD	HC	HS
Diamonds	DH	DD	DC	DS
Clubs	CH	CD	CC	CS
Spades	SH	SD	SC	SS

a $\frac{1}{4}$ b $\frac{3}{4}$

5 The game is fair as the probability of Ricky winning is $\frac{1}{2}$ as he wins in 2 of the 4 outcomes: HHH, HHT, HTT, TTT

6 a $\frac{1}{36}$ b $\frac{10}{36}=\frac{5}{18}$ c $\frac{25}{36}$

7 a $\frac{4}{24}=\frac{1}{6}$ b $\frac{6}{24}=\frac{1}{4}$

		Die score					
		1	2	3	4	5	6
spinner score	1	11	12	13	14	15	16
	2	21	22	23	24	25	26
	3	31	32	33	34	35	36
	4	41	42	43	44	45	46

8 15

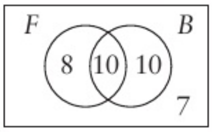
9 $1-\left(\frac{1}{2}\right)^n \geqslant 0.99$

$\therefore \left(\frac{1}{2}\right)^n \leqslant 0.01$

$\therefore n \geqslant 7$

Practice questions 18.2.2

1 **a**



bi, ci $\frac{18}{35}$ like fishing

bii, cii $\frac{8}{35}$ like fishing but not bowling

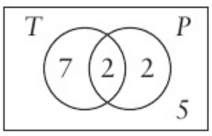
biii, ciii $\frac{17}{35}$ do not like fishing

2 **a** Walking, by bike, etc.

b

$P(F)$	$P(G \cap F)$	$P(G \cup F)$
Probability of using fossil fuels	Probability of using both fossil fuel and green power	Probability of using either fossil fuel and green power (or both)
$\frac{14}{24} = \frac{7}{12}$	$\frac{5}{24}$	$\frac{22}{24} = \frac{11}{12}$

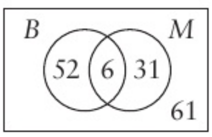
3 **a**



b $\frac{7}{16}$

c $\frac{5}{16}$

4 **a**

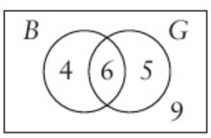


b $\frac{26}{75}$

c $\frac{61}{150}$

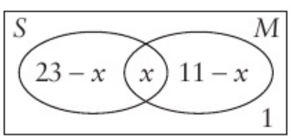
5 **a** 0.148 **b** 0.566

6



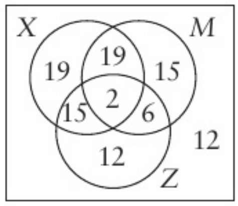
$\frac{3}{8}$

7



5 students do both.

8



a 12%

b 46%

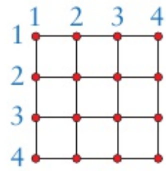
c 88%

Practice questions 18.2.3

1 $\frac{1}{4}$

2 $\frac{6}{36} = \frac{1}{6}$

3 **a** $\frac{1}{16}$

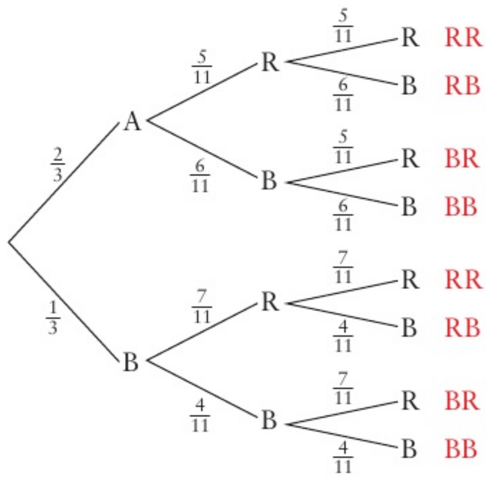


b $\frac{4}{16} = \frac{1}{4}$

c $\frac{3}{16}$

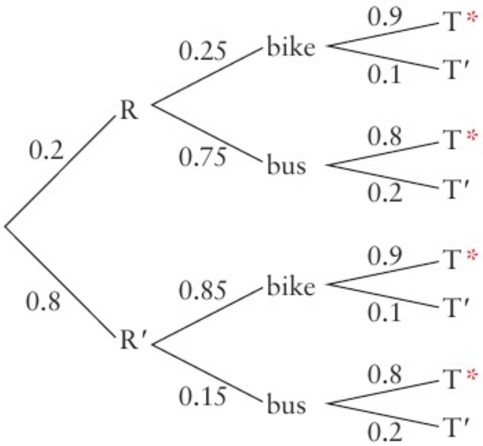
d $\frac{6}{16} = \frac{3}{8}$

4 The probability can be calculated as follows:



$$\frac{2}{3} \times \frac{5}{11} \times \frac{5}{11} + \frac{2}{3} \times \frac{6}{11} \times \frac{6}{11} + \frac{1}{3} \times \frac{7}{11} \times \frac{7}{11}$$
$$+ \frac{1}{3} \times \frac{4}{11} \times \frac{4}{11} = \frac{187}{363} = \frac{17}{33} = 0.515.$$

5 We can calculate the probability of Noa being at school on time on any given day by adding the probabilities of the 4 outcomes marked by a *:



$2 \times 0.25 \times 0.9 + 0.2 \times 0.75 \times 0.8 + 0.8 \times 0.85 \times 0.9 + 0.8 \times 0.15 \times 0.8 = 0.873$

6 a

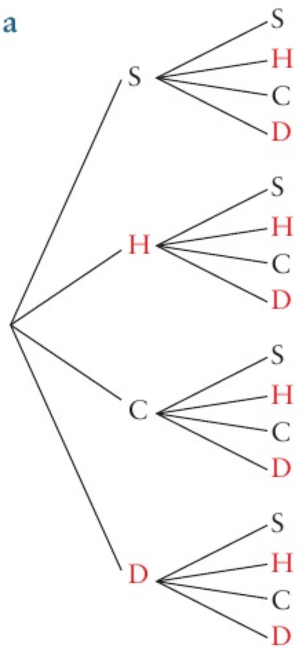
	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

b i $\frac{10}{36} = \frac{5}{18}$

ii $\frac{6}{36} = \frac{1}{6}$

iii $1 - \frac{12}{36} = \frac{24}{36} = \frac{2}{3}$

7 a



$P(SS) + P(HH) + P(CC) + P(DD) = 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}$

b $1 - \frac{1}{4} = \frac{3}{4}$

c $\frac{1}{2}$

Check your knowledge questions

1 a $\frac{11}{60}$ b $\frac{49}{60}$

c $\frac{5}{60} = \frac{1}{12}$ d $\frac{7}{60}$

2 a $\frac{6}{36} = \frac{1}{6}$ b $\frac{6}{36} = \frac{1}{6}$

c $\frac{3}{36} = \frac{1}{12}$ d $\frac{15}{36} = \frac{5}{12}$

3 a $\frac{8}{26} = \frac{4}{13}$ b 0

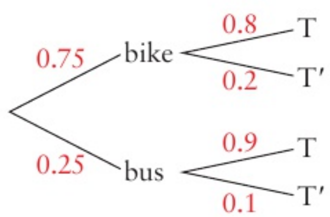
c $\frac{20}{26} = \frac{10}{13}$

4 $2^4 = 16$ possible outcomes:
HHHH, THHH, HTHH, HHTH, HHHT, TTHH, THTH, THHT, HTTH, HTHT, HHTT, HTTT, THTT, TTHT, TTTH, TTTT

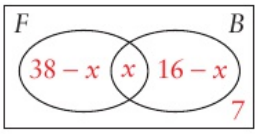
a $\frac{1}{16}$ b $\frac{6}{16} = \frac{3}{8}$ c $\frac{5}{16}$

5 a 0.56 b 0.06 c 0.94

6 a



b 0.825

7 a 

b 11 students play both sports.

8 Let B = brown eyes and G = glasses:

$P(B) = \frac{43}{80}, P(G) = \frac{35}{80} = \frac{7}{16}$

$\therefore P(B) \times P(G) = \frac{43}{80} \times \frac{7}{16} = 0.235$

However $P(B \cap G) = \frac{18}{80} = 0.225$

Therefore $P(B \cap G) \neq P(B) \times P(G)$ and therefore eye colour and wearing glasses are NOT independent.

Chapter 19 answers

Do you recall?

1 a 6 b 2 c 15.56 (2 d.p.)

2 a 4 b 4.39 (2 d.p.)

3 a $20 \leqslant x < 30$ b 25.18 (2 d.p.)

Practice questions 19.1

1 a Sample otherwise all the sim cards would be destroyed.

b Census as the population size is manageable.

c Sample as the population size is too large.

d Census as the population size is manageable.

2 a Random sampling – Advantage: it is easy to do. Disadvantage: it is not suitable for very large populations.

b Systematic sampling – Advantage: suitable for larger populations. Disadvantage: can introduce bias as the list is ordered.

c Stratified sampling – Advantage: it represents the population structure. Disadvantage: sometimes the strata can overlap.

d Convenience sampling – Advantage: it is easy to do. Disadvantage: it is easy to introduce bias.

e Quota sampling – Advantage: it is easy to do. Disadvantage: it is easy to introduce bias.

3 Calculate the number of each section that needs to be included in the sample to ensure the sample is in the same proportion. Calculated as in the table, with the rounded integer value in bold as the number in the sample.

Gender	Age Range (years)		
	0–15	16–19	20+
Males	$\frac{384}{12\,096} \times 200$ $= 6.35$ **6**	$\frac{811}{12\,096} \times 200$ $= 13.41$ **13**	$\frac{4750}{12\,096} \times 200$ $= 78.54$ **79**
Females	$\frac{328}{12\,096} \times 200$ $= 5.42$ **5**	$\frac{879}{12\,096} \times 200$ $= 14.53$ **15**	$\frac{4944}{12\,096} \times 200$ $= 81.75$ **82**

4 a i Kahoru should generate a random sample of 50 numbers between 1 and 4000. Using the Student Welfare spreadsheet he should select the corresponding student and send them a survey link.

ii As he has 4000 to choose from and wants to select 50 he should choose every $\frac{4000}{50}$ (80^{th}) data value from the spreadsheet.

b To ensure randomness he can use a random number generator to determine where to start and choose every 80^{th} data value from that point.

5 a One advantage is that the sample is easy to do, she can use the students in her group. One disadvantage is that she is introducing bias by just using her friends and not using a representative sample of the population.

b The risk of bias would be reduced if a different sampling method was used, for example a stratified sample of her grade rather than just asking her class.

6 a 90% accept

b 91.43% accept

c 60% reject and conduct a new survey

d 64% reject and conduct a new survey

Practice questions 19.2

1 a

$0 \leqslant x < 4$	$4 \leqslant x < 9$	$9 \leqslant x < 14$	$14 \leqslant x < 19$
24	32	3	11

b i $4 \leqslant x < 9$ ii 7.24 iii $4 \leqslant x < 9$

c The values correspond as they are similar values ≈ 7.

2 a $0 \leqslant P < 1$ b 1.97

c $1 \leqslant P < 2$

3 a $0 \leqslant P < 5$ b 8.47

c $5 \leqslant P < 10$.

d The mean value has increased over the time period from 1970 to 2010 and the modal group width has increased. The median has also increased over the time period. The data appears to show more countries have a higher percentage of students with a university education.

4 a 42.61 b $40 \leqslant P < 45$

c The mean and the median are similar in value which suggests the data is symmetrically spread around the centre. There is less frequency at either end of the age ranges, but this could be due to retirement or education.

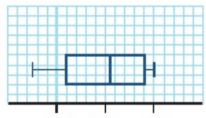
5 a 41.78

b $25 \leqslant P < 50$

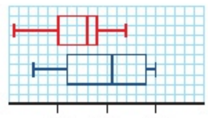
c The grouping is misleading because the younger age ranges group width is smaller, the proportion of these is higher than represented in the table.

Practice questions 19.3.1

1

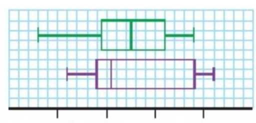


2



The second column data, represented by the red box plot, has data values that are less than the first column data, represented by the blue plot. The median values support this with 16 and 21. The range of the data is similar, 23 and 25. The IQR is less for the second column data than the first, 8 and 16, identifying less spread in the centre of the data for the second data set, represented by the red box plot.

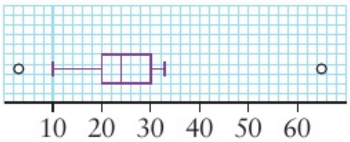
3



The second column data, represented by the green box plot, has data values that are less than the first column data, represented by the purple plot. The median values, however, do not support this with 25 and 21. This suggests the second set of data, represented by the green box plot is concentrated around the middle of the data values. The range of the data is similar, 32 and 30. The IQR is less for the second column data than the first, 13 and 20, identifying less spread in the centre of the data for the second data set, represented by the green box plot.

4 The key statistics are: minimum = 10, Q1 = 20, median = 24, Q3 = 30, maximum = 33

Outliers are 3 and 65

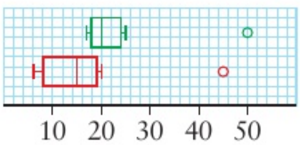


5 1st data set (red box plot): The key statistics are minimum = 6, Q1 = 8, median = 15, Q3 = 19, maximum = 20

Outlier is 45

2nd data set (green box plot): The key statistics are minimum = 17, Q1 = 18, median = 20, Q3 = 24, maximum = 25

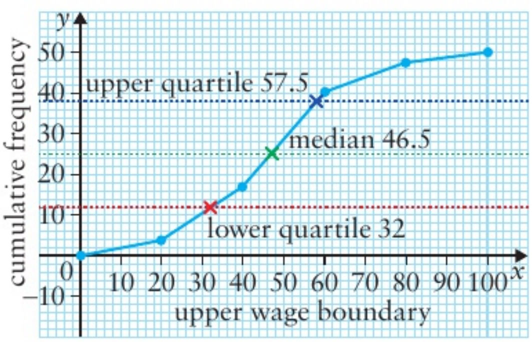
Outliers are 6 and 50



6 The key statistics are minimum = 10, Q1 = 22, median = 36, Q3 = 38, maximum = 42, no outliers.

Practice questions 19.3.2

1 a

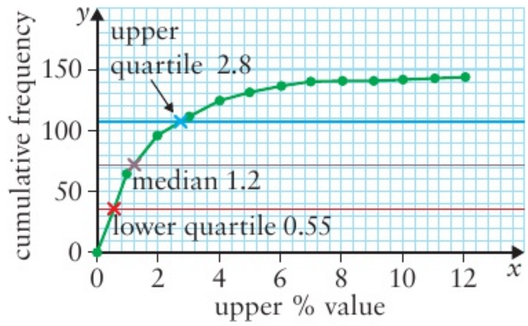


b median is approximately 46.5

c Lower quartile is approximately 32, upper quartile is approximately 57.5

d IQR = 25.5

2 a



b Median estimate is 1.2

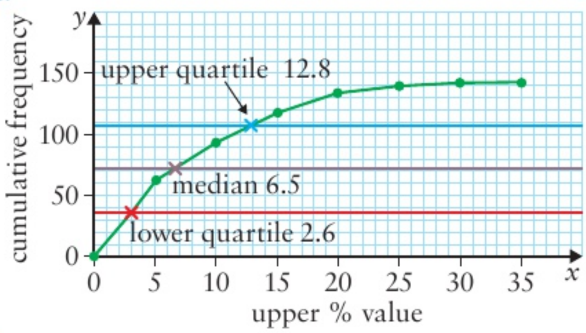
c The median estimated by this method, 1.2, is slightly smaller, but close in value to the median value for the midpoint method, 1.5.

d Upper quartile is 2.8, lower quartile is 0.55

e IQR is 2.25

f Outliers are any % population above 6.18, therefore data including the groups, $6.18 \leqslant P < 12$

3 a



b Median estimate is 6.5

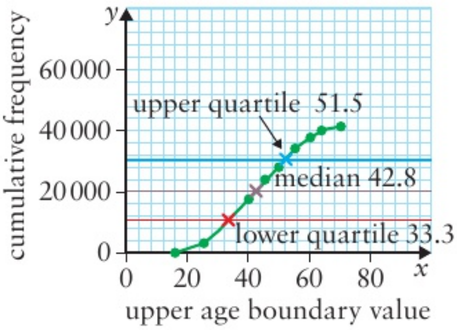
c The median estimated by this method, 6.5, is slightly smaller, but close in value to the median value for the midpoint method, 7.5. It is in the same group but towards the lower end.

d IQR = 12.8 − 2.9 = 9.9.

e Outliers are any % population above 27.65, therefore data including the groups, $27.65 \leqslant P < 35$

f The data values for 2010 show a general increase in the % of the population with university level education compared to 1970. The measures of central tendency have all increased in 2010, however the IQR has also increased.

4 a

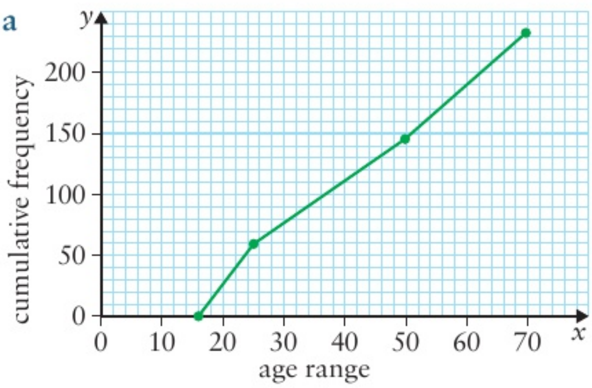


b Median estimate of the data is 42.8

c The median age estimated by this method, 42.8, is very close to the median value for the midpoint method, 42.5

d 20^{th} percentile is $\frac{20}{100} \times 40\,513 = 8102.6$th data value, using the graph this gives ~ 31,
80^{th} percentile is $\frac{80}{100} \times 40\,513 = 32\,410.4$th data value, this gives ~ 53.5.

5 a

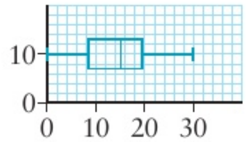


b Median estimate is 41.5

c 40^{th} percentile is $\frac{40}{100} \times 231.9 = 92.76$th data value, using the graph this gives ~35
60^{th} percentile is $\frac{60}{100} \times 231.9 = 139.14$th data value, using the graph this gives ~48.5

6

Key Statistics	
Minimum	0
Lower Quartile	8.5
Median	15.5
Upper Quartile	19.5
Maximum	30



7 a Minimum is \$10,000, maximum is \$70,000

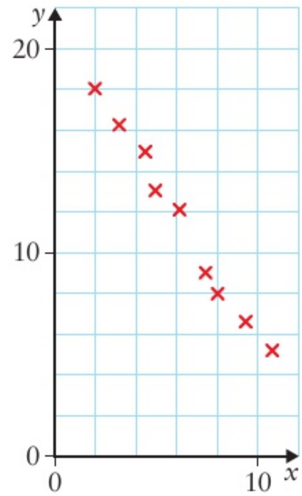
b median estimate is \$42,500

c less than \$30,000 is ~9 countries

d greater than \$60,000, therefore 35 − 31 = 4 countries

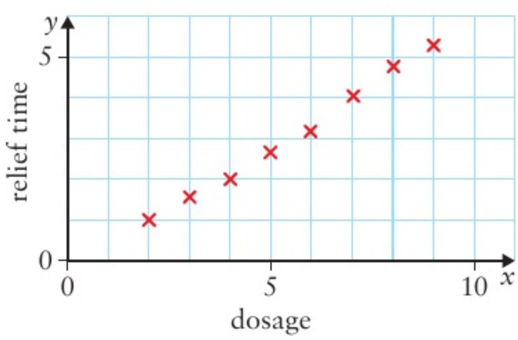
Practice questions 19.4.1

1



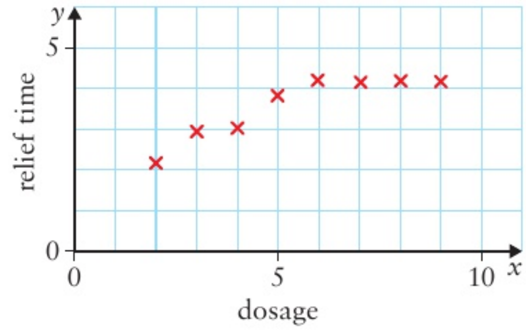
The y values decrease as the x values increase.

2 a



b As the dosage increases to 9 mg, the time a patient has relief from pain increases.

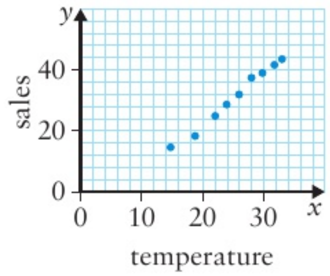
3 a



b As the dosage increases with time, the patient's relief from pain increases, however at 6mg and upwards, the relief appears to remain at 4.2 hours.

c Drug A lasts for just over an hour longer, however drug B lasts longer for a lower dosage. If the patient does not have adverse effects from the dosage level of drug A, then this is a better drug for longer lasting pain relief. If the patient does need a lower dose due to ill effects then drug B is better, but it may need to be administered more frequently.

4 a

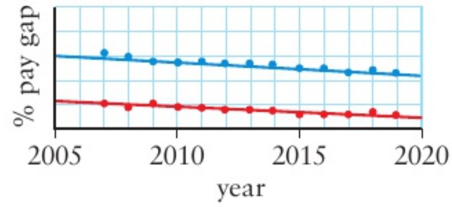


b As the temperature rises, the number of ice creams sold increases.

5 The graph appears to show that as the amount of cheese consumed increases, then the golf course revenue in the US also increases. However, this is a clear example of how an apparent relationship between numbers in a data set does not actually mean a relationship between the variables. We cannot claim that eating more cheese means golf course revenues increase.

Practice questions 19.4.3

1 a



Line of best fit for Norway is:

$y = -0.390x + 791.7$

Line of best fit for Japan is:

$y = -0.641x + 1317.5$

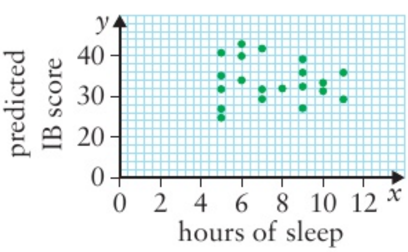
b Both countries have a pay gap that is decreasing with time, at the moment Norway's pay gap is lower than Japan's

c From the line of best fit, we can see that the rate of decrease for Japan's data is greater (-0.641) than that of Norway's (-0.390).

d Using the line of best fit, Norway should have a 0% pay gap in 2030 and Japan should be in 2055. However, this assumes the trend continues in the same way for both countries.

e Emily cannot claim there is a difference in trend for countries with female leaders because she has only explored the data for one country with a female leader and one country without a female leader. She can only comment on the difference between Norway and Japan.

2 a



b There appears to be no relationship between the data values.

c Elizabeth can conclude that there appears to be no relationship between the amount of sleep a student has and their predicted IB grade.

3 The data values appear to show a positive correlation. As the number of votes Trump received increases, so does the percentage of the population that own a rabbit. However, Theo cannot assume that the number of votes President Trump gains determines the number of rabbits people have. There is correlation between the data values, but Theo cannot claim causation.

4 a The number of popular votes in a state appears to be positively correlated with the number of Electoral College votes the state has. It appears to be a fair system as most of the data points are close to the line of best fit. This implies the Electoral College votes are in proportion to the population of the state.

b For Florida in 2008, using the equation of the line of best fit they should have: $V = 1.44(18.3) + 2.03$, 28.38, so 28 electoral college votes. In this case, Florida should have one more Electoral College vote. It appears unfair, however the population may have increased rapidly after the census to determine the number of Electoral College votes to be assigned.

c For the 2020 election Florida had 29 Electoral College votes which were allocated to President Trump, however the overall distribution of Electoral College votes was **306 Biden** to **232 Trump**, the additional 2 Electoral College votes did not affect the result of the 2020 election.

5 **President Trump gained 306** electoral college votes, **Ms Clinton gained 232**, however President Trump gained 62,984,825 individual people votes and Ms Clinton gained 65,853,516 individual votes. Exploring the breakdown of the votes it appears as if the states with the largest electoral votes also had close numbers of population votes, for example Texas has 38 electoral college votes, and the difference between the candidate's population votes was 807,179. So, Ms Clinton gained a lot of individual votes but did not win the state. For the states with a smaller number of electoral college votes, the percentage difference between the candidate's population votes was greater, for example for District of Columbia has 3 electoral college votes and Ms Clinton won 92.8% of the population votes. Ms Clinton gained a large share of the population votes, but President Trump won the close contests in the states with the larger number of electoral college votes.

An interesting political discussion about this can be found here:

Check your knowledge questions

1 a Sample, as the testing is destructive.

b Sample, as the population is too large to conduct a census easily.

c Census, the population is small enough to be able to ask everyone.

2 She would calculate the fraction of each group in the population and multiply this by the sample number. This will ensure that the sample proportion represents the population proportion.

	Girls	Boys
Grade 11	$\frac{65}{250} \times 50 = 13$	$\frac{70}{250} \times 50 = 14$
Grade 12	$\frac{54}{250} \times 50 = 10.8, 11$	$\frac{61}{250} \times 50 = 12.2, 12$

3 **a**

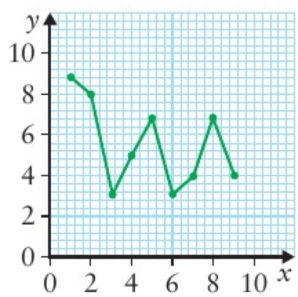
Number	1	2	3	4	5	6	7	8	9
Frequency	9	8	3	5	7	3	4	7	4

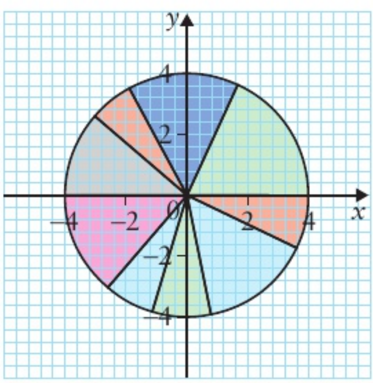
b 1

c 4.5

d 4.54

4 The data is discrete, but there is only one stem, 0, so a stem-and-leaf is not suitable, a bar chart, line graph, a pie chart or a box plot could be used:





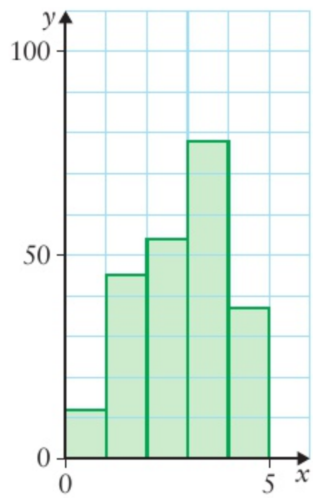
5 Median is 41, mode is 13

6 Red: Median is 45, mode is 24, mean is 47.18 (2 d.p.), range is 77.

Blue: Median is 77, mode is 77, mean is 72.74 (2 d.p.), range is 57.

The statistics values imply that the blue class has scored more than the red class as the mean, median and mode are greater for the blue class than the red. The measures of central tendency for the blue class are all similar, implying that the class scores are symmetrical, with an even spread around the centre; this is supported by the range for the blue class which is less than the red class.

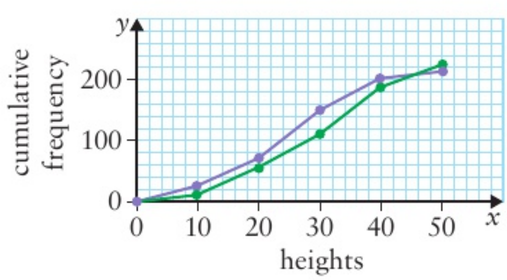
7



8

Height of plants	Frequency
$0 \leqslant P < 10$	24
$10 \leqslant P < 20$	48
$20 \leqslant P < 30$	78
$30 \leqslant P < 40$	51
$40 \leqslant P < 50$	15

9 **a**

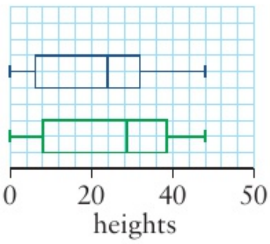


b Alex: median is 30.5, LQ is 19.5 UQ is 38, IQR is 18.5 (estimates from the graph)
Raul: median is 25, LQ is 16, UQ is 32, IQR is 16 (estimates from the graph)

c Below 32 cm there are approximately 125 plants.

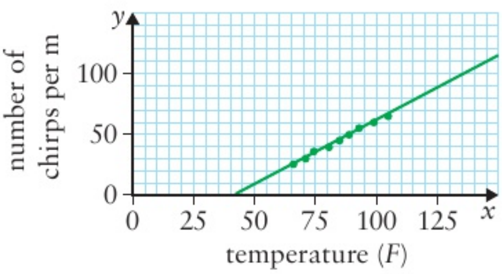
d Above 12 cm there are approximately $216 - 33 = 183$ plants.

10



Alex's plants appear to be slightly taller than Raul's plants as the median and quartile values are greater. The spread of the height values appears to be similar for both sets of plant.

11 a

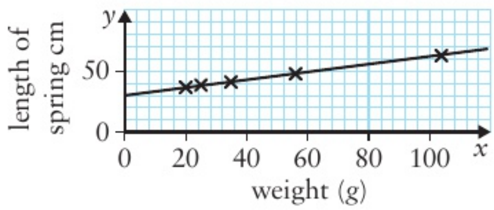


The line of best fit is: $C = 1.04413T - 43.52$ where T is the temperature (° F) and C is the number of cricket chirps

b As the temperature increases the number of chirps per minute also increases.

c This is an example of correlation and causation. It is a strong positive correlation because the scatter graph shows the data points lie close to the trend line. Researching the possible relationship between the two data values identifies this as a causation. The temperature outside directly affects the rate of chemical processes within the cricket that cause the chirping. A higher temperature means the process can happen at a greater rate and hence more chirps.

12 a



Line of best fit: $L = 0.0328w + 2.99$, where L is the length of the spring and w is the weight added.

b As the weight added increases the amount of stretch of the spring increases.

c This is an example of correlation and causation. It is a strong positive correlation because the scatter graph shows the data points lie close to the trend line. Researching identifies the established principle of Hooke's Law, therefore this is also a causation.

d The original length is when $w = 0$, therefore $L = 2.99$ cm.

e No. A weight of 1 kg is out of the range of values used to determine the line of best fit. When a value of 1000 g is substituted into the line of best fit it generates a length of 35.79 cm, this is not a reasonable stretch for a spring with initial length of 2.99 cm.

Chapter 20 answers

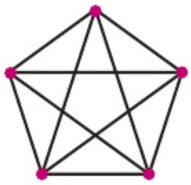
Do you recall?

1 a 3

b A complete graph, is a simple, connected graph, in which every vertex is connected to all other vertices.

2 True

3



4 $\dfrac{n(n-1)}{2}$

5 A subgraph is made out of a subset of the vertices and edges of the original graph.

6 2

Practice questions 20.1

1 a i 7 ii 5 iii 3, 2, 4, 2, 3

iv $14 = 2 \times 7$

b i 8 ii 7

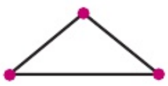
iii 3, 2, 2, 2, 2, 3, 2 iv $16 = 2 \times 8$

2 a subgraph with a cycle

b any tree, for instance

c with at least 1 disconnected vertex

d



3 a No, there is a cycle *ABCF*. Also there are 6 edges for 6 vertices, but a tree only needs 5 edges $(6 - 1)$.

b It needs to be connected, else the signal cannot reach the house (vertex).

c 1 edge can be removed: *AB*, *BC*, *AF* or *CF*.

4 Solution in Ted−ED video.

5

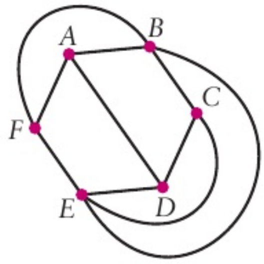
Diagram	E – edges	V – vertices	F – Faces
I	3	3	2
II	6	4	4
III	12	6	8

$E + 2 = V + F$ (or equivalent).

6 11

7 The vertices can be subdivided into odd $\{1, 3, 5, 7, 9\}$ and even $\{2, 4, 6, 8\}$. There are no edges connecting the vertices of either set. Note that the graph is not complete.

8 a subgraph with a cycle

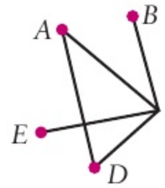


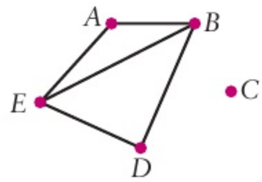
b $v = 6, e = 11, f = 7$

c $6 + 7 - 11 = 2$

d $v + f - e = 2$ or equivalent: Euler's Formula for planar graphs.

9 a, b

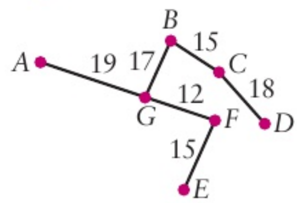




c K_5 the complete graph with 5 vertices.

Practice questions 20.2.1

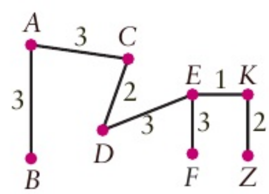
1 a



b, c Same result, which makes complete sense, as we are choosing the same edges, but in a different order.

2 Jason: 9 mins; Justin: 12 mins.

3



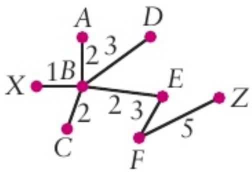
Total weight: $3 + 3 + 2 + 3 + 1 + 2 + 3 = 17$.

Alternative solution: FZ instead of EF.

4 a $E \to A \to C \to B \to D \to E$: $8+10+3+7+4 = 32$

b *EB*, *BC*, *ED*, *EA*: $2 + 3 + 4 + 8 = 17$

5 Shortest: $X \to A \to B \to C \to F \to E \to D \to Z$: total 28
Longest: $X \to C \to F \to E \to B \to A \to D \to Z$: total 32
Difference: $32 - 28 = 4$

6 

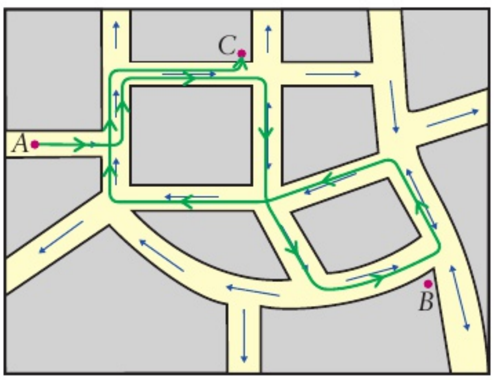
Total weight: $1 + 2 + 2 + 2 + 3 + 3 + 5 = 18$.

Practice questions 20.2.2

1 a $B \to C \to F$ or $B \to C \to D \to E \to F$
b $A \to B \to C \to F$, $A \to B \to C \to D \to E \to F$, $A \to C \to F$, $A \to C \to D \to E \to F$

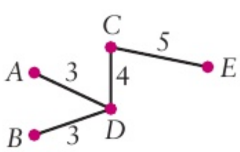
2 a 6 b 9
c i $A \to C$ ii $A \to D$, $A \to C \to D$

3 a Check
b Change of direction between G and H, F and H
c No, 4 vertices of odd degree (NB no Eulerian circuit, see MYP3 Ch.11).

4 

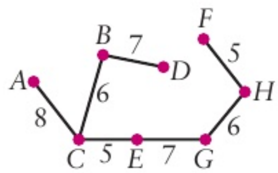
5 B and E: 5; C and D: 10.

Practice questions 20.3.1

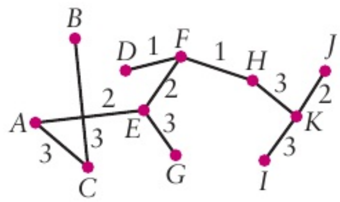
1 

2 Same result:

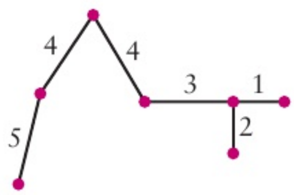
$A \xrightarrow{3} D \xrightarrow{3} B$, $D \xrightarrow{4} C \xrightarrow{5} E$

3 

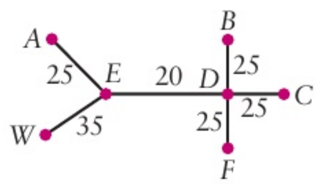
Total weight: 44

4 

Total weight: 23

5 

Total length: 19

6 

Total length: $25 + 20 + 25 + 25 + 25 + 35 = 155$ m

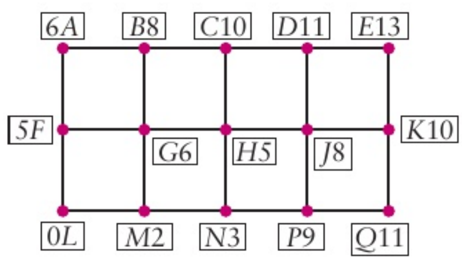
Practice questions 20.3.2

1 9

2 $A \to B \to C \to G$

3 $A \to H \to I \to F \to E$ or $A \to H \to F \to E$: 10 minutes

4 $A \to B \to C \to F \to G$ or $A \to B \to C \to E \to G$: 22 days

5 

The shortest distance is 13.

6 $A \rightarrow B \rightarrow E \rightarrow F \rightarrow D$ gives time of 32

Practice questions 20.4.1

1 5525

2 a 11648

b Algorithm should include working from right to left and carrying.

3 a 4698

b Algorithm should include working from right to left and borrowing.

4 Algorithm should include working from left to right, comparing card to the one next to it: swap when the card on the right is smaller, else keep existing order of cards. Cascading effect when a swap takes place.

5 Algorithm should distinguish slide and jump: slide when empty lily pad directly next to frog or newt and jump when empty lily pad on other side of frog or newt. Avoid getting 2 frogs or newts next to each other.

6 Algorithm should distinguish between odd and even amount of rings. E.g. if we start with 3 rings, label them 3, 2, 1 from big to small and pegs A, B, C from left to right. You can then record the moves as follows: C1, B2, B1, C3, A1, C2, C1.

Practice questions 20.4.2

1 a 12 b $\sqrt{5}$

c 2 d no (real) solutions

2 a $x = \pm 3$ b $x = \pm 9$

3 a $q = 7, r = 2$ b $q = 12, r = 5$

c $q = 14, r = 1$

4 a $37 = 5 \times 7 + 2$ b $89 = 7 \times 12 + 5$

c $183 = 13 \times 14 + 1$

5 93

6 No, the quotient can be smaller, e.g. 40 divided by 9. Any counterexample will do.

7 Yes, by definition the remainder can equal 0, 1, 2 when divided by 3, since $0 \leqslant r < 3$

8 p can be written as $3k$, $3k + 1$ or $3k + 2$. In the first case, p will be divisible by 3, in the second $p + 2$ and in the third $p + 4$.

Practice questions 20.4.3

1 a i 32: 1, 2, 4, **8**, 16, 32; 56: 1, 2, 4, 7, **8**, 14, 28, 56
so GCD(32, 56) = 8

ii $32 = 2^5, 56 = 2^3 \times 7$
so $\text{GCD}(32, 56) = 2^3 = 8$

iii $56 = 1(32) + 24$
$32 = 1(24) + 8$
$24 = 3(8) + 0$
so GCD(32, 56) = 8

b i 35: 1, 5, **7**, 35; 98: 1, 2, **7**, 14, 49, 98
so GCD(35, 98) = 7

ii $35 = 5 \times 7, 98 = 2 \times 7^2$
so GCD(35, 98) = 7

iii $98 = 2(35) + 28$
$35 = 1(28) + 7$
$28 = 4(7) + 0$
so GCD(35, 98) = 7

c i 70: 1, 2, 5, 7, 10, 14, **35**, 70; 105: 1, 3, 5, 7, 15, 21, **35**, 105
so GCD(70, 105) = 35

ii $70 = 2 \times 5 \times 7, 105 = 3 \times 5 \times 7.$
so $\text{GCD}(70, 105) = 5 \times 7 = 35$

iii $105 = 1(70) + 35$
$70 = 2(35) + 0$
so GCD(70, 105) = 35

2 GCD(182, 378) = 14

3 $1071 = 2(425) + 221$
$425 = 1(221) + 204$
$221 = 1(204) + 17$
$204 = 12(17) + 0$
GCD(425,1071) = 17

4 There are many different combinations: if $a = mg$, then $b = (m + 1)g$.

5 Agree, using $a + b = mg + ng = (m + n)g$.

6 $14 = 378 - 2(182)$

7 a GCD(124, 279) = 31, so 31 bouquets.

b $\frac{124}{31} = 4$ roses and $\frac{279}{31} = 9$ carnations.

Check your knowledge questions

1 a GCD(300, 195) = 15, so the largest square tile has a side of 15 cm.

b 260

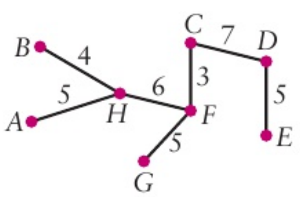
2 a

b 4

c 10

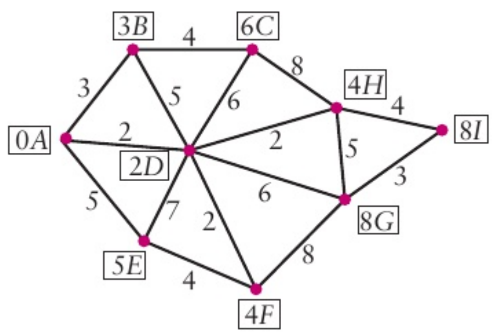
d Yes, that works as well: $n - 1$ edges leave each of the n vertices, so the sum of the degrees is $n(n - 1)$ and therefore the number of edges equals $\frac{n(n-1)}{2}$.

3



Total weight: $5 + 4 + 6 + 3 + 5 + 7 + 5 = 35$.

4



The shortest path is $A \rightarrow D \rightarrow H \rightarrow I$ with a distance of 8.

5 a $1080 = 2(384) + 312$

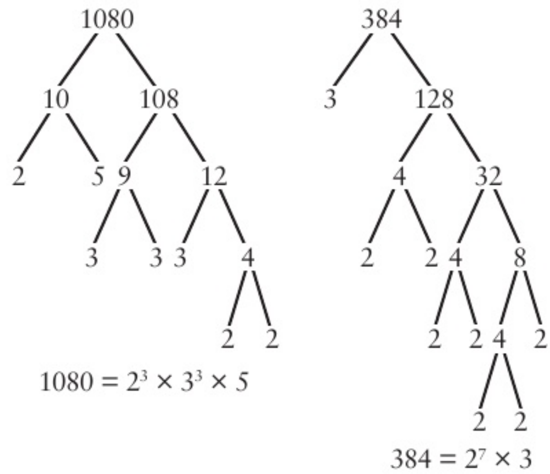
$384 = 1(312) + 72$

$312 = 4(72) + 24$

$72 = 3(24) + 0$

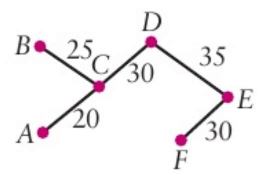
Therefore GCD(384, 1080) = 24

b



GCD(1080, 384) = $2^3 \times 3 = 24$.

6 a Total weight: 20 +25 + 30 + 35 + 30 = 140 m



b *ABCDEFCA* is the shortest and most logical route.

c Build path *AF*, which creates a complete circuit.

7 a *MONQSQPRPM*

b *QS* and *PR* would be travelled twice.

c Total hours: $6 + 4 + 7 + 4 + 4 + 9 + 5 + 5 + 6 = 50$ hours